The Keller-Segel model and the logarithmic Hardy-Littlewood-Sobolev inequality

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March 18, 2012

Kaust

A – The two-dimensional parabolic-elliptic Keller-Segel model

Outline

- Introduction: the two-dimensional parabolic-elliptic Keller-Segel model, the $M < 8\pi$ regime, scalings, etc
- The asymptotic behaviour of the solutions of the Keller-Segel model for small mass

References on the general theory of Keller-Segel systems:
[Horstmann]

Disclaimer: many references !

Introduction

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \ t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \ t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| \, u(t,y) \, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \, u(t,y) \, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x) \, dx = 0$

Blow-up

$M = \int_{\mathbb{R}^2} n_0\,dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2\, n_0\,dx < \infty$: blow-up in finite time
a solution u of

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\,\nabla v)$$

satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^2} |x|^2\, u(t,x)\;dx$$

$$= -\underbrace{\int_{\mathbb{R}^2} 2\,x\cdot\nabla u\;dx}_{-4\,M} + \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} \underbrace{\tfrac{2x\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\;dx\,dy}_{\frac{(x-y)\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\;dx\,dy}$$

$$= 4\,M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi$$

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \le 8\pi$: global existence [Jäger, Luckhaus], [JD, Perthame], [Blanchet, JD, Perthame], [Blanchet, Carrillo, Masmoudi]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u \, (\nabla \, (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \; dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \; dx$$

satisfies

$$\frac{d}{dt} F[u(t, \cdot)] = - \int_{\mathbb{R}^2} u \, |\nabla \, (\log u) - \nabla v|^2 \; dx$$

Log HLS inequality [Carlen, Loss]: F is bounded from below if $M < 8\pi$

The dimension $d = 2$

- In dimension d, the norm $L^{d/2}(\mathbb{R}^d)$ is critical. If $d = 2$, the mass is critical
- Scale invariance: if (u, v) is a solution in $\mathbb{R}^2$ of the parabolic-elliptic Keller and Segel system, then

$$\left(\lambda^2\, u(\lambda^2\, t, \lambda\, x)\,,\ v(\lambda^2\, t, \lambda\, x)\right)$$

  is also a solution
- For $M < 8\pi$, the solution vanishes as $t \to \infty$, but saying that "diffusion dominates" is not correct: to see this, study "intermediate asymptotics"

The existence setting

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2\,,\ t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\ t > 0 \\ u(\cdot, t = 0) = n_0 \ge 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx)\ , \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\ , \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx$ for any $t \ge 0$, see [Jäger-Luckhaus], [Blanchet, JD, Perthame]

$v = -\frac{1}{2\pi} \log|\cdot| * u$

Time-dependent rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[Blanchet, JD, Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means "intermediate asymptotics" in original variables:

$$\left\|u(x,t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right)\right\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M\,\frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- Radial symmetry [Naito]
- Uniqueness [Biler, Karch, Laurençot, Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [Blanchet, JD, Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M\to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[Joseph, Lundgreen] [JD, Stańczy]

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla\Big[n\,(\log n - x + \nabla c)\Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \; dx + \int_{\mathbb{R}^2} \frac{1}{2}\,|x|^2\, n \; dx - \frac{1}{2}\int_{\mathbb{R}^2} n\, c \; dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = -\int_{\mathbb{R}^2} n \left|\nabla\,(\log n) + x - \nabla c\right|^2 \, dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2\, n(\lambda\, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2)\; dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2}-1}{2}\,|x|^2\, n\; dx + \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\, \log\frac{1}{\lambda}\; dx\, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi}\right)}_{>0 \text{ if } M<8\pi} \log\lambda + \frac{\lambda^{-2}-1}{2}\int_{\mathbb{R}^2} |x|^2\, n\; dx$$

First result on rates

Theorem

There exists a positive constant M^* such that, for any initial data $n_0 \in L^2(n_\infty^{-1}\,dx)$ of mass $M < M^*$ satisfying the above assumptions, there is a unique solution $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ for any $\tau > 0$

Moreover, there are two positive constants, C and δ, such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \,\frac{dx}{n_\infty} \le C\,e^{-\,\delta\,t} \quad \forall\, t > 0$$

As a function of M, δ is such that $\lim_{M\to 0_+} \delta(M) = 1$

The condition $M \le 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- *Spectral gap* of a linearised operator $\mathcal{L}$

Proof of the first result on rates

- First step: the trap
- Second step: weighted H^1 estimates
- Third step: linearization and spectral gap
- Fourth step: collecting the estimates

The parabolic-elliptic Keller and Segel system

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First step: the trap

Decay Estimates of $u(t)$ in $L^\infty(\mathbb{R}^2)$

Lemma

For any $M < M_1$, there exists $C = C(M)$ such that, for any solution $u \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+_{\rm loc} \times \mathbb{R}^2)$

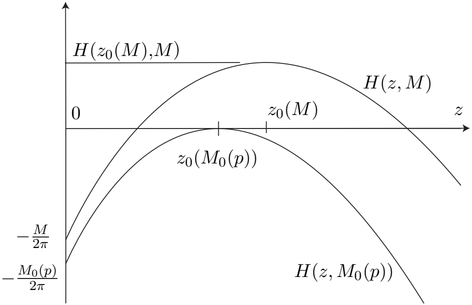
$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C\, t^{-1} \quad \forall\, t > 0$$

The *method of the trap...* prove that

$$H(\psi(t), M) \leq 0 \quad \text{where} \quad \psi(t) := t\,\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}$$

where $z \mapsto H(z, M)$ is a continuous function which is
- negative on $[0, z_1)$
- positive on (z_1, z_2) for some z_1, z_2 such that $0 < z_1 < z_2 < \infty$

ψ is continuous and $\psi(0) = 0 \Longrightarrow \psi(t) \leq z_1 \leq z_0(M)$ for any $t \geq 0$ if $H(z_0(M), M) = \sup_{z \in [z_1, z_2]} H(z, M) \geq 0$



The *method of the trap* amounts to prove that $H(z,M) \leq 0$ implies that $z = \psi(t)$ is bounded by $z_0(M)$ as long as $H(z_0(M),M) > 0$

Duhamel's formula:

$$u(x, t_0 + t) - \int_{\mathbb{R}^2} N(x - y, t)\, u(y, t_0)\, dy$$
$$= \int_0^t \int_{\mathbb{R}^2} N(x - y, t - s)\, \nabla \cdot [u(y, t_0 + s)\, \nabla v(y, t_0 + s)]\; dy\, ds$$

where $N(x, t) = \frac{1}{4\pi t}\, e^{-|x|^2/(4t)}$. Let $\kappa_\sigma = \|\partial N/\partial x_i\, (\cdot, 1)\|_{L^\sigma(\mathbb{R}^2)}$

$$\|u(\cdot, t_0 + t)\|_{L^\infty(\mathbb{R}^2)} - \frac{1}{4\pi t}\, \|u(\cdot, t_0)\|_{L^1(\mathbb{R}^2)}$$
$$\leq \sum_{i=1,2} \int_0^t \left\| \frac{\partial N}{\partial x_i}(\cdot, t - s) * \left[\left(u\, \frac{\partial v}{\partial x_i} \right)(\cdot, t_0 + s) \right] \right\|_{L^\infty(\mathbb{R}^2)} ds$$
$$\leq \sum_{i=1,2} \kappa_\sigma \int_0^t (t - s)^{-(1-\frac{1}{\sigma})-\frac{1}{2}} \left\| \left(u\, \frac{\partial v}{\partial x_i} \right)(\cdot, t_0 + s) \right\|_{L^\rho(\mathbb{R}^2)} ds$$

HLS inequality + Hölder and take $t_0 = t$

$$2t\,\|u(\cdot,2t)\|_{L^\infty(\mathbb{R}^2)} - \frac{M}{2\pi}$$
$$\le \frac{2\,\kappa_\sigma\,C_{\rm HLS}}{\pi}\,M^{\frac{1}{p}+\frac{1}{r}}\,t\int_0^t (t-s)^{\frac{1}{\sigma}-\frac{3}{2}}\,(t+s)^{\frac{1}{p}+\frac{1}{r}-2}\left[\psi(t)\right]^{2-\frac{1}{p}-\frac{1}{r}}ds$$

with $\psi(t) := \sup_{0\le s\le t} 2s\,\|u(\cdot,2s)\|_{L^\infty(\mathbb{R}^2)}$ and

$$t\int_0^t (t-s)^{\frac{1}{\sigma}-\frac{3}{2}}\,(t+s)^{\frac{1}{p}+\frac{1}{r}-2}ds \;=\; \frac{\sigma}{2-\sigma}$$

$$\psi(t) \le \frac{M}{2\pi} + C_0\,\big(\psi(t)\big)^\theta \quad \text{with} \quad C_0 = \frac{2\,\kappa_\sigma\,C_{\rm HLS}}{\pi}\,M^{\frac{1}{p}+\frac{1}{r}}\,\frac{\sigma}{2-\sigma}\;, \quad \theta = 2-\frac{1}{p}-\frac{1}{r}$$

Choice: $H(z,M) = z - C_0\,z^\theta - M/(2\pi)$

How small is the mass ?

The exponents σ, ρ, p, q and r are related by

$$\begin{cases} \frac{1}{\sigma} + \frac{1}{\rho} = 1 \,, & 1 < \sigma < 2 \\ \frac{1}{p} + \frac{1}{q} = \frac{1}{\rho} \,, & p \,,\; q > 2 \\ \frac{1}{r} - \frac{1}{q} = \frac{1}{2} \,, & r > 1 \end{cases}$$

For the choice $r = 4/3$, $q = 4$, $C_{\rm HLS} = 2\sqrt{\pi}$

$$C_0 = \tfrac{4\kappa_\sigma}{\sqrt{\pi}} \, M^{\frac{1}{p}+\frac{1}{4}} \, \frac{\sigma}{2-\sigma} \text{ with } \sigma = \tfrac{4p}{3p-4}$$

... there exists $M_0(p)$ such that $H(z_0(M), M) > 0$ if and only if $M < M_0(p)$ and $\sup_{p\in(4,+\infty)} M_0(p) = \lim_{p\to+\infty} M_0(p) \approx 0.822663$ $< 8\pi \approx 25.1327$

Other norms: interpolation

Corollary

For any mass $M < M_1$ *and all* $p \in [1, \infty]$, *there exists a positive constant* $C = C(p, M)$ *with* $\lim_{M \to 0_+} C(p, M) = 0$, *such that*

$$\|u(t)\|_{L^p(\mathbb{R}^2)} \leq C\, t^{-(1-\frac{1}{p})} \quad \forall\, t > 0$$

*Remark*The above rates are optimal as can easily be checked using the self-similar solutions $(n_\infty,\ c_\infty)$

Second step: weighted H^1 estimates

L^p and H^1 estimates in the self-similar variables

Consider the solution of

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t>0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t>0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

For any $p \in (1, \infty]$

$$\|n(t)\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \forall\ t>0$$

for some positive constant C_1, and for $p>2$

$$2\pi\ \|\nabla c(t)\|_{L^\infty} \leq \underbrace{\sup_{x\in\mathbb{R}^2} \int_{|x-y|\geq 1} \frac{n(t,y)}{|x-y|}\,dy}_{\leq M} + \underbrace{\sup_{x\in\mathbb{R}^2} \int_{|x-y|\leq 1} \frac{n(t,y)}{|x-y|}\,dy}_{\leq \left(2\pi\,\frac{p-1}{p-2}\right)^{\frac{p}{p-1}}\,\|n\|_{L^p(\mathbb{R}^2)}}$$

$$\|n(t)\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \text{and} \quad \|\nabla c(t)\|_{L^\infty(\mathbb{R}^2)} \leq C_2 \quad \forall\, t > 0$$

Lemma

The constants C_1 and C_2 depend on M and are such that

$$\lim_{M \to 0_+} C_i(M) = 0 \quad i = 1\,,\ 2$$

Exponential weights

With $K = K(x) = e^{|x|^2/2}$, let us rewrite the equation for n as

$$\frac{\partial n}{\partial t} - \frac{1}{K}\,\nabla\cdot(K\,\nabla n) = -\nabla c\cdot\nabla n + 2n + n^2$$

Proposition

For any mass $M \in (0, M_1)$, there is a positive constant C such that

$$\|n(t)\|_{H^1(K)} \le C \quad \forall\, t > 0$$

First ingredient [M. Escobedo and O. Kavian]: for any $q > 2$ and $\varepsilon > 0$, there exists a positive constant $C(\varepsilon, q)$ such that

$$\int_{\mathbb{R}^2} n^2\;K\,dx \le \varepsilon \int_{\mathbb{R}^2} |\nabla n|^2\;K\,dx + C(\varepsilon, q)\,\|n\|^2_{L^q(\mathbb{R}^2)}$$

$L^2(K)$ estimate

Multiply the equation by $n\,K$ and integrate by parts

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^2}|n|^2\;K\,dx+\int_{\mathbb{R}^2}|\nabla n|^2\;K\,dx$$
$$=-\int_{\mathbb{R}^2}n\nabla c\cdot\nabla n\;K\,dx+2\int_{\mathbb{R}^2}n^2\;K\,dx+\int_{\mathbb{R}^2}n^3\;K\,dx$$
$$\le\varepsilon\int_{\mathbb{R}^2}|\nabla n|^2\;K\,dx+C$$

and so

$$\frac{1}{2}\,\frac{d}{dt}\int_{\mathbb{R}^2}|n|^2\;K\,dx+(1-\varepsilon)\underbrace{\int_{\mathbb{R}^2}|\nabla n|^2\;K\,dx}_{\ge\int_{\mathbb{R}^2}|n|^2\;K\,dx}\le C$$

(expand the square in $\int_{\mathbb{R}^2}|\nabla(n\,K)|^2\,K^{-1}\,dx\ge 0$)

$H^1(K)$ estimate (1/2)

Let $S(t)$ be the semi-group generated by $-K^{-1}\,\nabla\cdot(K\;\nabla\cdot)$ on $L^2(K)$

$$n(t,x) = S(t)\,n_0(x) - \int_0^t S(t-s)\,(\nabla c\cdot\nabla n)(s)\;ds + \int_0^t S(t-s)\,(2n+n^2)(s)\;ds$$

$$\|n(t)\|_{H^1(K)} - \|S(t)\,n_0\|_{H^1(K)}$$
$$\le \int_0^t \|S(t-s)\,(\nabla c\cdot\nabla n)(s)\|_{H^1(K)}\;ds + \int_0^t \|S(t-s)\,(2n+n^2)(s)\|_{H^1(K)}\;ds$$

Second ingredient: $\|S(t)\,h\|_{H^1(K)} \le \kappa\,(1+t^{-1/2})\,\|h\|_{L^2(K)}$

$$\frac{1}{\kappa}\left(\|n(t)\|_{H^1(K)} - \|S(t)\,n_0\|_{H^1(K)}\right)$$
$$\le C_2\int_0^t \left(1+\tfrac{1}{\sqrt{t-s}}\right)\|\nabla n(s)\|_{L^2(K)}\;ds + (2+C_1)\int_0^t \left(1+\tfrac{1}{\sqrt{t-s}}\right)\|n(s)\|_{L^2(K)}$$

$H^1(K)$ estimate (2/2)

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)} \le \left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1 + C_3\int_0^t\left(1+\tfrac{1}{\sqrt{t-s}}\right)\|n(s+\tau)\|_{H^1(K)}\,ds$$

Let $H(T)=\sup_{t\in(0,T)}\int_0^t\left(1+\frac{1}{\sqrt{t-s}}\right)\|n(s+\tau)\|_{H^1(K)}\,ds$ and choose $T>0$ such that $\frac{1}{2\kappa}=C_3\int_0^T\left(1+\frac{1}{\sqrt{T-s}}\right)ds=C_3\big(T+2\sqrt{T}\big)$

$$\frac{1}{\kappa}\,H(T)\le\left(\pi+4\sqrt{T}+T\right)\,C_1+\frac{1}{2\kappa}\,H(T)\implies H(T)\le 2\left(\pi+4\sqrt{T}+T\right)\kappa\,C$$

For any $t\in(0,T)$

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)}\le\left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1+C_3\,H(T)\le\left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1+2\left(\pi+4\sqrt{T}+T\right.$$

Conclusion: bound in $H^1(K)$

The estimate

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)} \leq \left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1+C_3\,H(T) \leq \left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1+2\left(\pi+4\sqrt{T}+T\right)$$

for any $t \in (0,T)$ gives a bound on $\|n(T+\tau)\|_{H^1(K)}$ for any $\tau > 0$

Lemma

$$\|n(t)\|_{H^1(K)} \leq C\,\max\left\{1, \frac{\sqrt{T}}{\sqrt{t}}\right\} \quad \forall\; t > 0$$

Actually $n(t)$ can be bounded also in $H^1(n_\infty^{-1})$ but further estimates are needed...

Third step: linearization and spectral gap

A linearized operator

Introduce f and g defined by

$$n(x,t) = n_\infty(x)(1 + f(x,t)) \quad \text{and} \quad c(x,t) = c_\infty(x)(1 + g(x,t))$$

(f, g) is solution of the non-linear problem

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mathcal{L}(t,x,f,g) = -\dfrac{1}{n_\infty} \nabla \cdot [f\, n_\infty \nabla (g\, c_\infty)] & x \in \mathbb{R}^2\,,\ t > 0 \\ -\Delta(c_\infty\, g) = f\, n_\infty & x \in \mathbb{R}^2\,,\ t > 0 \end{cases}$$

where $\mathcal{L}$ is the linear operator given by

$$\mathcal{L}(t,x,f,g) = \frac{1}{n_\infty} \nabla \cdot [n_\infty \nabla (f - g\, c_\infty)]$$

The conservation of mass is replaced here by $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$

A spectral gap estimate

Proposition

For any $M \in (0, M_2)$, for any $f \in H^1(n_\infty\,dx)$ such that

$$\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0 \quad \Longrightarrow \quad \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx \geq \Lambda(M) \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

for some $\Lambda(M) > 0$ and $\lim_{M\to 0_+} \Lambda(M) = 1$

Let $h = \sqrt{n_\infty}\, f = \sqrt{\lambda}\, e^{-|x|^2/4 + c_\infty/2}\, f$

$$\lambda\,|\nabla f|^2\, n_\infty = |\nabla h|^2 + \frac{|x|^2}{4}\, h^2 + \frac{1}{4}\,|\nabla c_\infty|^2\, h^2 + h\,\nabla h \cdot (x - \nabla c_\infty) - \frac{1}{2}\, x \cdot \nabla c_\infty\, h^2$$

(integrations by parts)

$$\int_{\mathbb{R}^2} h\,\nabla h \cdot x\, dx = -\int_{\mathbb{R}^2} h^2\, dx$$
$$\int_{\mathbb{R}^2} h\,\nabla h \cdot \nabla c_\infty\, dx = \tfrac{1}{2} \int_{\mathbb{R}^2} h^2\,(-\Delta c_\infty)\, dx \leq \tfrac{1}{2}\, \|n_\infty\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} h^2\, dx$$
$$\tfrac{1}{2} \int_{\mathbb{R}^2} x \cdot \nabla c_\infty\, h^2\, dx \leq \tfrac{\sigma^2 - 1}{\sigma^2} \int_{\mathbb{R}^2} \tfrac{|x|^2}{4}\, h^2\, dx + \tfrac{1}{4}\, \tfrac{\sigma^2}{\sigma^2 - 1} \int_{\mathbb{R}^2} |\nabla c_\infty|^2\, h^2\, dx$$

$H^1(n_\infty^{-1})$ estimate

Assume that $n_0/n_\infty \in L^2(n_\infty)$
There exists a constant $C > 0$ such that

$$|x| > 1 \quad \Longrightarrow \quad \Big| c_\infty + M/(2\pi) \log|x| \Big| \le C$$

$n_\infty\, K = e^{c_\infty}$ behaves like $O(|x|^{-M/(2\pi)})$ as $|x| \to \infty$

$$\frac{\partial n}{\partial t} - n_\infty\, \nabla \cdot \left(\frac{1}{n_\infty} \nabla n \right) = (\nabla c_\infty - \nabla c) \cdot \nabla n + 2n + n^2$$

Corollary

If $M < M_2$, then any solution n is bounded in

$$L^\infty(\mathbb{R}^+, L^2(n_\infty^{-1}\, dx)) \cap L^\infty((\tau, \infty), H^1(n_\infty^{-1}\, dx))$$

for any $\tau > 0$

Fourth step: collecting the estimates

Proof of the exponential rate of convergence

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mathcal{L}(t,x,f,g) = -\dfrac{1}{n_\infty}\,\nabla\cdot\left[f\,n_\infty\,\nabla\,(g\,c_\infty)\right] & x\in\mathbb{R}^2\,,\ t>0 \\ -\Delta(c_\infty\,g) = f\,n_\infty & x\in\mathbb{R}^2\,,\ t>0 \end{cases}$$

Multiply by $f\,n_\infty$ and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|f|^2\,n_\infty\;dx + \int_{\mathbb{R}^2}|\nabla f|^2\,n_\infty\;dx$$
$$= \underbrace{\int_{\mathbb{R}^2}\nabla f\cdot\nabla\,(g\,c_\infty)\;n_\infty\;dx}_{=\mathrm{I}} + \underbrace{\int_{\mathbb{R}^2}\nabla f\,\cdot\,\nabla(g\,c_\infty)\,f\,n_\infty\;dx}_{=\mathrm{II}}$$

Cauchy-Schwarz' inequality

$$\mathrm{I} = \int_{\mathbb{R}^2}\nabla f\cdot\nabla\,(g\,c_\infty)\;n_\infty\;dx \le \|\nabla f\|_{L^2(n_\infty\,dx)}\,\|\nabla(g\,c_\infty)\|_{L^2(n_\infty\,dx)}$$

first term

Hölder's inequality (with $q > 2$)

$$\|\nabla(g\,c_\infty)\|_{L^2(n_\infty\,dx)} \le M^{1/2-1/q}\,\|n_\infty\|_{L^\infty(\mathbb{R}^2)}^{1/q}\,\|\nabla(g\,c_\infty)\|_{L^q(\mathbb{R}^2)}$$

HLS inequality (with $1/p = 1/2 + 1/q$)

$$\|\nabla(g\,c_\infty)\|_{L^q(\mathbb{R}^2)} \le \frac{1}{2\pi}\left(\int_{\mathbb{R}^2}\left|(f\,n_\infty)*\frac{1}{|\cdot|}\right|^q dx\right)^{\frac{1}{q}} \le \frac{C_{\rm HLS}}{2\pi}\,\|f\,n_\infty\|_{L^p(\mathbb{R}^2)}$$

Hölder's inequality: $\|f\,n_\infty\|_{L^p(\mathbb{R}^2)} \le \|f\|_{L^2(n_\infty\,dx)}\,\|n_\infty\|_{L^{q/2}(\mathbb{R}^2)}^{1/2}$

$$\mathrm{I} = \int_{\mathbb{R}^2}\nabla f\cdot\nabla(g\,c_\infty)\,f\,n_\infty\,dx \le C_*(M)\,\|f\|_{L^2(n_\infty\,dx)}\,\|\nabla f\|_{L^2(n_\infty\,dx)}$$

$$C_*(M) := C_{\rm HLS}\,(2\pi)^{-1}\,M^{1/2-1/q}\,\|n_\infty\|_{L^{q/2}(\mathbb{R}^2)}^{1/2}\,\|n_\infty\|_{L^\infty(\mathbb{R}^2)}^{1/q} \to 0 \text{ as } M \to 0$$

Use $g\,c_\infty = c - c_\infty$ and the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^2} \nabla f \cdot \nabla (g\,c_\infty)\, f\, n_\infty\, dx \le \underbrace{\|\nabla c - \nabla c_\infty\|_{L^\infty(\mathbb{R}^2)}}_{\le 2\,C_2(M) \searrow 0} \|f\|_{L^2(n_\infty\,dx)} \|\nabla f\|_{L^2(n_\infty\,dx)}$$

Spectral gap estimate

$$\underbrace{\sqrt{\Lambda(M)}}_{\to 1} \|f\|_{L^2(n_\infty\,dx)} \le \|\nabla f\|_{L^2(n_\infty\,dx)}$$

With $\gamma(M) := \frac{C_*(M)+2\,C_2(M)}{\sqrt{\Lambda(M)}} \searrow 0$,

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx \le -\,[1-\gamma(M)] \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx$$

Uniqueness

If n_1 and n_2 are two solutions in $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ for any $\tau > 0$, with $f = (n_2 - n_1)/n_\infty$ we also get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx \le -\,[1-\gamma(M)]\,\Lambda(M)\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

As a consequence, if the initial condition is the same, then $n_1 = n_2$

B – Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \leq \mathsf{C}_{p,d} \, \|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)} \, \|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}$$

with $\theta = \theta(p) := \frac{p-1}{p} \, \frac{d}{d+2-p\,(d-2)}$

- $1 < p \leq \frac{d}{d-2}$ if $d \geq 3$
- $1 < p < \infty$ if $d = 2$

[M. del Pino, J.D.] equality holds in if $f = F_p$ with

$$F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

and that all extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling.

- If $d \geq 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality [T. Aubin, G. Talenti]
- When $p \to 1$, we recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form [F. Weissler]
- If $d = 2$ and $p \to \infty$...

Onofri's inequality as a limit case

When $d = 2$, Onofri's inequality can be seen as an endpoint case of the family of the Gagliardo-Nirenberg inequalities [J.D.]

Proposition

[J.D.] *Assume that* $g \in \mathcal{D}(\mathbb{R}^d)$ *is such that* $\int_{\mathbb{R}^2} g \, d\mu = 0$ *and let*

$$f_p := F_p(1 + \frac{g}{2p})$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$, *and* $d\mu(x) := \mu(x)\,dx$, *we have*

$$1 \le \lim_{p\to\infty} \mathsf{C}_{p,2} \frac{\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx}}{\int_{\mathbb{R}^2} e^g\,d\mu}$$

The standard form of the euclidean version of Onofri's inequality is

$$\log\left(\int_{\mathbb{R}^2} e^g\,d\mu\right) - \int_{\mathbb{R}^2} g\,d\mu \le \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx$$

Legendre duality: Onofri and log HLS

Legendre's duality: $F^*[v] := \sup \left(\int_{\mathbb{R}^d} u\, v\, dx - F[u] \right)$

$$F_1[u] := \log \left(\int_{\mathbb{R}^2} e^u \, d\mu \right) \quad \text{and} \quad F_2[u] := \frac{1}{16\,\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u\, \mu \, dx$$

Onofri's inequality amounts to $F_1[u] \le F_2[u]$ with $d\mu(x) := \mu(x)\, dx$, $\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2}$

Proposition

For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} v\, dx = 1$, such that $v \log v$ and $(1 + \log |x|^2)\, v \in \mathrm{L}^1(\mathbb{R}^2)$, we have

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v \log \left(\frac{v}{\mu} \right) dx - 4\,\pi \int_{\mathbb{R}^2} (v - \mu)\,(-\Delta)^{-1}(v - \mu)\, dx \ge 0$$

[Carlen-Loss, Beckner, Calvez-Corrias]

A puzzling result of Carlen, Carrillo and Loss ($d \geq 3$)

[E. Carlen, J.A. Carrillo and M. Loss] The fast diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with exponent $m = d/(d+2)$, when $d \geq 3$, is such that

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

obeys to

$$\frac{1}{2}\,\frac{d}{dt}\mathsf{H}_d[v(t,\cdot)] = \frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \tfrac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d\,\|u\|^{4/(d-1)}_{\mathrm{L}^{q+1}(\mathbb{R}^d)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^{2q}_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

with $u = v^{(d-1)/(d+2)}$ and $q = \frac{d+1}{d-1}$. If $\frac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d = (\mathsf{C}_{q,\,d})^{2q}$, the r.h.s. is nonnegative. Optimality is achieved simultaneously in both functionals (Barenblatt regime): the Hardy-Littlewood-Sobolev inequalities can be improved by an integral remainder term

... and the two-dimensional case

Recall that $(-\Delta)^{-1} v = G_d * v$ with

- $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$ if $d \ge 3$
- $G_2(x) = \frac{1}{2\,\pi}\,\log|x|$ if $d = 2$

Same computation in dimension $d = 2$ with $m = 1/2$ gives

$$\frac{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}{8}\,\frac{d}{dt}\left[\frac{4\,\pi}{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}\int_{\mathbb{R}^2} v\,(-\Delta)^{-1} v\;dx - \int_{\mathbb{R}^2} v\,\log v\;dx\right]$$
$$= \|u\|^4_{\mathrm{L}^4(\mathbb{R}^2)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} - \pi\,\|v\|^6_{\mathrm{L}^6(\mathbb{R}^2)}$$

The r.h.s. is one of the Gagliardo-Nirenberg inequalities ($d = 2$, $q = 3$): $\pi\,(\mathsf{C}_{3,2})^6 = 1$
The l.h.s. is bounded from below by the logarithmic Hardy-Littlewood-Sobolev inequality and achieves its minimum if $v = \mu$ with

$$\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2} \quad \forall\, x \in \mathbb{R}^2$$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall \, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \; dx \quad \forall \, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$. Can we recover this using a nonlinear flow approach ? Can we improve it ?

Keller-Segel model: another motivation [Carrillo, Carlen and Loss] and [Blanchet, Carlen and Carrillo]

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\;dx + \mathsf{S}_d\left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\;dx\right)^{\frac{2}{d}}\int_{\mathbb{R}^d} \nabla v^m\cdot\nabla v^{\frac{d-2}{d+2}}\;dx$$

where $v = v(t,\cdot)$ is a solution of (**??**). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2d}{d+2}$

A first statement

Proposition

[J.D.] *Assume that* $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (**??**) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\;dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (**??**) if v is optimal for (**??**)

By integrating along the flow defined by (**??**), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (**??**), but only when $d \geq 5$ for integrability reasons

Theorem

[J.D.] *Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. delPino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]
For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot)-1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Improved inequality: proof (1/2)

$\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa\,T = \frac{2\,d}{d+2}\,\frac{T}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2}$$

Improved inequality: proof (2/2)

By the **Cauchy-Schwarz inequality**, we have

$$\frac{\mathsf{J}'^2}{(m+1)^2} = \|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2$$
$$\le \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx = Cst\,\mathsf{J}''\,\mathsf{J}$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q} - 1)\,, \quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \le \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \le 0$$

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1 - e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, the proof is completed □

$$\mathrm{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\;dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log\left(v/\mu\right) \quad t > 0\,, \quad x \in \mathbb{R}^2$$

Proposition

If $v = \mu\, e^{u/2}$ *is a solution with nonnegative initial datum* v_0 *in* $\mathrm{L}^1(\mathbb{R}^2)$ *such that* $\int_{\mathbb{R}^2} v_0\;dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ *and* $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, *then*

$$\begin{aligned}\frac{d}{dt}\mathrm{H}_2[v(t,\cdot)] &= \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\;dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\;d\mu\\ &\geq \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\;dx + \int_{\mathbb{R}^2} u\;d\mu - \log\left(\int_{\mathbb{R}^2} e^u\;d\mu\right) \geq 0\end{aligned}$$

C: Keller-Segel model

1. Small mass results
2. Spectral analysis
3. Collecting estimates: towards exponential convergence

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

A parametrization of the solutions and the linearized operator

[Campos, JD]

$$-\Delta c = M\,\frac{e^{-\frac{1}{2}\,|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,|x|^2+c}\,dx}$$

Solve

$$-\phi'' - \frac{1}{r}\,\phi' = e^{-\frac{1}{2}\,r^2+\phi}\,, \quad r > 0$$

with initial conditions $\phi(0) = a$, $\phi'(0) = 0$ and get

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}\,r^2+\phi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}\,r^2+\phi_a(r)}}{2\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}\,r^2+\phi_a}\,dx} = e^{-\frac{1}{2}\,r^2+\phi_a(r)}$$

With $-\Delta\,\varphi_f = n_a\,f$, consider the operator defined by

$$\mathcal{L}\,f := \frac{1}{n_a}\nabla\cdot\left(n_a(\nabla(f-\varphi_f))\right)\,, \quad x \in \mathbb{R}^2$$

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

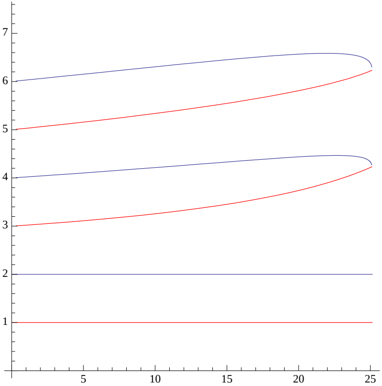


Figure: The lowest eigenvalues of $-\mathcal{L}$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta\, f_0 = n_\infty\, f_0$$

and observe that $g_0 = f_0 / c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f_0 - c_\infty\, g_0) \big) =: \mathcal{L}\, f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty} \frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty} \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_1 = f_1$

With $D := x \cdot \nabla$, let$f_2 = 1 + \frac{1}{2}\, D \log n_\infty = 1 + \frac{1}{2\, n_\infty}\, D\, n_\infty$. Then

$$-\Delta\, (D\, c_\infty) + 2\, \Delta c_\infty = D\, n_\infty = 2\, (f_2 - 1)\, n_\infty$$

and so $g_2 := \frac{1}{c_\infty}\, (-\Delta)^{-1} (n_\infty\, f_2)$ is such that $-\mathcal{L}\, f_2 = 2\, f_2$

Functional setting...

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\; dx$$

achieves its minimum for $n = n_\infty$ according to log HLS and

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\, F[n_\infty(1 + \varepsilon\, f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\; dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma

For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ such that $\int_{\mathbb{R}^2} f\, n_\infty\; dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\; dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\; dx$$

... and eigenvalues

With g such that $-\Delta(g\, c_\infty) = f\, n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\, f_2\, n_\infty\, dx - \int_{\mathbb{R}^2} f_1\, n_\infty\, (g_2\, c_\infty)\, dx$$

on the orthogonal to f_0 in $L^2(n_\infty\, dx)$ and with $G_2(x) := -\frac{1}{2\pi} \log|x|$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\, c_\infty)|^2\, n_\infty\, dx \quad \text{with} \quad g = \frac{1}{c_\infty}\, G_2 * (f\, n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L} f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma

In this setting, $\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is positive

An interpolation inequality induced by log HLS

Lemma

For any $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$ *such that* $\int_{\mathbb{R}^2} f\, f_0\, n_\infty\, dx = 0$ *holds, we have*

$$-\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, n_\infty(x)\, \log|x-y|\, f(y)\, n_\infty(y)\, dx\, dy \le \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

where $g\, c_\infty = G_2 * (f\, n_\infty)$ *and, if* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$ *holds,*

$$\int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, dx \le \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

Equalities in the above inequality then holds if and only if $f = 0$

A new Onofri type inequality

- The spectral gap inequality of $\mathcal{L}$ is a refined version of

Theorem (Onofri type inequality)

For any $M \in (0, 8\pi)$, if $n_\infty = M \frac{e^{c_\infty - \frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}|x|^2}\,dx}$ with $c_\infty = (-\Delta)^{-1} n_\infty$, $d\mu_M = \frac{1}{M} n_\infty\, dx$, we have the inequality

$$\log\left(\int_{\mathbb{R}^2} e^{\phi}\, d\mu_M\right) - \int_{\mathbb{R}^2} \phi\, d\mu_M \le \frac{1}{2\,M}\int_{\mathbb{R}^2} |\nabla\phi|^2\, dx \quad \forall\, \phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Back to Keller-Segel: exponential convergence for any mass $M \leq 8\pi$

- [Campos, JD] Uniform convergence of $n(t,\cdot)$ to n_∞ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [Diaz, Nagai, Rakotoson]

- Spectral gap of $\mathcal{L}$ can be established (Persson's lemma or concentration-compactness methods). An improved interpolation inequality also holds:

$$
-\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, n_\infty(x)\, \log|x-y|\, f(y)\, n_\infty(y)\, dx\, dy \leq (1-\varepsilon)\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx
$$

- Exponential convergence of the relative entropy follows [Campos, JD, work in progress] but estimates are delicate

Thank you for your attention !