From entropy methods to symmetry and symmetry breaking in interpolation inequalities

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Topics in Applied Nonlinear Analysis:
Recent Advances and New Trends
in honor of David Kinderlehrer's 75th Birthday

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- **Entropy methods, gradient flows and rates of convergence**
  - ▷ The Bakry-Emery method
  - ▷ Gradient flow interpretations
- **Flows and sharp interpolation inequalities on the sphere**
  - ▷ Rigidity, Γ_2 framework, and flows
  - ▷ Linear versus nonlinear flows
  - ▷ Constraints and improved inequalities
  - ▷ Onofri inequalities, Riemannian manifolds, Lin-Ni problems
- **Fast diffusion equation: global and asymptotic rates of convergence**
  - ▷ Gagliardo-Nirenberg inequalities: optimal constants and rates
  - ▷ Asymptotic rates of convergence, Hardy-Poincaré inequality
  - ▷ The Rényi entropy powers approach
- **Symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities**
  - ▷ The symmetry issue in the critical case
  - ▷ Flow, rigidity and symmetry
  - ▷ The subcritical case

Entropy methods, gradient flows and rates of convergence

- ▷ The Bakry-Emery method
- ▷ Gradient flow interpretation

The Fokker-Planck equation

The linear Fokker-Planck equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \, \nabla \phi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \, \nabla \phi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \phi \cdot \nabla v =: \mathcal{L} \, v$$

The unique stationary solution (with mass normalized to 1) is

$$\gamma = \frac{e^{\phi}}{\int_\Omega e^{\phi} \, dx}$$

The Bakry-Emery method

With v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$, the *q-entropy* is defined by

$$\mathcal{E}_q[v] := \frac{1}{q-1}\int_\Omega \left(v^q - 1 - q\,(v-1)\right) d\gamma$$

Under the action of (OU), with $w = v^{q/2}$, $\mathcal{I}_q[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$,

$$\frac{d}{dt}\mathcal{E}_q[v(t,\cdot)] = -\,\mathcal{I}_q[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\left(\mathcal{I}_q[v] - \,2\,\lambda\,\mathcal{E}_q[v]\right) \le 0$$

with $\quad \lambda := \inf_{w \in H^1(\Omega, d\gamma)\setminus\{0\}} \dfrac{\int_\Omega \left(2\,\frac{q-1}{q}\,\|\mathrm{Hess}\,w\|^2 + \mathrm{Hess}\,\phi :\,\nabla w \otimes \nabla w\right) d\gamma}{\int_\Omega |w|^2\,d\gamma}$

Proposition

(Bakry, Emery, 1984) (JD, Nazaret, Savaré, 2008) *Let Ω be convex. If $\lambda > 0$ and v is a solution of (OU), then $\mathcal{I}_q[v(t,\cdot)] \le \mathcal{I}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ and $\mathcal{E}_q[v(t,\cdot)] \le \mathcal{E}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ for any $t \ge 0$ and, as a consequence,*

$$\mathcal{I}_q[v] \ge \,2\,\lambda\,\mathcal{E}_q[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma)$$

Remarks, consequences and applications

- Grisvard's lemma: by convexity of Ω, boundary terms have the right sign
- $q=2$: $p=2/q=1$, Poincaré inequality

$$\mathcal{E}_2[v]:=\int_\Omega\left(w^2-\bar{w}^2\right)d\gamma\le\frac{1}{2\,\lambda}\,\mathcal{I}_2[v]=\frac{1}{\lambda}\int_\Omega|\nabla w|^2\,d\gamma$$

- Limit case $q=1$: $p=2/q=2$, logarithmic Sobolev inequality

$$\mathcal{E}_1[v]:=\int_\Omega v\,\log v\,d\gamma\le\frac{2}{\lambda}\,\mathcal{I}_1[v]=\frac{2}{\lambda}\int_\Omega|\nabla\sqrt{v}|^2\,d\gamma$$

- Improvements based on remainder terms: (Arnold, JD), (Arnold, Bartier, JD), (Bartier, JD, Illner, Kowalczyk),...
- Applications:

▷ Brownian ratchets (JD, Kinderlehrer, Kowalczyk), (Blanchet, JD, Kowalczyk)

▷ Keller-Segel models: (Blanchet, Carrillo, Kinderlehrer, Kowalczyk, Laurençot, Lisini)

Gradient flow interpretations

A question by F. Poupaud (1992)... Let ϕ s.t. $\operatorname{Hess}\phi \ge \lambda\,\mathsf{I}$, $\mu := e^{-\phi}\,\mathscr{L}^d$

Entropy : $\mathcal{E}(\rho) := \int_{\mathbb{R}^d} \psi(\rho)\,d\mu$
Action density : $\phi(\rho, \boldsymbol{w}) := \frac{|\boldsymbol{w}|^2}{h(\rho)}$
Action functional : $\Phi(\rho, \boldsymbol{w}) := \int_{\mathbb{R}^d} \phi(\rho, \boldsymbol{w})\,d\gamma$
$\Gamma(\mu_0, \mu_1)$: $(\mu_s, \boldsymbol{\nu}_s)_{s\in[0,1]}$ is an *admissible path* connecting μ_0 to μ_1 if there is a solution $(\mu_s, \boldsymbol{\nu}_s)_{s\in[0,1]}$ to the *continuity equation*

$$\partial_s\mu_s + \nabla\cdot\boldsymbol{\nu}_s = 0\,, \quad s\in[0,1]$$

h-Wasserstein distance

$$W_h^2(\mu_0,\mu_1) := \inf\left\{\int_0^1 \Phi(\mu_s,\boldsymbol{\nu}_s)\,ds\;:\;(\mu,\boldsymbol{\nu})\in\Gamma(\mu_0,\mu_1)\right\}$$

(JD, Nazaret, Savaré): (OU) is the gradient flow of $\mathcal{E}$ w.r.t. W_h.

Flows and sharp interpolation inequalities on the sphere

- ▷ Rigidity, Γ_2 framework, and flows
- ▷ Linear versus nonlinear flows
- ▷ Constraints and improved inequalities
- ▷ Onofri inequalities, Riemannian manifolds, Lin-Ni type problems

(Bakry, Emery, 1984)
(Bidault-Véron, Véron, 1991), (Bakry, Ledoux, 1996)
(Demange, 2008), (JD, Esteban, Loss, 2014 & 2015)
(JD, Esteban, Kowalczyk, Loss, 2013-15), (JD, Kowalczyk, 2016)

The interpolation inequalities

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{S}^d)}^2 + \frac{d}{p-2}\,\|u\|_{\mathrm{L}^2(\mathbb{S}^d)}^2 \ge \frac{d}{p-2}\,\|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^2 \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \neq 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{S}^d)}^2 \ge \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \,\log\left(\frac{|u|^2}{\|u\|_{\mathrm{L}^2(\mathbb{S}^d)}^2}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathcal{E}_p[\rho] := \tfrac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\tfrac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \le -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \le 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \ge d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \le 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

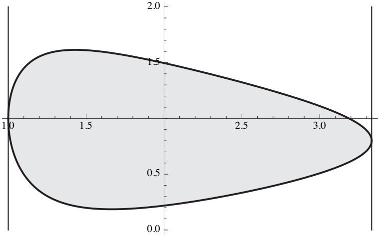
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{1}$$

(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - \, d\, \mathcal{E}_p[\rho]\Big) \leq 0$$



(p, m) admissible region, $d = 5$

The rigidity point of view (nonlinear flow)

In cylindrical coordinates with $z \in [-1,1]$, let

$$\mathcal{L}\,f := (1-z^2)\,f'' - d\,z\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f'$$

be the *ultraspherical operator* and consider

$$-\,\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

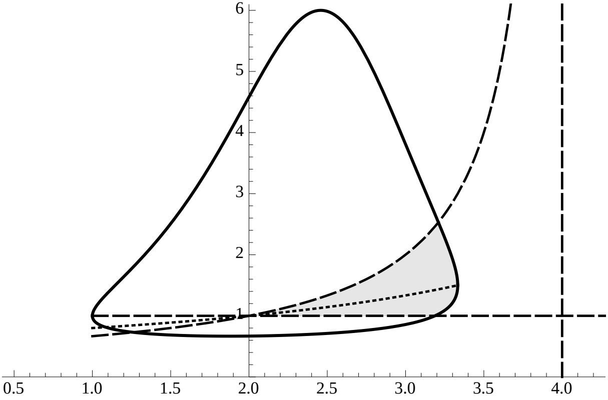
$$\int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d = 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

Improved functional inequalities

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

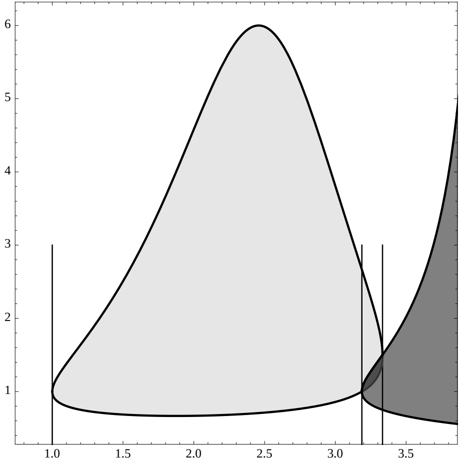
$$\rho = |u|^{\beta p}$$

$$m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$$



(p,β) representation of the admissible range of parameters when $d = 5$
(JD, Esteban, Kowalczyk, Loss)

Can one prove Sobolev's inequalities with a heat flow ?



(p, β) representation when $d = 5$. In the dark grey area, the functional is not monotone under the action of the heat flow (JD, Esteban, Loss)

Integral constraints

With the heat flow...

Proposition

For any $p\in(2,2^\#)$, the inequality

$$\int_{-1}^1|f'|^2\,\nu\,d\nu_d+\frac{\lambda}{p-2}\,\|f\|_2^2\ge\frac{\lambda}{p-2}\,\|f\|_p^2$$

$$\forall\,f\in\mathrm H^1((-1,1),d\nu_d)\;s.t.\;\int_{-1}^1 z\,|f|^p\,d\nu_d=0$$

holds with

$$\lambda\ge d+\frac{(d-1)^2}{d\,(d+2)}\,(2^\#-p)\,(\lambda^\star-d)$$

... and with a nonlinear diffusion flow ?

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

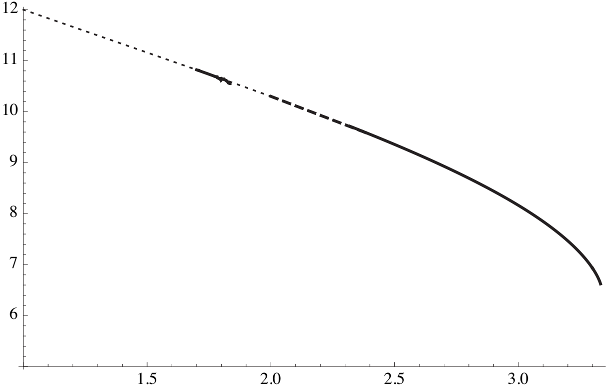
If $p \in (1,2) \cup (2, 2^*)$, *we have*

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{p-2} \left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1} \right] \left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ *with antipodal symmetry. The limit case* $p=2$ *corresponds to the improved logarithmic Sobolev inequality*

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{2} \frac{(d+3)^2}{(d+1)^2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu$$

The optimal constant in the antipodal framework



Branches of antipodal solutions: numerical computation of the optimal constant when $d=5$ and $1 \le p \le 10/3 \approx 3.33$. The limiting value of the constant is numerically found to be equal to
$\lambda_\star = 2^{1-2/p}\, d \approx 6.59754$ *with* $d = 5$ *and* $p = 10/3$

Onofri inequalities, Riemannian manifolds, Lin-Ni problems

- The extension to **Riemannian manifolds** (JD, Esteban, Loss, 2013): for any $p \in (1,2) \cup (2,2^*)$ or $p = 2^*$ if $d \geq 3$, the FDE equations provides a lower bound for Λ in

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\Lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

- **Onofri inequality** (JD, Esteban, Jankowiak, 2015): the flow

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\, e^{-f/2}$$

determines a rigidity interval for smooth solutions to $-\,\frac{1}{2}\,\Delta_g u + \lambda = e^u$

- **Lin-Ni problems** (JD, Kowalczyk): rigidity interval for solutions with Neumann homogeneous boundary conditions on bounded convex domain
- **Keller-Lieb-Thirring inequalities** on manifolds: estimates for Schrödinger operator that (really) differ from the semi-classical

Fast diffusion equation: global and asymptotic rates of convergence

- ▷ Gagliardo-Nirenberg inequalities: optimal constants and rates
- ▷ Asymptotic rates of convergence, Hardy-Poincaré inequality
- ▷ The Rényi entropy powers approach

The fast diffusion equation

The fast diffusion equation corresponds to $m < 1$

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

(Friedmann, Kamin) Barenblatt (self-similar functions) attract all solutions as $t \to +\infty$

▷ *Entropy methods allow to measure the speed of convergence of any solution to $\mathfrak{U}$ in norms which are adapted to the equation*
▷ *Entropy methods provide explicit constants*

- The *Bakry-Emery method* (Carrillo, Toscani), (Juengel, Markowich, Toscani), (Carrillo, Juengel, Markowich, Toscani, Unterreiter), (Carrillo, Vázquez)
- The variational approach and Gagliardo-Nirenberg inequalities: (del Pino, JD)
- Mass transportation and gradient flow issues: (Otto et al.)
- Large time asymptotics and the spectral approach: (Blanchet, Bonforte, JD, Grillo, Vázquez), (Denzler, Koch, McCann), (Seis)
- Refined relative entropy methods

Time-dependent rescaling, free energy

- *Time-dependent rescaling*: Take $u(\tau,y)=R^{-d}(\tau)\,v\,(t,y/R(\tau))$ where

$$\frac{dR}{d\tau}=R^{d(1-m)-1}\;,\quad R(0)=1\;,\quad t=\log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t}=\Delta v^m+\nabla\cdot(x\,v)\;,\quad v_{|\tau=0}=u_0$$

- (Ralston, Newman, 1984) Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[v]:=\int_{\mathbb{R}^d}\left(\frac{v^m}{m-1}+\frac{1}{2}\,|x|^2\,v\right)\,dx-\mathcal{F}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[v]=-\mathcal{I}[v]\;,\quad \mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\frac{\nabla v^m}{v}+x\right|^2\,dx$$

Relative entropy and entropy production

- *Stationary solution:* choose C such that $\|\mathfrak{B}\|_{L^1} = \|u\|_{L^1} = M > 0$

$$\mathfrak{B}(x) := \left(C + \tfrac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[\mathfrak{B}] = 0$

- *Entropy – entropy production inequality*

Theorem

$d \ge 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \ge 2\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[u_0]\,e^{-\,2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\mathcal{F}[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \mathcal{F}_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2}\,\mathcal{I}[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} \, dx - K \geq 0$$

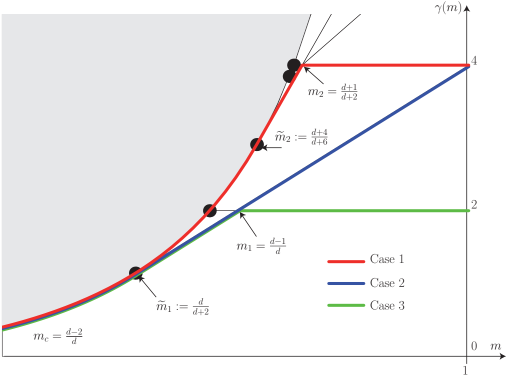
Theorem

[Del Pino, J.D.] *With $1 < p \leq \frac{d}{d-2}$ (fast diffusion case) and $d \geq 3$*

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq \mathcal{C}^{\mathrm{GN}}_{p,d} \, \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)} \, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

Improved asymptotic rates

(Denzler, McCann), (Denzler, Koch, McCann), (Seis)
(Blanchet, Bonforte, J.D., Grillo, Vázquez), (Bonforte, J.D., Grillo, Vázquez), (J.D., Toscani)



A Hardy-Poincaré inequality : $\int_{\mathbb{R}^d} |\nabla f|^2 \, \mathfrak{B} \, dx \ge \Lambda \int_{\mathbb{R}^d} |f|^2 \, \mathfrak{B}^{2-m} \, dx$

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(x, t=0) = u_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} u_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx \quad \text{with} \quad \mathsf{P} = \frac{m}{m-1} \, u^{m-1}$$

If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{P}}{\partial t} = (m-1) \, \mathsf{P} \, \Delta \mathsf{P} + |\nabla \mathsf{P}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The concavity property

Theorem

(Toscani-Savaré) *Assume that* $m \geq 1 - \frac{1}{d}$ *if* $d > 1$ *and* $m > 0$ *if* $d = 1$. *Then* $F(t)$ *is increasing,* $(1-m)\,\mathsf{F}''(t) \leq 0$ *and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

(Dolbeault-Toscani) The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$

The proof

Lemma

If u solves $\frac{\partial u}{\partial t} = \Delta u^m$ with $\frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx = -\,2 \int_{\mathbb{R}^d} u^m \left(\|\mathrm{D}^2 \mathsf{P}\|^2 + (m-1)\,(\Delta \mathsf{P})^2 \right) dx$$

$$\|\mathrm{D}^2 \mathsf{P}\|^2 = \frac{1}{d}\,(\Delta \mathsf{P})^2 + \left\| \mathrm{D}^2 \mathsf{P} - \frac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id} \right\|^2$$

$$\begin{aligned}\frac{1}{\sigma\,(1-m)}\,\mathsf{E}^{2-\sigma}\,(\mathsf{E}^\sigma)'' = &\,(1-m)\,(\sigma-1) \left(\int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx \right)^2 \\ &-\,2 \left(\frac{1}{d} + m - 1 \right) \int_{\mathbb{R}^d} u^m\,dx \int_{\mathbb{R}^d} u^m\,(\Delta \mathsf{P})^2\,dx \\ &-\,2 \int_{\mathbb{R}^d} u^m\,dx \int_{\mathbb{R}^d} u^m \left\| \mathrm{D}^2 \mathsf{P} - \frac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id} \right\|^2 dx\end{aligned}$$

Symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

- ▷ The symmetry issue in the critical case
- ▷ Flow, rigidity and symmetry
- ▷ The subcritical case

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)

Critical Caffarelli-Kohn-Nirenberg inequalities

Let $\mathcal{D}_{a,b} := \left\{ w \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla w| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|w|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^{2\,a}}\,dx \quad \forall\, w \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

▷ *An optimal function among radial functions:*

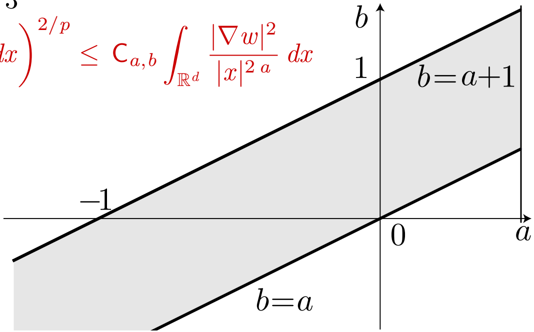
$$w_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,w_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla w_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d}\frac{|w|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le\mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla w|^2}{|x|^{2\,a}}\,dx$$



$a\le b\le a+1$ if $d\ge 3$

$a<b\le a+1$ if $d=2$, $a+1/2<b\le a+1$ if $d=1$

and $a<a_c:=(d-2)/2$

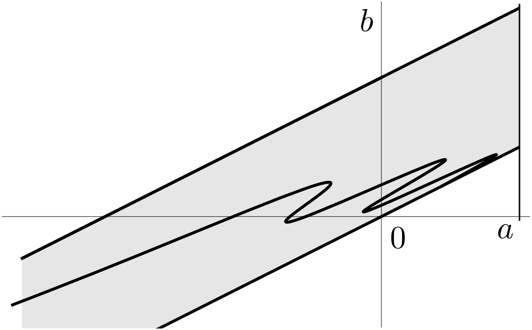
$$p=\frac{2\,d}{d-2+2\,(b-a)}$$

(Glaser, Martin, Grosse, Thirring (1976))

(F. Catrina, Z.-Q. Wang (2001))

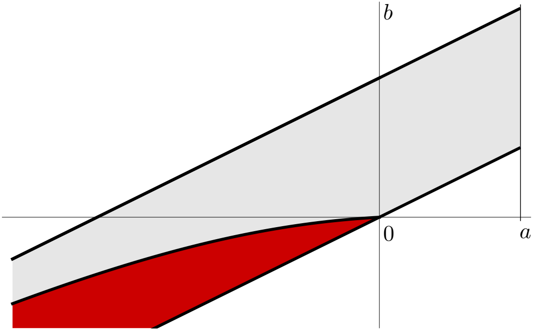
The threshold between symmetry and symmetry breaking

(F. Catrina, Z.-Q. Wang)



(JD, Esteban, Loss, Tarantello, 2009) There is a curve which separates the symmetry region from the symmetry breaking region, which is parametrized by a function $p \mapsto a + b$

Linear instability of radial minimizers: the Felli-Schneider curve

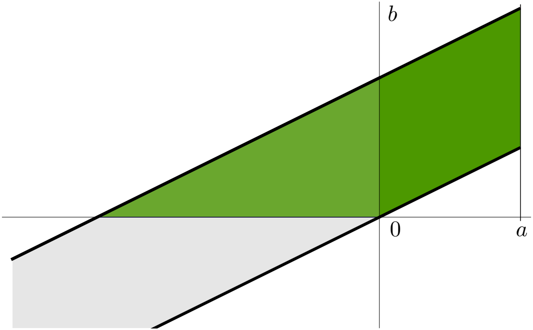


(Catrina, Wang), (Felli, Schneider) The functional

$$\mathsf{C}^\star_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla w|^2}{|x|^{2\,a}}\,dx-\left(\int_{\mathbb{R}^d}\frac{|w|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

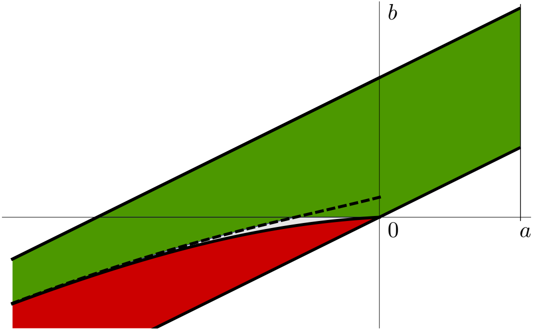
is linearly instable at $w=w_\star$

Moving planes and symmetrization



(Chou, Chu), (Horiuchi)
(Betta, Brock, Mercaldo, Posteraro)
+ Perturbation results: (CS Lin, ZQ Wang), (Smets, Willem), (JD, Esteban, Tarantello 2007), (JD, Esteban, Loss, Tarantello, 2009)

Direct spectral estimates

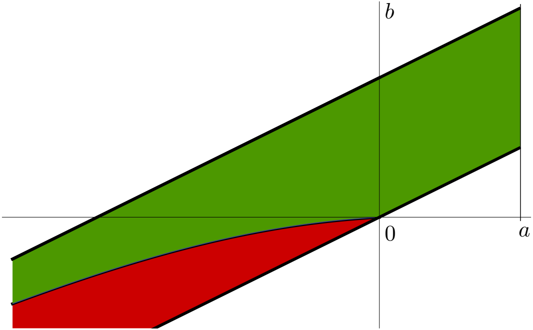


(JD, Esteban, Loss, 2011): sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

- Further numerical results (JD, Esteban, 2012) (coarse / refined / self-adaptive grids). Formal commutation of the non-symmetric branch near the bifurcation point (JD, Esteban, 2013)
- Asymptotic energy estimates (JD, Esteban, 2013)

Symmetry *versus* symmetry breaking: the sharp result in the critical case

A result based on entropies and nonlinear flows



(JD, Esteban, Loss (Inventiones 2016))

The symmetry result in the critical case

The Felli & Schneider curve is defined by

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

Proof (1/3): a change of variables

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x)=w(x)$ as

$$\|v\|_{\mathrm{L}^{\frac{2n}{n-2},d-n}(\mathbb{R}^d)}\le \mathsf{K}_{\alpha,n}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}$$

with the notations $\mathfrak{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac1s\,\nabla_\omega v\right)$, $s=|x|$, and

$$d\ge2\,,\quad \alpha>0\,,\quad n=\frac{d-b\,p}{\alpha}=\frac{d-2\,a-2}{\alpha}+2>d$$

By our change of variables, $p=\frac{2\,n}{n-2}$ is the critical Sobolev exponent associated with n and $w_\star$ is changed into

$$v_\star(x):=\left(1+|x|^2\right)^{-1/(p-1)}\quad\forall\,x\in\mathbb{R}^d$$

Fisher information $\mathcal{I}$ and *pressure function* P: with $u^{\frac12-\frac1n}=|w|$,

$$\mathcal{I}[u]:=\int_{\mathbb{R}^d}u\,|\mathfrak{D}_\alpha\mathsf{P}|^2\,d\mu\,,\quad \mathsf{P}=\frac{m}{1-m}\,u^{m-1}\quad\text{and}\quad m=1-\frac1n$$

Goal: prove that $\inf\mathcal{I}[u]$ under the mass constraint $\int_{\mathbb{R}^d}u\,d\mu=1$ is achieved by $\mathsf{P}(x)=1+|x|^2$

Proof (2/3): decay of $\mathcal{J}$ along the fast diffusion flow

$$\frac{\partial u}{\partial t} = \mathcal{L}_\alpha\, u^m = -\,\mathfrak{D}_\alpha^*\,\mathfrak{D}_\alpha\, u^m\,,\quad m = 1 - \frac{1}{n}$$

Barenblatt self-similar solutions:

$u_\star(t,r,\omega) = t^{-n}\left(\mathsf{c}_\star + \frac{r^2}{2\,(n-1)\,\alpha^2\,t^2}\right)^{-n}$

$$\frac{d}{dt}\mathcal{J}[u(t,\cdot)] = -\,2\,(n-1)^{n-1}\int_{\mathbb{R}^d}\mathsf{k}[\mathsf{P}]\,\mathsf{P}^{1-n}\,d\mu$$

and (long and painful computation !), with $\alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$,

$$\mathsf{k}[\mathsf{P}] = \alpha^4\left(1-\frac{1}{n}\right)\left[\mathsf{P}'' - \frac{\mathsf{P}'}{r} - \frac{\Delta\,\mathsf{P}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla\mathsf{P}' - \frac{\nabla\mathsf{P}}{r}\right|^2$$
$$+\,\frac{n-2}{r^4}\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla\mathsf{P}|^2\,\mathsf{P}^{1-n} + \frac{\zeta_\star\,(n-d)}{r^4}\,|\nabla\mathsf{P}|^4\,\mathsf{P}^{1-n}$$

▷ Boundary terms ! Regularity !

Proof (3/3): a perturbation argument and elliptic regularity

- If u is a critical point of $\mathcal{J}$ under the constraint $\int_{\mathbb{R}^d} u\,d\mu = 1$, then

$$0 = D\mathcal{J}[u] \cdot \mathcal{L}_\alpha\, u^m = -\,2\,(n-1)^{n-1} \int_{\mathbb{R}^d} \mathsf{k}[\mathsf{P}]\,\mathsf{P}^{1-n}\,d\mu$$

- Regularity issues and boundary terms: after an Emden-Fowler transformation, a critical point satisfies the Euler-Lagrange equation

$$-\,\partial_s^2\,\varphi - \,\Delta_\omega\,\varphi + \Lambda\,\varphi = \varphi^{p-1} \quad \text{in} \quad \mathcal{C} = \mathbb{R} \times \mathbb{S}^{d-1}$$

with $p < \frac{2\,d}{d-2}$: $C_1\,e^{-\sqrt{\Lambda}\,|s|} \le \varphi(s,\omega) \le C_2\,e^{-\sqrt{\Lambda}\,|s|}$

With $s = \log r$, one can prove, *e.g.*, that

$$\int_{\mathbb{S}^d} |\mathsf{P}''(r,\omega)|^2\,d\mu \le O(1/r^2)$$

- If $\alpha \le \alpha_{\rm FS}$, then $u = u_\star$

Some subcritical Caffarelli-Kohn-Nirenberg inequalities

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p} \, \|\nabla w\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}^{\vartheta} \, \|w\|_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta} \qquad \text{(CKN)}$$

$\mathsf{C}_{\beta,\gamma,p}$ is the optimal constant, $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$

$$d \ge 2\,, \quad \gamma-2 < \beta < \tfrac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star]\,, \quad p_\star := \tfrac{d-\gamma}{d-\beta-2}$$

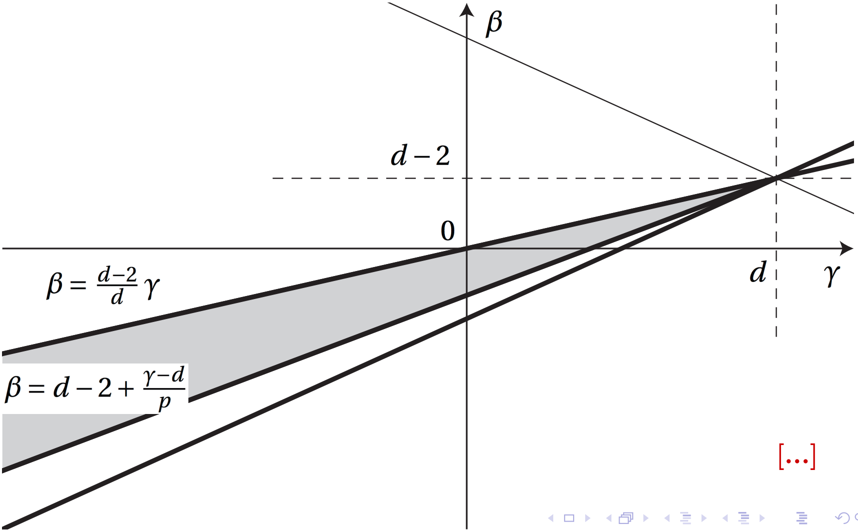
$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,(d+\beta+2-2\,\gamma-p\,(d-\beta-2))}$ is determined by the scalings

- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions ?

Range of the parameters



[...]

CKN and entropy – entropy production inequalities

If symmetry holds, (CKN) is equivalent to

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\mathcal{I}[v]$$

$p=1/(2m-1)$, and equality is achieved by
$\mathfrak{B}_{\beta,\gamma}=\left(1+|x|^{2+\beta-\gamma}\right)^{\frac{1}{m-1}}$

$$\mathcal{F}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\,\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\,\frac{dx}{|x|^\beta}\,.$$

If v solves the *Fokker-Planck type equation*

$$\partial_t v+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0$$

then the *free energy* and the *relative Fisher information* satisfy

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\,\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Decay of the free energy in the symmetry range

Proposition

Let $m=\frac{p+1}{2\,p}$ *and consider a solution to*

$$\partial_t v+|x|^\gamma\,\nabla\cdot\Big[|x|^{-\beta}\,v\,\nabla\big(v^{m-1}-|x|^{2+\beta-\gamma}\big)\Big]=0$$

with nonnegative initial datum $u_0\in\mathcal L^{\,1,\gamma}(\mathbb R^d)$ *such that* $\|u_0^m\|_{\mathrm L^{1,\gamma}(\mathbb R^d)}$ *and* $\int_{\mathbb R^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ *are finite. Then*

$$\mathcal F[v(t,\cdot)]\le\mathcal F[u_0]\,e^{-\,(2+\beta-\gamma)^2\,t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:

(i) either u_0 *is a.e. radially symmetric*

(ii) or symmetry holds in (CKN)

Symmetry and symmetry breaking

(JD, Esteban, Loss, Muratori, 2016)

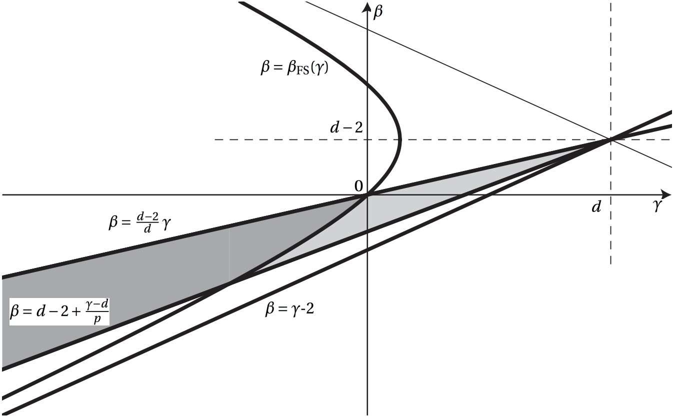
Let us define $\beta_{\mathrm{FS}}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad \textit{and} \quad \beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$, $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal.



The grey area corresponds to the admissible cone. The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2 - (\beta-d+2)^2 - 4\,(d-1) = 0$$

Some concluding remarks

The so-called *entropy methods* in PDEs address various questions which are relevant for applications in physics and biology

- **Symmetry and symmetry breaking**

▷ Sharp conditions for symmetry breaking, phase transitions, *etc.*
▷ Characterization of the rates of convergence (for the evolution equation) in terms of the symmetry of the optimal functions
▷ Power law non-linearities or weights make sense to explore some limiting case (blow-up, large scale)

- **Rates of convergence, identification of the optimal constants**

▷ are crucial for some applications (*e.g.*, in astrophysics)
▷ raise questions: global vs. asymptotic rates ? corrections (delays) ?
▷ identify worst case scenarios: numerics, obstructions

- **Functional inequalities**: useful not only for *a priori* estimates, but also for providing extreme cases and enlightening structures

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
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Thank you for your attention !