

Functional inequalities and rates of convergence in nonlinear diffusions and kinetic equations

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Stability results based on entropy and diffusion equations

- Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$
- Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$
- Constructive stability results for (log-)Sobolev inequalities

Rényi entropy powers, inequalities and flow, a formal approach on $\mathbb{R}^d$

[Toscani, Savaré, 2014]

[JD, Toscani, 2016]

[JD, Esteban, Loss, 2016]

▷ *How do we relate Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$*

$$\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\text{GNS}} \|f\|_{2p} \quad (\text{GNS})$$

and the fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m \quad (\text{FDE})$$

Entropy growth rate as a consequence of (GNS)

• Entropy functional and Fisher information functional

$$E[u] := \int_{\mathbb{R}^d} u^m dx \quad \text{and} \quad I[u] := \frac{m^2}{1-m} \int_{\mathbb{R}^d} u |\nabla u^{m-1}|^2 dx$$

If u solves (FDE) $\frac{\partial u}{\partial t} = \Delta u^m$, then $E' = I$

• Gagliardo-Nirenberg-Sobolev inequalities, $p = \frac{1}{2m-1} \iff m = \frac{p+1}{2p}$

$$\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \geq C_{\text{GNS}} \|f\|_{2p} \quad (\text{GNS})$$

$u = f^{2p}$ so that $u^m = f^{p+1}$ and $m^2 u |\nabla u^{m-1}|^2 = (p-1)^2 |\nabla f|^2$

$$\mathcal{M} = \|f\|_{2p}^{2p}, \quad E[u] = \|f\|_{p+1}^{p+1}, \quad I[u] = \frac{p-1}{2p} (p+1)^2 \|\nabla f\|_2^2$$

• Best growth estimate in (FDE) means best constant in (GNS)

$$E' \geq \frac{p-1}{2p} (p+1)^2 C_{\text{GNS}}^{\frac{2}{\theta}} \|f\|_{2p}^{\frac{2}{\theta}} \|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}} = C_0 E^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d} u^m(t, x) dx \geq \left(\int_{\mathbb{R}^d} u_0^m dx + \frac{(1-m)C_0}{m-m_c} t \right)^{\frac{1-m}{m-m_c}} \quad \forall t \geq 0$$

Exponents

Gagliardo-Nirenberg-Sobolev inequalities

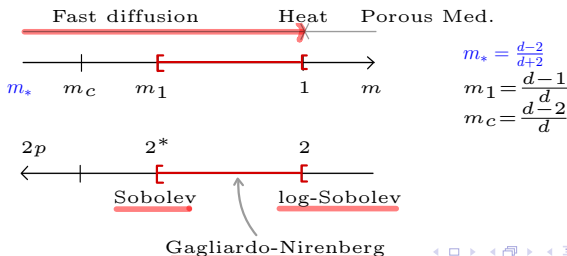
$$\|\nabla f\|_2^\theta \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\text{GNS}}(p) \|f\|_{2p} \quad (\text{GNS})$$

$$u = f^{2p} \text{ so that } u^m = f^{p+1} \text{ and } u |\nabla u^{m-1}|^2 = (p-1)^2 |\nabla f|^2$$

$$p = \frac{1}{2m-1} \iff m = \frac{p+1}{2p} \in [m_1, 1)$$

Fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m \quad (\text{FDE})$$



Self-similar solutions

If u solves (FDE) $\frac{\partial u}{\partial t} = \Delta u^m$

$$\int_{\mathbb{R}^d} u^m(t, x) dx \geq \left(\int_{\mathbb{R}^d} u_0^m dx + \frac{(1-m)C_0}{m-m_c} t \right)^{\frac{1-m}{m-m_c}} \quad \forall t \geq 0$$

Equality case is achieved if and only if, up to a translation

$$u(t, x) = \frac{c_1}{R(t)^d} \mathcal{B} \left(\frac{c_2 x}{R(t)} \right)$$

where $\mathcal{B}$ is the *Barenblatt self-similar solution*

$$\mathcal{B}(x) := (1 + |x|^2)^{\frac{1}{m-1}}$$

The *Aubin-Talenti profile* $\varphi(x) = (1 + |x|^2)^{-\frac{1}{p-1}}$ such that $\varphi^{2p} = \mathcal{B}$ is optimal for (GNS)

► Integrations by parts and completion of squares: with $m_1 = 1 - \frac{1}{d}$

where $P := \frac{m}{1-m} u^{m-1}$ is the *pressure variable*

$$\lim_{t \rightarrow +\infty} \frac{\mathbb{I}[u(t, \cdot)]^\theta \mathbb{E}[u(t, \cdot)]^{2\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\theta}{p}}} = \frac{\mathbb{I}[\mathcal{B}]^\theta \mathbb{E}[\mathcal{B}]^{2\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_1^{\frac{2\theta}{p}}} = (p+1)^{2\theta} \mathcal{C}_{\text{GNS}}^{2\theta}$$
$$I[u]^\theta E[u]^{2\frac{1-\theta}{p+1}} \geq (p+1)^{2\theta} (\mathcal{C}_{\text{GNS}})^{2\theta} \mathcal{M}^{\frac{2\theta}{p}}$$

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

Joint papers with M. Bonforte, B. Nazaret and N. Simonov
***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows,
regularity and the entropy method***
[arXiv:2007.03674](https://arxiv.org/abs/2007.03674), *Memoirs of the AMS* 308 (2025)

***Constructive stability results in interpolation inequalities and
explicit improvements of decay rates of fast diffusion equations***
DCDS, 43 (3&4): 1070–1089, 2023

Entropy – entropy production inequality

The fast diffusion equation on $\mathbb{R}^d$ in *self-similar variables*

$$\frac{\partial v}{\partial t} + \nabla \cdot [v (\nabla v^{m-1} - 2x)] = 0 \quad (\text{FDE})$$

admits a stationary Barenblatt solution $\mathcal{B}(x) := (1 + |x|^2)^{\frac{1}{m-1}}$

$$\frac{d}{dt} \mathcal{F}[v(t, \cdot)] = -\mathcal{I}[v(t, \cdot)]$$

Generalized entropy (free energy) and Fisher information

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} (v^m - \mathcal{B}^m - m \mathcal{B}^{m-1} (v - \mathcal{B})) \, dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v |\nabla v^{m-1} - \nabla \mathcal{B}^{m-1}|^2 \, dx$$

are such that $\mathcal{I}[v] \geq 4 \mathcal{F}[v] \iff (\text{GNS})$ [del Pino, JD, 2002] so that

$$\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0] e^{-4t}$$

Asymptotic regime as $t \rightarrow +\infty$, spectral gap

Linearized free energy

& linearized Fisher information

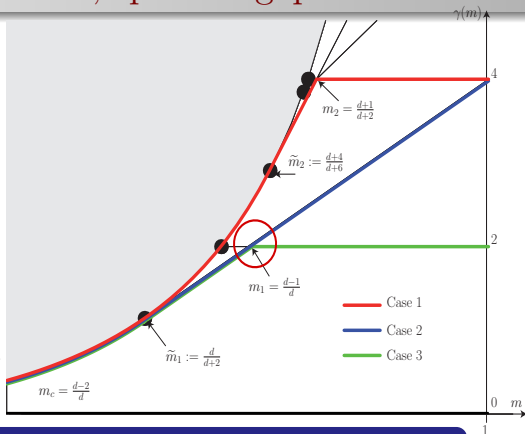
With $f_\varepsilon := \mathcal{B}(1 + \varepsilon \mathcal{B}^{1-m} w)$,
 expand $\mathcal{F}[f_\varepsilon]$, $\mathcal{I}[f_\varepsilon]$ at order $O(\varepsilon^2)$

$$F[w] := \frac{m}{2} \int_{\mathbb{R}^d} w^2 \mathcal{B}^{2-m} dx$$

$$I[w] := m(1-m) \int_{\mathbb{R}^d} |\nabla w|^2 \mathcal{B} dx$$

[Denzler, McCann, 2005], [BB-DGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015], [Denzler, Koch, McCann, 2015]

$m \in [m_1, 1)$ if $d \geq 3$, $m \in (1/2, 1)$ if $d = 2$, $m \in (1/3, 1)$ if $d = 1$



Proposition (Hardy-Poincaré inequality)

If $w \in L^2(\mathbb{R}^d, \mathcal{B}^{2-m} dx)$ with $\nabla w \in L^2(\mathbb{R}^d, \mathcal{B} dx)$, $\int_{\mathbb{R}^d} w \mathcal{B}^{2-m} dx = 0$

$$I[w] \geq 4\alpha F[w]$$

with $\alpha = 1$, or $\alpha = 2 - d(1-m)$ if $\int_{\mathbb{R}^d} x w \mathcal{B}^{2-m} dx = 0$

Asymptotic and initial time layers

• The asymptotic time layer improvement

Proposition

If $\int_{\mathbb{R}^d} v \, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x \cdot v \, dx = 0$ and
 $(1 - \varepsilon) \mathcal{B} \leq v \leq (1 + \varepsilon) \mathcal{B}$ for some $\varepsilon \in (0, \eta m / (266 + 56 m))$,
 then $\mathcal{I}[v] \geq (4 + \eta) \mathcal{F}[v]$,

• The initial time layer improvement. As a consequence of the *carré du champ* method, if v solves (FDE), $\mathcal{Q}[v] := \mathcal{I}[v] / \mathcal{F}[v]$, then

$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}(\mathcal{Q} - 4)$$

Lemma

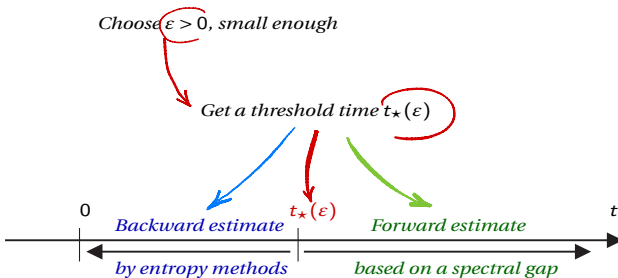
If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star, \cdot)] \geq 4 + \eta$, then

$$\mathcal{Q}[v(t, \cdot)] \geq 4 + \frac{4\eta e^{-4t_\star}}{4 + \eta - \eta e^{-4t_\star}} \quad \forall t \in [0, t_\star]$$

Uniform convergence in relative error: threshold time

Theorem

[Bonforte, JD, Nazaret, Simonov, 2023-25] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$ and $A > 0$ be given. There exists an explicit **threshold time** $t_\star \geq 0$ such that, if v solves (FDE) with $A[v(t=0, \cdot)] = \sup_{r>0} \frac{\int_{|x|>r} v(t=0, \cdot) dx}{\int_{|x|>r} B dx} \leq A < \infty$, then

$$(1 - \varepsilon) \mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon) \mathcal{B} \text{ for any } t \geq t_\star$$


Two consequences (subcritical case)

- ▶ Improved decay rate for (FDE) in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, and $A > 0$. If v solves (FDE) with initial datum $v_0 \in L^1_+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} v_0 dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x v_0 dx = 0$ and v_0 satisfies (H_A) , then

$$\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0] e^{-(4+\zeta)t} \quad \forall t \geq 0$$

- ▶ *Stability of the entropy - entropy production inequality*

$$\mathcal{I}[v] - 4\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta} \mathcal{I}[v]$$

- ▶ Critical case (*Sobolev inequality*)
 ... a refinement is needed to take care of the scale invariance

Constructive stability results in subcritical Gagliardo-Nirenberg inequalities on $\mathbb{S}^d$

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \geq d \mathcal{E}_p[F] := \frac{d}{p-2} \left(\|F\|_{L^p(\mathbb{S}^d)}^2 - \|F\|_{L^2(\mathbb{S}^d)}^2 \right)$$

for any $p \in [1, 2) \cup (2, 2^*)$
with $2^* := \frac{2d}{d-2}$ if $d \geq 3$ and $2^* = +\infty$ if $d = 1$ or 2

Joint paper with G. Brigati and N. Simonov
*Logarithmic Sobolev and interpolation inequalities on the sphere:
constructive stability results*

Annales IHP, Analyse non linéaire, 362, 2023. [arXiv: 2211.13180](https://arxiv.org/abs/2211.13180)

Gagliardo-Nirenberg inequalities: stability

An improved inequality under orthogonality constraint (Π_1 is a projection on some positive spherical harmonic functions) and the stability inequality arising from the *carré du champ* method can be combined *in the subcritical case* as follows

Theorem (Brigati, JD, Simonov)

Let $d \geq 1$ and $p \in (1, 2^*)$. For any $F \in H^1(\mathbb{S}^d, d\mu)$, we have

$$\begin{aligned} \int_{\mathbb{S}^d} |\nabla F|^2 d\mu - d \mathcal{E}_p[F] \\ \geq \mathcal{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|_{L^2(\mathbb{S}^d)}^4}{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \|F\|_{L^2(\mathbb{S}^d)}^2} + \|\nabla(\text{Id} - \Pi_1) F\|_{L^2(\mathbb{S}^d)}^2 \right) \end{aligned}$$

for some explicit stability constant $\mathcal{S}_{d,p} > 0$

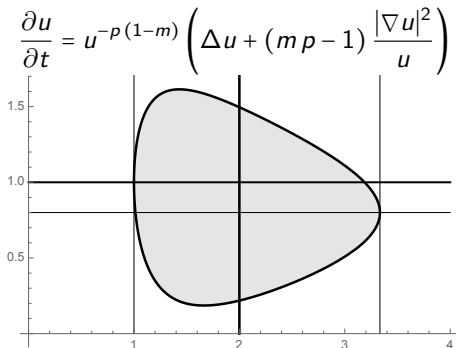
▷ The result holds true for the logarithmic Sobolev inequality ($p = 2$), again with an explicit constant $\mathcal{S}_{d,2}$, for any finite dimension d

Carré du champ & admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss, 2014] Monotonicity of the deficit

$$\delta[u] := \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[u] \quad \text{with} \quad \mathcal{E}_p[u] := \frac{d}{p-2} \left(\|u\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^2(\mathbb{S}^d)}^2 \right)$$

along the nonlinear diffusion flow



$d = 5$: admissible $p \in (2, 2^*)$ and m (horizontal axis: p , vertical axis: m)

- If $\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 / \|F\|_{L^p(\mathbb{S}^d)}^2 \geq \vartheta_0 > 0$, by the convexity of ψ

$$\geq \frac{d}{\vartheta_0} \psi\left(\frac{\vartheta_0}{d}\right) \|\nabla F\|_{L^2(\mathbb{S}^d)}^2$$

- Take $\|F\|_{L^p(\mathbb{S}^d)} = 1$, assume that $\frac{d\vartheta_0}{d-(p-2)\vartheta_0} > 0$ and deduce from the Poincaré inequality that

$$1 - \frac{\vartheta}{d} < \left(\int_{\mathbb{S}^d} F d\mu \right)^2 \leq 1$$

- + a Taylor expansion using a partial decomposition on spherical harmonics

The “far away” regime and the “neighborhood” of $\mathcal{M}$

▷ If $\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 / \|F\|_{L^p(\mathbb{S}^d)}^2 \geq \vartheta_0 > 0$, by the convexity of ψ

$$\begin{aligned} \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[F] &\geq d \|F\|_{L^p(\mathbb{S}^d)}^2 \psi\left(\frac{1}{d} \frac{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2}{\|F\|_{L^p(\mathbb{S}^d)}^2}\right) \\ &\geq \frac{d}{\vartheta_0} \psi\left(\frac{\vartheta_0}{d}\right) \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \end{aligned}$$

▷ From now on, we assume that $\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 < \vartheta_0 \|F\|_{L^p(\mathbb{S}^d)}^2$, take $\|F\|_{L^p(\mathbb{S}^d)} = 1$, learn that

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 < \vartheta := \frac{d \vartheta_0}{d - (p-2) \vartheta_0} > 0$$

from the standard interpolation inequality and deduce from the Poincaré inequality that

$$\frac{d - \vartheta}{d} < \left(\int_{\mathbb{S}^d} F d\mu \right)^2 \leq 1$$

Partial decomposition on spherical harmonics

$$\mathcal{M} = \Pi_0 F \text{ and } \Pi_1 F = \varepsilon \mathcal{Y} \text{ where } \mathcal{Y}(x) = \sqrt{\frac{d+1}{d}} x \cdot \nu, \nu \in \mathbb{S}^d$$

$$F = \mathcal{M} (1 + \varepsilon \mathcal{Y} + \eta G)$$

Apply $c_{p,d}^{(-)} \varepsilon^6 \leq \|1 + \varepsilon \mathcal{Y}\|_{L^p(\mathbb{S}^d)}^p - (1 + a_{p,d} \varepsilon^2 + b_{p,d} \varepsilon^4) \leq c_{p,d}^{(+)} \varepsilon^6$ (with explicit constants) to $u = 1 + \varepsilon \mathcal{Y}$ and $r = \eta G$ the estimate

$$\begin{aligned} & \|u + r\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^p(\mathbb{S}^d)}^2 \\ & \leq \frac{2}{p} \|u\|_{L^p(\mathbb{S}^d)}^{2-p} \left(p \int_{\mathbb{S}^d} u^{p-1} r \, d\mu + \frac{p}{2} (p-1) \int_{\mathbb{S}^d} u^{p-2} r^2 \, d\mu \right. \\ & \quad \left. + \sum_{2 < k < p} C_k^p \int_{\mathbb{S}^d} u^{p-k} |r|^k \, d\mu + K_p \int_{\mathbb{S}^d} |r|^p \, d\mu \right) \end{aligned}$$

Estimate $\int_{\mathbb{S}^d} (1 + \varepsilon \mathcal{Y})^{p-1} G \, d\mu$, $\int_{\mathbb{S}^d} (1 + \varepsilon \mathcal{Y})^{p-k} |G|^k \, d\mu$, etc. to obtain (under the condition that $\varepsilon^2 + \eta^2 \sim \vartheta$)

$$\begin{aligned} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - d \mathcal{E}_p[F] & \geq \mathcal{M}^2 \left(A \varepsilon^4 - B \varepsilon^2 \eta + C \eta^2 - \mathcal{R}_{p,d} (\vartheta^p + \vartheta^{5/2}) \right) \\ & \geq \mathcal{C} \left(\frac{\varepsilon^4}{\varepsilon^2 + \eta^2 + 1} + \eta^2 \right) \end{aligned}$$

Large dimensional limit of (GNS) on $\mathbb{S}^d$

... based on the Maxwell-Poincaré lemma [McKean, 1973]
 Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1, 2)$

$$\|\nabla u\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 \geq \frac{d}{p-2} \left(\|u\|_{L^p(\mathbb{S}^d, d\mu_d)}^2 - \|u\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 \right)$$

Proposition

Let $v \in H^1(\mathbb{R}^n, dx)$ with compact support, $d \geq n$ and

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \dots, \omega_n/\sqrt{d}\right)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\pi)^{-n/2} e^{-\frac{1}{2}|y|^2} dy$,

$$\begin{aligned} \lim_{d \rightarrow +\infty} d \left(\|\nabla u_d\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 - \frac{d}{2-p} \left(\|u_d\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 - \|u_d\|_{L^p(\mathbb{S}^d, d\mu_d)}^2 \right) \right) \\ = \|\nabla v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \frac{1}{2-p} \left(\|v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \|v\|_{L^p(\mathbb{R}^n, d\gamma)}^2 \right) \end{aligned}$$

Stability results for the logarithmic Sobolev inequality

Joint papers with M.J. Esteban, A. Figalli, R. Frank, M. Loss

*Sharp stability for Sobolev and log-Sobolev inequalities, with
optimal dimensional dependence*

[arXiv: 2209.08651](#), Cambridge J. Math. 2025

Joint papers with G. Brigati and N. Simonov

Stability for the logarithmic Sobolev inequality

Journal of Functional Analysis, 287, oct. 2024

[arXiv: 2411.13271](#)

*Logarithmic Sobolev inequalities:
a review on stability and instability results*

La Matematica 5, 2026

[arXiv: 2504.08658](#)

▷ *Entropy methods, with constraints*

A stability result for the logarithmic Sobolev inequality

Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\begin{aligned} \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - \frac{1}{4} d(d-2) \left(\|F\|_{L^{2^*}(\mathbb{S}^d)}^2 - \|F\|_{L^2(\mathbb{S}^d)}^2 \right) \\ \geq \frac{\beta}{d} \inf_{G \in \mathcal{M}(\mathbb{S}^d)} \left(\|\nabla F - \nabla G\|_{L^2(\mathbb{S}^d)}^2 + \frac{1}{4} d(d-2) \|F - G\|_{L^2(\mathbb{S}^d)}^2 \right) \end{aligned}$$

Rescale by $\sqrt{d}$, consider a function depending only on n coordinates and take the limit as $d \rightarrow +\infty$ to approximate the Gaussian measure $d\gamma = e^{-\pi|x|^2} dx$

Corollary (JD, Esteban, Figalli, Frank, Loss)

With $\beta > 0$ as in the result for the Sobolev inequality

$$\begin{aligned} \|\nabla u\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \pi \int_{\mathbb{R}^n} u^2 \log \left(\frac{|u|^2}{\|u\|_{L^2(\mathbb{R}^n, d\gamma)}^2} \right) d\gamma \\ \geq \frac{\beta \pi}{2} \inf_{a \in \mathbb{R}^n, c \in \mathbb{R}} \int_{\mathbb{R}^n} |u - c e^{a \cdot x}|^2 d\gamma \end{aligned}$$

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon x$ in the limit as $\varepsilon \rightarrow 0$

$$d(u_\varepsilon, 1)^2 = \|u'_\varepsilon\|_{L^2(\mathbb{R}, d\gamma)}^2 = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathcal{M}} d(u_\varepsilon, w)^\alpha \leq \frac{1}{2} \varepsilon^4 + O(\varepsilon^6)$$

$\mathcal{M} := \{w_{a,c} : (a, c) \in \mathbb{R}^d \times \mathbb{R}\}$ where $w_{a,c}(x) = c e^{-a \cdot x}$

Proposition

For all $u \in H^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_2 = 1$ and $\|x u\|_2^2 \leq d$, we have

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \geq \frac{1}{2d} \left(\int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \right)^2$$

and, with $\psi(s) := s - \frac{d}{4} \log(1 + \frac{4}{d} s)$,

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \geq \psi \left(\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 \right)$$

Unconditional stability with worse exponents

[Bakry, Ledoux '06], [Toscani '14], [JD, Toscani '16]

Lemma

Let $d \geq 1$. With $\varphi(t) := \frac{d}{4} \left[\exp\left(\frac{2t}{d}\right) - 1 - \frac{2t}{d} \right]$

$$\begin{aligned} \int_{\mathbb{R}^d} |\nabla v|^2 d\gamma - \frac{1}{2} \int_{\mathbb{R}^d} |v|^2 \log |v|^2 d\gamma \\ \geq \varphi \left(\int_{\mathbb{R}^d} |v|^2 \log |v|^2 d\gamma + \frac{d}{2} - \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 |v|^2 d\gamma \right) \end{aligned}$$

for any $v \in H^1(\mathbb{R}^d, d\gamma)$ such that $\|v\|_{L^2(\mathbb{R}^d, d\gamma)} = 1$

Counter-examples to the H^1 stability if $\|x u\|_2^2 > d$

[Indrei, Marcon '14], [Kim '18], [Kim, Indrei '21], [Indrei '21-'23],
 [Brigati, JD, Simonov '25]

Stability under log-concavity

Theorem

For all $u \in H^1(\mathbb{R}^d, d\gamma)$ such that $u^2 \gamma$ is log-concave and such that

$$\int_{\mathbb{R}^d} (1, x) |u|^2 d\gamma = (1, 0) \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 |u|^2 d\gamma \leq K$$

we have

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \frac{\mathcal{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \geq 0$$

$$\mathcal{C}_\star = 1 + \frac{1}{432K} \approx 1 + \frac{0.00231481}{K}$$

Self-improving Poincaré inequality and stability for LSI

[Fathi, Indrei, Ledoux, '16]

Stability under weak decay conditions

Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathcal{C} > 1$ depending only on ε such that, for any $u \in H^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x) |u|^2 d\gamma = (1, 0), \quad \int_{\mathbb{R}^d} |u|^2 e^{\varepsilon |x|^2} d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 \geq \frac{\mathcal{C}}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma$$

with $\mathcal{C} = 1 + \frac{\mathcal{C}_*(K_*) - 1}{1 + R^2 \mathcal{C}_*(K_*)}$, $K_* := \max\left(d, \frac{(d+1)R^2}{1+R^2}\right)$ if $\text{supp}(u) \subset B(0, R)$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

Fundamental solution of a nonlinear kinetic equation

We consider the nonlinear kinetic equation

$$\partial_t f + v \cdot \nabla_x f = \Delta_v f^m \quad (t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \quad (\text{kFP})$$

The *pressure* variable

$$P := \frac{m}{1-m} f^{m-1}$$

solves

$$\partial_t P = (1-m) P \Delta_v P - |\nabla_v P|^2 - v \cdot \nabla_x P$$

Special (fundamental) solution: with $A = \frac{1+d-dm}{3-d+dm}$

$$P_*(t, x, v) = \beta(t) + \frac{(1+A)}{2(1-A)t} \left(\left| v - \frac{x}{(1-A)t} \right|^2 + A \left| \frac{x}{(1-A)t} \right|^2 \right)$$

$$f_*(t, x, v) = \left(\frac{1-m}{m} P_*(t, x, v) \right)_+^{\frac{1}{m-1}}$$

$$m_1 := 1 - \frac{1}{d} < m < 1 \quad \text{or} \quad 1 < m < m_2 := 1 + \frac{1}{d}$$

Intermediate asymptotics

The *fundamental solution* $f_\star$ governs the large time behaviour of all solutions of (kFP) with initial datum f_0 such that

$$f_0 \in L^1(\mathbb{R}^d \times \mathbb{R}^d), \quad \|f_0\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 1$$

$$g_{\gamma_1}(x, v - x) \leq f_0(x, v) \leq g_{\gamma_2}(x, v - x)$$

for any $(x, v) \in \mathbb{R}^d \times \mathbb{R}^d$, with $(1 - m)\gamma_2 > (1 - m)\gamma_1 > 0$ and

$$g_\gamma(x, v) := \left(\frac{1-m}{m} \left(\gamma + \frac{1+A}{2} |v|^2 + \frac{1+A}{2} A |x|^2 \right) \right)_+^{\frac{1}{m-1}}$$

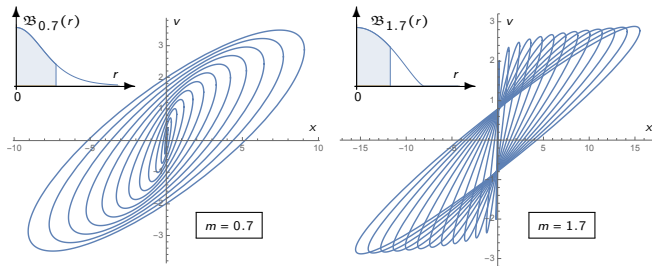
Theorem

With $d \geq 1$ and $m \in (m_1, 1) \cup (1, m_2)$,

$$\lim_{t \rightarrow +\infty} \|f(t, \cdot, \cdot) - f_\star(t, \cdot, \cdot)\|_{L^1(\mathbb{R}^d \times \mathbb{R}^d)} = 0$$

Level lines on the phase space

The level lines of the *fundamental solution* $f_\star$ are ellipses, which rotate and expand in the phase space $\mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$ as t increases



Formal diffusion limit / overdamped regime

With $\varepsilon \rightarrow 0$ and $2(d m - d + 1)\eta = 3$,

$$f(t, x, v) = \left(\frac{\varepsilon^{2\eta}}{R(\varepsilon t)} \right)^d h_\varepsilon \left(\varepsilon^2 \tau(\varepsilon t), \varepsilon^{\eta+1} x, \frac{\varepsilon^{\eta-1} v}{R(\varepsilon t)} \right)$$

$$R(s) = (1 + s/\alpha)^\alpha \quad \text{with } 1/\alpha = d m - d + 2 = d(m - m_c)$$

so that $(\tau, x, v) \mapsto h_\varepsilon(\tau, x, v)$ solves

$$\varepsilon \frac{d\tau}{ds}(s) \partial_\tau h_\varepsilon + R(s) v \cdot \nabla_x h_\varepsilon = \frac{\sigma(s)}{\varepsilon} (\Delta_v h_\varepsilon^m + \nabla_v \cdot (v h_\varepsilon))$$

$\sigma(s) := (1 + s/\alpha)^{-1}$, $\tau = \tau(s)$ to be chosen, and $s = \varepsilon t$

Formal *Hilbert expansion*

$$h_\varepsilon(\tau, x, v) \approx H(\tau, x, v) := \left(\mu(\tau, x) + \frac{1-m}{2m} |v|^2 \right)_+^{\frac{1}{m-1}}$$

$$\text{where } \rho(\tau, x) = \int_{\mathbb{R}^d} \left(\mu(\tau, x) + \frac{1-m}{2m} |v|^2 \right)_+^{\frac{1}{m-1}} dv = \left(\frac{\mu(\tau, x)}{\mu_1} \right)^{\frac{d}{2} + \frac{1}{m-1}}$$

$$\mu(\tau, x) = \mu_1 (\rho(\tau, x))^{k-1}, \quad k = 1 + 2\alpha(m-1)$$

🟢 *local mass conservation* : $\rho_\varepsilon(\tau, x) := \int_{\mathbb{R}^d} h_\varepsilon(\tau, x, v) dv$ solves

$$\frac{1}{R} \frac{d\tau}{ds} \partial_\tau \rho_\varepsilon + \nabla_x \cdot j_\varepsilon = 0$$

🟢 *flux* $j_\varepsilon(\tau, x) := \frac{1}{\varepsilon} \int_{\mathbb{R}^d} v h_\varepsilon(\tau, x, v) dv$

$$\frac{\varepsilon}{R} \frac{d\tau}{ds} \partial_\tau j_\varepsilon + \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v h_\varepsilon(\tau, x, v) dv = -\frac{\sigma}{R} j_\varepsilon$$

Standard heuristics apply: we drop the $O(\varepsilon)$ term

$$-\frac{\sigma(s)}{R(s)} j_\varepsilon(\tau, x) \approx \nabla_x \cdot \int_{\mathbb{R}^d} v \otimes v H(\tau, x, v) dv = \nu_1 \nabla_x (\rho(\tau, x)^k)$$

for some explicit numerical constant $\nu_1 > 0$

• $\rho_\varepsilon \approx \rho$ solves the nonlinear diffusion equation

$$\partial_\tau \rho = \Delta_x \rho^k \quad \text{where} \quad \frac{1}{k-1} = \frac{1}{m-1} + \frac{d}{2}$$

in the limit as $\varepsilon \rightarrow 0$ if we choose the time scale $s \mapsto \tau(s)$ such that

$$\frac{d\tau}{ds} = \nu_1 \frac{R(s)^2}{\sigma(s)}$$

with the condition $\tau(0) = 0$, i.e.,

$$\tau(t) = \frac{\alpha \nu_1}{2(1+\alpha)} \left((1 + t/\alpha)^{2(1+\alpha)} - 1 \right)$$

• Self-similar Barenblatt-Pattle solution, with $1/\beta = d(k-1) + 2$

$$\rho(\tau, x) = (\tau/\beta)^{-d\beta} \rho_*((\tau/\beta)^{-\beta} x)$$

▷ Assuming that α is positive, that is, $m > m_c$, we notice that $k-1$ and $m-1$ have the same sign because $k-1 = 2\alpha(m-1)$

▷ This formal computation requires at least a second moment

$$m > \tilde{m}_1 = d/(d+1) \iff k > d/(d+2)$$

Existence and diffusion limits

Joint paper with *Emeric Bouin & Antoine Mellet*

Nonlinear kinetic Fokker-Planck equations: existence and diffusion limits. [arXiv: 2606.31899](https://arxiv.org/abs/2606.31899)

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \Delta_v f^m + \nabla_v \cdot (v f) \quad \text{with } t > 0, x \in \Omega, v \in \mathbb{R}^d$$

$$m_c := (d-2)/d, m_0 := d/(d+2) \text{ and } m_1 := (d-1)/d$$

Theorem

Let $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$ with $d \geq 1$ and assume that $m \in (m_c, 1) \cup (1, +\infty)$. For any initial condition $f_{\text{in}} \in L^1_+ \cap L^\infty(\Omega \times \mathbb{R}^d)$ + (x, v) moments, Equation (kFP) has a unique solution $f \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^1 \cap L^\infty(\Omega \times \mathbb{R}^d))$ with $\nabla_v f^{q/2} \in L^2_{\text{loc}}(\mathbb{R}^+ \times \Omega \times \mathbb{R}^d)$ for all $q > m$. Moreover, if $m > m_0$, the solution f satisfies the entropy dissipation inequality

$$\mathcal{F}[f(t)] + \int_0^t \iint_{\Omega \times \mathbb{R}^d} f \left| v + \frac{m}{m-1} \nabla_v f^{m-1} \right|^2 dx dv dt \leq \mathcal{F}[f_{\text{in}}]$$

Diffusion limit

$$\varepsilon^2 \frac{\partial f}{\partial t} + \varepsilon v \cdot \nabla_x f = \Delta_v f^m + \nabla_v \cdot (v f) \quad \text{with } t > 0, x \in \Omega, v \in \mathbb{R}^d \text{ (kFP}_\varepsilon)$$

... limit as $\varepsilon \rightarrow 0_+$? Local equilibrium with spatial density ρ

$$G[\rho](v) := \left(\mu_1 \rho^{k-1} - \frac{m-1}{2m} |v|^2 \right)_+^{\frac{1}{m-1}}$$

Theorem

Let $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$ with $d \geq 1$ and assume that $m \in (m_1, 1) \cup (1, +\infty)$. Let $f_{\text{in}} \in L^1(\Omega \times \mathbb{R}^d)$ be such that $\mathcal{F}[f_{\text{in}}]$ is finite and $0 \leq f_{\text{in}} \leq G[M]$ a.e. for some constant $M > 0$ and, if $\Omega = \mathbb{R}^d$, $\mathcal{M}_{a,0}[f_{\text{in}}]$ is finite for some $a \in (0, 1)$ if $m < 1$ and $a = 2$ if $m > 1$. For all $T > 0$, the solution f_ε to (kFP $_\varepsilon$) with initial condition f_{in} converges strongly in $L^1((0, T) \times \Omega \times \mathbb{R}^d)$ to $G[\rho]$ and $\rho_\varepsilon := \int_{\mathbb{R}^d} f_\varepsilon(\cdot, \cdot, v) dv$ converges strongly in $L^1((0, T) \times \Omega)$ to the solution ρ to

$$\partial_t \rho = \nu_1 \Delta \rho^k \quad \text{in } [0, T] \times \Omega \quad \text{where} \quad \frac{1}{k-1} = \frac{1}{m-1} + \frac{d}{2}$$

with initial condition $\rho_{\text{in}} = \int_{\mathbb{R}^d} f_{\text{in}}(\cdot, v) dv$

Gradient flows

Nonlinear kinetic Fokker–Planck equation ($m > 1$, porous medium case only)

$$\partial_t f + v \cdot \nabla_x f = \nabla_v \cdot (f (\nabla_v P_m(f) + v)) \quad (\text{kFP})$$

time $t \geq 0$, phase space $\Gamma := \mathbb{R}^d \times \mathbb{R}^d \ni (x, v)$, *pressure variable* $P_m(f)$

$$P_m(f) := \frac{m}{m-1} f^{m-1} \quad (\text{sign convention !})$$

$P_1(f) := \log f$ (case $m = 1$): linear Vlasov–Fokker–Planck equation corresponding to the second-order Langevin stochastic

$$\frac{d}{dt} \mathcal{F}_m(f(t, \cdot, \cdot)) = -\mathcal{I}_m(f)$$

where

$$\mathcal{F}_m(f) := \iint_{\Gamma} \left(\frac{1}{m} P_m(f) + \frac{1}{2} |v|^2 \right) f \, dx \, dv, \quad \mathcal{I}_m(f) := \iint_{\Gamma} |\nabla_v P_m(f) + v|^2 f \, dx \, dv$$

Joint paper with *Giovanni Brigati, Guillaume Carlier & Filippo Quattrocchi*
Nonlinear kinetic Fokker–Planck equations as gradient flows of the free energy [arXiv: 2606.16315](https://arxiv.org/abs/2606.16315)

Transport by Vlasov, discrepancy and speed

Newton's equations associated to the force field $F_t(x, v)$

$$x'(t) = v(t), \quad v'(t) = F_t(x(t), v(t))$$

Vlasov equation

$$\partial_t \mu_t + v \cdot \nabla_x \mu_t + \nabla_v \cdot (F_t \mu_t) = 0 \quad \text{in} \quad \mathcal{D}'((0, T) \times \Gamma) \quad (\text{V})$$

$\mathcal{P}_2(\Gamma)$: set of probability measures μ with $\iint_{\Gamma} (|x|^2 + |v|^2) d\mu < \infty$
Discrepancy between μ and $\nu \in \mathcal{P}_2(\Gamma)$

$$d_T^2(\mu, \nu) := \inf_{(\mu_t, F_t)_{t \in [0, T]}} T \int_0^T \|F_t\|_{L^2(\mu_t)}^2 dt, \quad \mu_0 = \mu, \quad \mu_T = \nu$$

[Brigati, Maas, Quattrocchi] Optimisers are d_T -regular curves of measures $(\mu_t)_t$ and one can define a *speed* by

$$|\mu'_t|_d := \lim_{h \rightarrow 0_+} \frac{d_h(\mu_t, \mu_{t+h})}{h}$$

Chain rule for the free energy and gradient flow

For a smooth function f of the Vlasov equation (V)

$$\nabla_v \cdot (f \nabla_v P_m(f)) = \Delta_v f^m, \quad \sqrt{f} \nabla_v P_m(f) = \frac{m}{m-1/2} \nabla_v (f^{m-1/2})$$

$$\mathcal{I}_m(f) = \iint_{\Gamma} \left(\left(\frac{2m}{2m-1} \right)^2 |\nabla_v (f^{m-1/2})|^2 + |v|^2 f - 2 d f^m \right) dx dv$$

Weak solutions $0 \leq f \in L^1_{\text{loc}}(\mathbb{R}^+; L^1(1 + |v|^2, \Gamma)) \cap L^m_{\text{loc}}(\mathbb{R}^+; L^m(\Gamma))$

Theorem

Let $m \in [1, 3/2]$ and $T > 0$. If $\mu_t = f(t, \cdot, \cdot)$ solves (V) in $\mathcal{P}_2(\Gamma)$ with $\int_0^T (\mathcal{F}_m(\mu_t) + \mathcal{I}_m(\mu_t)) dt + \int_0^T \|F_t\|_{L^2(\mu_t)}^2 dt < \infty$, for $0 < a < b < T$ a.e.

$$\begin{aligned} \mathcal{F}_m(\mu_b) - \mathcal{F}_m(\mu_a) &= \int_a^b \iint_{\Gamma} F_t \cdot (\nabla_v P_m(\mu_t) + v) \mu_t dx dv dt \\ &\geq -\frac{1}{2} \int_a^b (|\mu'_t|_{\mathcal{d}}^2 + \mathcal{I}_m(\mu_t)) dt \end{aligned}$$

with equality iff f solves (kFP) on $[0, T]$, in which case $|\mu'_t|_{\mathcal{d}}^2 = \mathcal{I}_m(\mu_t)$

Minimising-movement scheme and maximal dissipation

A JKO type scheme $\mu_0^{(h)} = \mu_0 \in \mathcal{P}_2(\Gamma)$ with $\mathcal{F}_m(\mu_0) < \infty$

$$\begin{aligned} \mu_0^{(h)} &= \mu_0 \in \mathcal{P}_2(\Gamma) \quad \text{with} \quad \mathcal{F}_m(\mu_0) < \infty \\ \mu_{k+1}^{(h)} &\in \arg \min_{\nu \in \mathcal{P}_2(\Gamma)} \left(\mathcal{F}_m(\nu) + \frac{1}{2h} d_h^2(\mu_k^{(h)}, \nu) \right) \end{aligned} \quad (\text{JKO})$$

Piecewise constant interpolation: $\mu_t^{(h)}$

Theorem

Let $m \geq 1$

- (JKO) is well-posed for any $h \in (0, 1)$ and $k \in \mathbb{N}$
- $(\mu_t^{(h)})_h$ relatively compact in $L_{\text{loc}}^1(\mathbb{R}^+; L^1(\Gamma))$
- Up to the extraction of a sequence, any limit $\mu = \lim_{h \rightarrow 0_+} \mu^{(h)}$ solves (kFP) in $\mathcal{D}'(\mathbb{R}^+ \times \Gamma)$ with initial datum μ_0

Hypocoercivity methods in kinetic equations

- 🔵 Global overview of hypocoercivity methods Fokker-Planck and Poincaré inequality
- 🔵 Classification of the rates of convergence and further results

*Based on the joint paper with Clément Mouhot and Christian Schmeiser
J. Dolbeault, C. Mouhot, and C. Schmeiser. Hypocoercivity for
linear kinetic equations conserving mass. Trans. Amer. Math. Soc.,
367 (6): 3807-3828, 2015*

and extensions obtained with Emeric Bouin and various other collaborators

*Joint paper with Emeric Bouin and Luca Ziviani
Weighted L^2 bounds for kinetic Fokker-Planck equations and
hypocoercivity. arXiv: 2608.28120*

• H^1 hypocoercivity: an example

$$\frac{\partial f}{\partial t} + \mathsf{T}f = \Delta_v f + \nabla_v \cdot (vf), \quad \mathsf{T}f := v \cdot \nabla_x f - x \cdot \nabla_v f \quad (\text{VFP})$$

[JD, X. Li] take $h = (f/f_*)^{2/p}$, $p \in (1, 2)$

$$\frac{\partial h}{\partial t} + \mathsf{T}h = \mathsf{L}h + \frac{2-p}{p} \frac{|\nabla_v h|^2}{h}, \quad \mathsf{L}h := \Delta_v h - v \cdot \nabla_v h$$

Twisted Fisher information

$$\mathcal{J}_\lambda[h] = (1-\lambda) \int_{\mathbb{R}^d} |\nabla_v h|^2 d\mu + (1-\lambda) \int_{\mathbb{R}^d} |\nabla_x h|^2 d\mu + \lambda \int_{\mathbb{R}^d} |\nabla_x h + \nabla_v h|^2 d\mu$$

Theorem

For some $t \mapsto \lambda(t)$, there is a function $t \mapsto \rho(t) > 1$ a.e.

$$\frac{d}{dt} \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq -\rho(t) \mathcal{J}_{\lambda(t)}[h(t, \cdot)] \quad \forall t \geq 0$$

and $\mathcal{J}_{\lambda(t)}[h(t, \cdot)] \leq \mathcal{J}_{1/2}[h_0] \exp\left(-\int_0^t \rho(s) ds\right)$

[C. Villani] [JD, X. Li] + coupling with a Beckner type inequality

🟢 H^{-1} hypocoercivity: a brief summary

- ▷ [S. Armstrong, J.-C. Mourrat, 2019], [D. Albritton, S. Armstrong, J.-C. Mourrat, M. Novack, 2024], [Y. Cao, J. Lu, L. Wang, 2020-23]. Variational methods for the kinetic Fokker-Planck equation.
- ▷ [G. Brigati, 2021], [G. Brigati & *al.*]. Time averages for kinetic Fokker-Planck equations

Consider the *kinetic-Ornstein-Uhlenbeck equation*

$$\partial_t h + v \cdot \nabla_x h = \Delta_\alpha h := \Delta_v h - \alpha v \langle v \rangle^{\alpha-2} \cdot \nabla_v h, \quad h(0, \cdot, \cdot) = h_0 \quad (\text{OU})$$

on $\mathbb{R}^+ \times \mathbb{T}^d \times \mathbb{R}^d$ with local equilibrium $\gamma_\alpha(v) = Z_\alpha^{-1} e^{-\langle v \rangle^\alpha}$

Theorem (G. Brigati)

Let $\alpha \geq 1$ and $\tau > 0$. There exists a constant $\lambda > 0$ such that, for all $h_0 \in L^2(dx d\gamma_\alpha)$ with zero-average,

$$\int_t^{t+\tau} \|h(s, \cdot, \cdot)\|_{L^2(dx d\gamma_\alpha)}^2 ds \leq \|h_0\|_{L^2(dx d\gamma_\alpha)}^2 e^{-\lambda t} \quad \forall t \geq 0$$

Averaging lemma + Poincaré inequality (based on JL Lions' lemma)

• L^2 hypocoercivity: the strategy

[JD, C. Mouhot, C. Schmeiser] Π is the orthogonal projection on $\text{Ker}(L)$

$$\varepsilon \frac{dF}{dt} + \mathbb{T}F = \frac{1}{\varepsilon} LF$$

$$F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3) \text{ as } \varepsilon \rightarrow 0_+, \quad u = F_0 = \Pi F_0$$

$$\partial_t u + (\mathbb{T}\Pi)^* (\mathbb{T}\Pi) u = 0$$

▷ The *microscopic coercivity* assumption

$$\frac{1}{2} \frac{d}{dt} \|F\|^2 = \langle LF, F \rangle \leq -\lambda_m \|(1 - \Pi)F\|^2$$

is not enough to conclude that $\|F(t, \cdot)\|^2$ decays exponentially and a *macroscopic coercivity* assumption (Poincaré inequality) is needed

$$\|\mathbb{T}\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2$$

$A := (1 + (\mathbb{T}\Pi)^* \mathbb{T}\Pi)^{-1} (\mathbb{T}\Pi)^*$ is such that $\langle A \mathbb{T}\Pi F, F \rangle \geq \frac{\lambda_M}{1 + \lambda_M} \|\Pi F\|^2$ and we can use the L^2 entropy / Lyapunov functional

$$H[F] := \frac{1}{2} \|F\|^2 + \delta \text{Re} \langle A F, F \rangle$$

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$
 $*$ denotes the adjoint with respect to $\langle \cdot, \cdot \rangle$

Π is the orthogonal projection onto the null space of L

The estimate

$$\frac{1}{2} \frac{d}{dt} \|F\|^2 = \langle \mathsf{L}F, F \rangle \leq -\lambda_m \|(1 - \Pi)F\|^2$$

is not enough to conclude that $\|F(t, \cdot)\|^2$ decays exponentially

$\Leftarrow$ *microscopic coercivity*

Formal macroscopic / diffusion limit

$F = F(t, x, v)$, $T = v \cdot \nabla_x$, L good collision operator. Scaled evolution equation

$$\varepsilon \frac{dF}{dt} + TF = \frac{1}{\varepsilon} LF$$

on the Hilbert space $\mathcal{H}$. With $F_\varepsilon = F_0 + \varepsilon F_1 + \varepsilon^2 F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \rightarrow 0_+$

$$\varepsilon^{-1} : \quad LF_0 = 0,$$

$$\varepsilon^0 : \quad TF_0 = LF_1,$$

$$\varepsilon^1 : \quad \frac{dF_0}{dt} + TF_1 = LF_2$$

▷ the first equation reads as $u = F_0 = \Pi F_0$

▷ the second equation is simply solved by $F_1 = -(\Pi T) F_0$

▷ after projection, the third equation is $\frac{d}{dt}(\Pi F_0) - \Pi T(\Pi T) F_0 = \Pi L F_2 = 0$

$$\partial_t \rho + (\Pi T)^* (\Pi T) \rho = 0$$

is such that $\frac{d}{dt} \|\rho\|^2 = -2 \|(\Pi T) \rho\|^2 \leq -2 \lambda_M \|\rho\|^2$

← *macroscopic coercivity*

The assumptions in the compact case

λ_m , λ_M , and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity*

$$-\langle LF, F \rangle \geq \lambda_m \|(1 - \Pi)F\|^2 \quad (\text{H1})$$

▷ *macroscopic coercivity*

$$\|\Pi F\|^2 \geq \lambda_M \|\Pi F\|^2 \quad (\text{H2})$$

▷ *parabolic macroscopic dynamics:*

$$\Pi T \Pi F = 0 \quad (\text{H3})$$

▷ *bounded auxiliary operators:*

$$\|AT(1 - \Pi)F\| + \|ALF\| \leq C_M \|(1 - \Pi)F\| \quad (\text{H4})$$

Equivalence and entropy decay: an abstract result

For some $\delta > 0$ to be chosen, the L^2 entropy / Lyapunov functional is defined by

$$H[F] := \frac{1}{2} \|F\|^2 + \delta \operatorname{Re}\langle AF, F \rangle$$

▷ *norm equivalence* of $H[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4} \|F\|^2 \leq H[F] \leq \frac{2+\delta}{4} \|F\|^2$$

Entropy decay: $\frac{d}{dt} H[F] = -D[F]$

▷ *entropy decay rate*: for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$D[F] \geq \lambda H[F]$$

[JD, C. Mouhot, C. Schmeiser, 2015]

Theorem

Under (H1)–(H4), for any $t \geq 0$,

$$H[F(t, \cdot)] \leq H[F_0] e^{-\lambda t}$$

$$\|F(t, \cdot)\|^2 \leq \mathcal{C} \|F_0\|^2 e^{-\lambda t} \quad \text{with} \quad \mathcal{C} = \frac{2+\delta}{2-\delta}$$

Details of the proof

$A := (1 + (\mathsf{T}\Pi)^* \mathsf{T}\Pi)^{-1} (\mathsf{T}\Pi)^*$ is such that $\langle A \mathsf{T}\Pi F, F \rangle \geq \frac{\lambda_M}{1 + \lambda_M} \|\Pi F\|^2$

With $X := \|(\text{Id} - \Pi)F\|$ and $Y := \|\Pi F\|$, we can prove that

$$\begin{aligned} \frac{1}{2} \|F\|^2 + \delta \operatorname{Re} \langle A F, F \rangle &=: H[F] \leq \frac{1}{2} (X^2 + Y^2) + \frac{\delta}{2} X Y \\ D[F] &\geq (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y \end{aligned}$$

The largest value of $\lambda = \lambda(\delta)$ for which $D[F] - \lambda H[F] \geq 0$ can be estimated by the largest value of λ for which

$$\begin{aligned} Q(X, Y) &:= (\lambda_m - \delta) X^2 + \frac{\delta \lambda_M}{1 + \lambda_M} Y^2 - \delta C_M X Y - \frac{\lambda}{2} (X^2 + Y^2) - \frac{\lambda}{2} \delta X Y \\ &= \left(\lambda_m - \delta - \frac{\lambda}{2} \right) X^2 - \delta \left(C_M + \frac{\lambda}{2} \right) X Y + \left(\frac{\delta \lambda_M}{1 + \lambda_M} - \frac{\lambda}{2} \right) Y^2 \end{aligned}$$

is a nonnegative quadratic form

A classical example with confinement

- Vlasov-Fokker-Planck equation (harmonic potential)

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \Delta_v f + \nabla_v \cdot (v f) \quad (\text{VFP})$$

With $\rho := \int_{\mathbb{R}^d} F(v) dv$, $\Pi F := \rho \mathcal{M}(v)$, $\phi(x) = \frac{1}{2} |x|^2$

▷ *microscopic coercivity*: Gaussian Poincaré inequality in v

$$\int_{\mathbb{R}^d} |F(v) - \rho \mathcal{M}(v)|^2 \frac{dv}{\mathcal{M}(v)} \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_v F(v)|^2 \frac{dv}{\mathcal{M}(v)} \quad (\text{H1})$$

▷ *macroscopic coercivity*: Gaussian Poincaré inequality in x

$$\int_{\mathbb{R}^d} \left| \rho(x) - \frac{M e^{-|x|^2/2}}{(2\pi)^{d/2}} \right|^2 e^{\frac{|x|^2}{2}} dx \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\nabla_x \rho(x)|^2 e^{\frac{|x|^2}{2}} dx \quad (\text{H2})$$

▷ *parabolic macroscopic dynamics* (H3) and *bounded auxiliary operators* (H4) are consequences of elliptic estimates

Another example: without confinement

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = Lf, \quad f(0, x, v) = f_0(x, v)$$

L is the *Fokker-Planck operator* L_1 or the *linear BGK operator* L_2

$$L_1 f := \Delta_v f + \nabla_v \cdot (v f) \quad \text{and} \quad L_2 f := \rho_f \mu - f$$

$\mu(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$ is the normalized Gaussian function

$\rho_f := \int_{\mathbb{R}^d} f \, dv$ is the spatial density

$$d\gamma := \gamma(v) \, dv \quad \text{where} \quad \gamma := \frac{1}{\mu}$$

Weighted norm

$$\|f\|_{L^2(dx \, d\gamma)}^2 := \iint_{\mathbb{R}^d \times \mathbb{R}^d} |f(x, v)|^2 \, dx \, d\gamma$$

On the whole Euclidean space, we can define the entropy

$$H[f] := \frac{1}{2} \|f\|_{L^2(dx d\gamma)}^2 + \delta \langle Af, f \rangle_{dx d\gamma}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|_{L^2(dx)}^2 \leq C_{\text{Nash}} \|u\|_{L^1(dx)}^{\frac{4}{d+2}} \|\nabla u\|_{L^2(dx)}^{\frac{2d}{d+2}}$$

proves that

$$H[f] \leq C \left(H[f_0] + \|f_0\|_{L^1(dx dv)}^2 \right) (1+t)^{-\frac{d}{2}}$$

[Bouin, JD, Mischler, Mouhot, Schmeiser, 2020]

Theorem

There exists a constant $C > 0$ such that, for any $t \geq 0$

$$\|f(t, \cdot, \cdot)\|_{L^2(dx d\gamma)}^2 \leq C \left(\|f_0\|_{L^2(dx d\gamma)}^2 + \|f_0\|_{L^2(d\gamma; L^1(dx))}^2 \right) (1+t)^{-\frac{d}{2}}$$

By the *Nash inequality* (1958)

$$\|u\|_{L^2(dx)}^2 \leq C_{\text{Nash}} \|u\|_{L^1(dx)}^{\frac{4}{d+2}} \|\nabla u\|_{L^2(dx)}^{\frac{2d}{d+2}} \quad \forall u \in L^1 \cap H^1(\mathbb{R}^d)$$

$$\|\Pi f\|^2 \leq \Phi^{-1}(2 \langle A \Pi f, f \rangle) \quad \text{with} \quad \Phi^{-1}(y) := y + \left(\frac{y}{c}\right)^{\frac{d}{d+2}} \quad \forall y \geq 0$$

$$\text{where } c = 2 \Theta C_{\text{Nash}}^{-1 - \frac{2}{d}} \|f_0\|_{L^1(dx dv)}^{-\frac{4}{d}}$$

We deduce the entropy decay inequality

$$D[F] = -\frac{d}{dt} H[f] \gtrsim \left(\|f_0\|_{L^2(dx d\gamma)}^{\frac{4}{d+2}} + \|f_0\|_{L^1(dx dv)}^{\frac{4}{d+2}} \right)^{-\frac{d+2}{d}} H[f]^{1+\frac{2}{d}}$$

The classical example revisited: Poincaré inequality

• *homogeneous* Fokker-Planck equation (no v dependence)

If $\phi(x) = \alpha^{-1} \langle x \rangle^\alpha$ with $\alpha \geq 1$, $\langle x \rangle := \sqrt{1 + |x|^2}$, and $d\mu = e^{-\phi} dx$, then

$$\int_{\mathbb{R}^d} |\nabla u|^2 d\mu \geq \lambda_M \int_{\mathbb{R}^d} |u|^2 d\mu \quad \forall u \in \mathcal{D}(\mathbb{R}^d) \quad \text{s.t.} \quad \int_{\mathbb{R}^d} u d\mu = 0$$

with $\lambda_M > 0$ (false if $\alpha < 1$). A solution $\rho = e^\phi u$ of

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \nabla \cdot (\rho \nabla \phi) \quad (\text{FP})$$

satisfies

$$\frac{d}{dt} \|\rho(t, \cdot)\|_{L^2(\mathbb{R}^d, d\mu)}^2 = -2 \|\nabla \rho(t, \cdot)\|_{L^2(\mathbb{R}^d, d\mu)}^2 \leq -2 \lambda_M \|\rho(t, \cdot)\|_{L^2(\mathbb{R}^d, d\mu)}^2$$

which yields the estimate

$$\|\rho(t, \cdot)\|_{L^2(\mathbb{R}^d, d\mu)}^2 \leq \|\rho_0\|_{L^2(\mathbb{R}^d, d\mu)}^2 e^{-2 \lambda_M t} \quad \forall t \geq 0$$

Rates of convergence (homogeneous case)

Potential	$\phi = 0$	$\phi(x) = \gamma \log \langle x \rangle$ $\gamma < d$	$\phi(x) = \gamma \log \langle x \rangle$ $\gamma > d$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \in (0, 1)$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn-Nirenberg	Hardy-Poincaré	Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-k/2}$ convergence	$t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence
References	[7]	[4]	[5]	[3]	(*)

Table 1 Alternative: *weak Poincaré inequalities*: see [8, Theorem 2.1], [1, Theorem 1.4] and [6]. (*) *Poincaré inequality*: [7], [2]

References [1] D. Bakry, P. Cattiaux, A. Guillin, 2008; [2] D. Bakry, I. Gentil, M. Ledoux, 2014; [3] E. Bouin, JD, L. Lafleche, C. Schmeiser, 2020; [4] E. Bouin, JD, C. Schmeiser, 2020; [5] E. Bouin, JD, L. Ziviani, 2024; [6] O. Kavian, S. Mischler, M. Ndao, 2021; [7] J. Nash, 1958; [8] M. Röckner, F.Y. Wang, 2001

Generalized kinetic Fokker-Planck equation

Generalized kinetic Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = e^{-\psi(v)} \nabla_v \left(e^{\psi(v)} \nabla_v f \right), \quad f_\star(x, v) = Z^{-1} \exp(-\phi(x) - \psi(v)) \quad (\text{gKFP})$$

Potential	$\phi = 0$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \in (0, 1)$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
$\psi(v) = \frac{1}{\beta} \langle v \rangle^\beta$ $\beta \geq 1$ Micro Poincaré	$t^{-d/2}$ decay [4]	e^{-t^b} , $b < 1$ $\beta = 2$ convergence [7]	$e^{-\lambda t}$ convergence [11, 13, 9, 10, 12, 1, 8, 5, 6]
$\psi(v) = \frac{1}{\beta} \langle v \rangle^\beta$ $\beta \in (0, 1)$	$t^{-\zeta}$ $\zeta = \min\{\frac{d}{2}, \frac{\ell}{2(1-\beta)}\}$ decay, [3]	$t^{-\zeta}$ convergence (*)	$t^{-\zeta}$ convergence (*)
Limit as $\beta \rightarrow 0_+$ $\psi(v) =$ $-(d + \varepsilon) \log \langle v \rangle$	$\varepsilon \in (0, 2)$ fractional dif- fusion limit, [2]	(*)	$t^{-\zeta}$ if $\varepsilon > 2$ convergence (*)

Table 2 Additional assumptions are needed: if $(\alpha, \beta) \in [1 + \infty) \times (0, 1)$, $f_0 \in L^2(\mathbb{R}^d \times \mathbb{R}^d, \langle v \rangle^{(1-\beta)\sigma} d\mu)$; if $\phi = 0$ and $\beta \in (0, 1)$, $f_0 \in L^2(\mathbb{R}^d \times \mathbb{R}^d, \langle v \rangle^{\ell/2} d\mu)$; if $(\alpha, \beta) \in (0, 1) \times (0, 1)$, uniform bounds (weak Poincaré inequality)

References [1] D. Albritton, S. Armstrong, J.C. Mourrat, M. Novack, 2024; [2] E. Bouin, JD, L. Lafleche, 2022; [3] E. Bouin, JD, L. Lafleche, C. Schmeiser, 2020; [4] E. Bouin, JD, S. Mischler, C. C. Mouhot Schmeiser, 2020; [5] G. Brigati, 2023; [6] G. Brigati, G. Stoltz, 2023; [7] C. Cao, 2019; [8] Y. Cao, J. Lu, L. Wang, 2023; [9] JD, C.C. Mouhot Schmeiser, 2009; [10] JD, C.C. Mouhot Schmeiser, 2015; [11] F. Hérau, 2006; [12] S. Mischler, C. Mouhot, 2016; [13] C. Mouhot, L. Neumann, 2006

With the transport operator defined by the Poisson bracket as

$$\mathcal{T}f := \nabla_v \mathcal{E} \cdot \nabla_x f - \nabla_x \mathcal{E} \cdot \nabla_v f$$

corresponding to the Hamiltonian energy $\mathcal{E}(x, v) := \frac{1}{\beta} \langle v \rangle^\beta + \phi(x)$
 [E. Bouin, JD, L. Ziviani, 2024] Under the condition

$$0 \leq f_0 \leq C f_\star, \quad f_\star = Z^{-1} e^{-\mathcal{E}(x, v)}$$

Potential	$\phi = 0$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \in (0, 1)$	$\phi(x) = \frac{1}{\alpha} \langle x \rangle^\alpha$ $\alpha \geq 1$
$\psi(v) = \frac{1}{\beta} \langle v \rangle^\beta$ $\beta \geq 1$ Micro Poincaré	$t^{-d/2}$ decay	$t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence
$\psi(v) = \frac{1}{\beta} \langle v \rangle^\beta$ $\beta \in (0, 1)$	$t^{-\min\{\frac{d}{2}, \frac{\ell}{2(1-\beta)}\}}$ convergence	$t^{-\min\{\frac{k}{2(1-\alpha)}, \frac{\ell}{2(1-\beta)}\}}$ convergence	$t^{-\frac{\ell}{2(1-\beta)}}$ convergence

Without the condition $0 \leq f_0 \leq C f_\star$, moments on the initial data are needed [E. Bouin, JD, L. Ziviani, 2026]... With $d\mu_\star := f_\star^{-1} dx dv \dots$

Theorem (E. Bouin, JD, L. Ziviani, 2026)

Let $f \in C^0(\mathbb{R}^+; L^1(dx dv) \cap L^2(d\mu_\star))$ be a solution with nonnegative initial datum $f_0 \in L^1(dx dv) \cap L^2(m^k d\mu_\star)$ with m such that

- (a) $m(x, v) = \mathcal{E}(x, v)$ if $\min\{\alpha, \beta\} > 2/3$, or $\alpha \geq 1$ and $\beta > 0$
- (b) $m(x, v) = \lfloor x \rfloor^{\max\{\alpha, 2\}} + \lfloor v \rfloor^{\max\{\beta, 2\}}$ if $\alpha > 0$ and $\beta > 1$
- (c) $m(x, v) = \mathcal{E}(x, v)$ if $\alpha \in (0, 1)$ and $\beta \in (0, 1)$

There is a constant $\mathcal{C}_\star > 0$ such that

$$\|f - f_\star\|_{L^2(d\mu_\star)}^2 \leq \mathcal{C}_\star (1+t)^{-\zeta/2} \|f_0 - f_\star\|_{L^2(d\mu_\star)}^2$$

with ζ such that:

- (a) $\zeta = k/\max\{1-\alpha, 1-\beta\}$ if $(\alpha, \beta) \in (2/3, 1)^2$,
 and $\zeta = k/(1-\beta)$ if $\alpha \geq 1$ and $\beta \in (0, 1)$
- (b) $\zeta = k/(1-\alpha)$ if $\alpha \in (0, 1)$ and $\beta > 1$
- (c) $\zeta = k/\max\{1-\alpha, 1-\beta\}$ if $(\alpha, \beta) \in (0, 1) \times (0, 1)$

L^2 Hypocoercivity: comments & further results

No *a priori* moment bounds or perturbative regimes: global estimates on weighted norms use two methods:

- ▷ a coupling of the weighted norms with entropy dissipation (DMS method)
- ▷ a Foster-Lyapunov condition approach if $(\alpha, \beta) \in (0, 1)^2 \times (0, 1]$

$$\nabla_v \psi \cdot \nabla_x m - \nabla_x \phi \cdot \nabla_v m + \Delta_v m - \nabla_v \psi \cdot \nabla_v m \leq C \mathbb{1}_{\{\mathcal{E} \leq R\}} - \frac{c_0 \kappa^{1+\sigma_*} m}{\{\ln(e + m)\}^{\sigma_*}}$$

with $\sigma_* = 2 \max\{\alpha^{-1} - 1, \beta^{-1} - 1\}$

Further results:

- 🟢 [Bouin, 2026] Same results for (gKFP)
- 🟢 Mode-by-mode methods [Bouin, JD, Mischler, Mouhot, Schmeiser, 2020], [A. Arnold, JD, C. Schmeiser, T. Wöhrer, 2021]
- 🟢 Fractional diffusion limits [Bouin, JD, Lafleche, 2022]
- 🟢 Macroscopic modes [K. Carapatoso, JD, F. Hérau, S. Mischler, C. Mouhot, C. Schmeiser, 2024]

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Thank you for your attention !