# Reverse Hardy-Littlewood-Sobolev inequalities

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*Joint work with J. A. Carrillo, M. G. Delgadino, R. Frank, F. Hoffmann*

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# Reverse Hardy-Littlewood-Sobolev inequality

# The reverse HLS inequality

For any $\lambda > 0$ and any measurable function $\rho \geq 0$ on $\mathbb{R}^N$, let

$$I_\lambda[\rho] := \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^\lambda \, \rho(x) \, \rho(y) \, dx \, dy$$

$$N \geq 1 \,, \quad 0 < q < 1 \,, \quad \alpha := \frac{2\,N - q\,(2\,N + \lambda)}{N\,(1-q)}$$

Convention: $\rho \in \mathrm{L}^p(\mathbb{R}^N)$ if $\int_{\mathbb{R}^N} |\rho(x)|^p \, dx$ for any $p > 0$

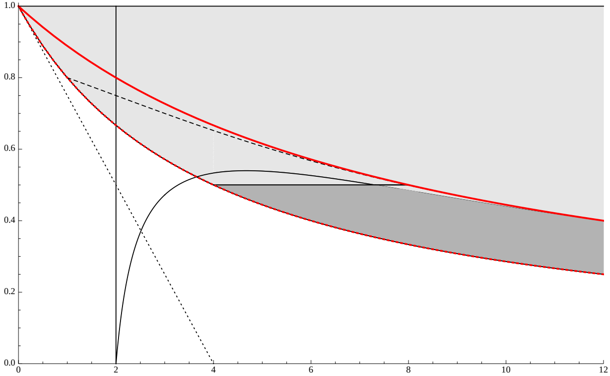
**Theorem**

*The inequality*

$$I_\lambda[\rho] \geq \mathcal{C}_{N,\lambda,q} \left( \int_{\mathbb{R}^N} \rho \, dx \right)^\alpha \left( \int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q} \tag{1}$$

*holds for any* $\rho \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ *with* $\mathcal{C}_{N,\lambda,q} > 0$ *if and only if* $q > N/(N+\lambda)$

*If either* $N = 1, 2$ *or if* $N \geq 3$ *and* $q \geq \min\left\{1 - 2/N \,,\, 2N/(2N+\lambda)\right\}$, *then there is a radial nonnegative optimizer* $\rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)$




*$N = 4$, region of the parameters $(\lambda, q)$ for which $\mathcal{C}_{N,\lambda,q} > 0$*
*Optimal functions exist in the light grey area*

# The conformally invariant case $q = 2N/(2N+\lambda)$

$$I_\lambda[\rho] = \iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy \ \geq \mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{2/q}$$

$$2N/(2N+\lambda) \quad \Longleftrightarrow \quad \alpha = 0$$

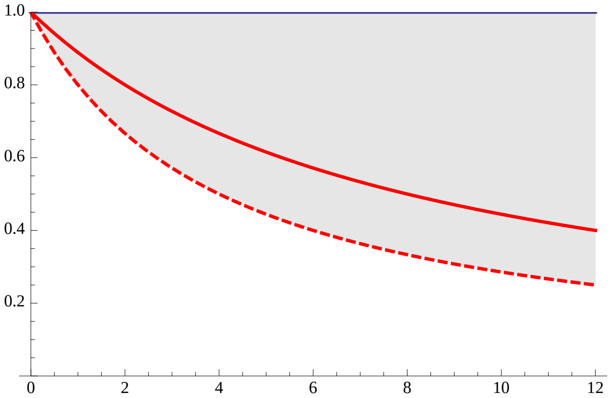
(Dou, Zhu 2015) (Ngô, Nguyen 2017)

The optimizers are given, up to translations, dilations and multiplications by constants, by

$$\rho(x) = \left(1+|x|^2\right)^{-N/q} \quad \forall\, x\in\mathbb{R}^N$$

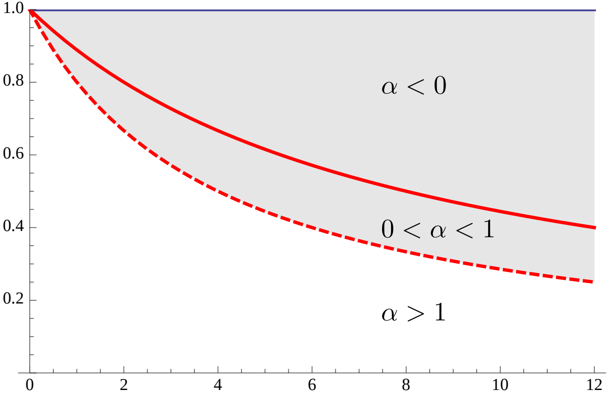
and the value of the optimal constant is

$$\mathcal{C}_{N,\lambda,q(\lambda)} = \frac{1}{\pi^{\frac{\lambda}{2}}}\,\frac{\Gamma\left(\frac{N}{2}+\frac{\lambda}{2}\right)}{\Gamma\left(N+\frac{\lambda}{2}\right)}\left(\frac{\Gamma(N)}{\Gamma\left(\frac{N}{2}\right)}\right)^{1+\frac{\lambda}{N}}$$




*$N = 4$, region of the parameters $(\lambda, q)$ for which $\mathcal{C}_{N,\lambda,q} > 0$*
*The plain, red curve is the conformally invariant case $\alpha = 0$*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} |x-y|^\lambda\,\rho(x)\,\rho(y)\,dx\,dy \ge \mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}$$



# A Carlson type inequality

## Lemma

*Let* $\lambda > 0$ *and* $N/(N+\lambda) < q < 1$

$$\left(\int_{\mathbb{R}^N} \rho\,dx\right)^{1-\frac{N\,(1-q)}{\lambda\,q}} \left(\int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx\right)^{\frac{N\,(1-q)}{\lambda\,q}} \geq c_{N,\lambda,q} \left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{\frac{1}{q}}$$

$$c_{N,\lambda,q} = \frac{1}{\lambda}\left(\frac{(N+\lambda)\,q-N}{q}\right)^{\frac{1}{q}} \left(\frac{N\,(1-q)}{(N+\lambda)\,q-N}\right)^{\frac{N}{\lambda}\,\frac{1-q}{q}} \left(\frac{\Gamma\left(\frac{N}{2}\right)\Gamma\left(\frac{1}{1-q}\right)}{2\,\pi^{\frac{N}{2}}\,\Gamma\left(\frac{1}{1-q}-\frac{N}{\lambda}\right)\Gamma\left(\frac{N}{\lambda}\right)}\right)^{\frac{1-q}{q}}$$

*Equality is achieved if and only if*

$$\rho(x) = \left(1+|x|^\lambda\right)^{-\frac{1}{1-q}}$$

*up to translations, dilations and constant multiples*

(Carlson 1934) (Levine 1948)

# An elementary proof of Carlson's inequality

$$\int_{\{|x|<R\}} \rho^q \, dx \le \left( \int_{\mathbb{R}^N} \rho \, dx \right)^q |B_R|^{1-q} = C_1 \left( \int_{\mathbb{R}^N} \rho \, dx \right)^q R^{N\,(1-q)}$$

and

$$\int_{\{|x|\ge R\}} \rho^q \, dx \le \left( \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \right)^q \left( \int_{\{|x|\ge R\}} |x|^{-\frac{\lambda\, q}{1-q}} \, dx \right)^{1-q}$$
$$= C_2 \left( \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \right)^q R^{-\lambda q + N\,(1-q)}$$

and optimize over $R > 0$

... existence of a radial monotone non-increasing optimal function; rearrangement; Euler-Lagrange equations

## Proposition

*Let $\lambda > 0$. If $N/(N+\lambda) < q < 1$, then $\mathcal{C}_{N,\lambda,q} > 0$*

By rearrangement inequalities: prove the reverse HLS inequality for symmetric non-increasing $\rho$'s so that

$$\int_{\mathbb{R}^N} |x-y|^\lambda \, \rho(y) \, dx \ge \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \quad \text{for all} \quad x \in \mathbb{R}^N$$

implies

$$I_\lambda[\rho] \ge \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \int_{\mathbb{R}^N} \rho \, dx$$

In the range $\frac{N}{N+\lambda} < q < 1$

$$\frac{I_\lambda[\rho]}{\left(\int_{\mathbb{R}^N} \rho(x) \, dx\right)^\alpha} \ge \left(\int_{\mathbb{R}^N} \rho \, dx \, dx\right)^{1-\alpha} \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx \ge c_{N,\lambda,q}^{2-\alpha} \left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{\frac{2-\alpha}{q}}$$

and conclude with Carlson's inequality

# The case $\lambda = 2$

**Corollary**

*Let $\lambda = 2$ and $N/(N+2) < q < 1$. Then the optimizers for (1) are given by translations, dilations and constant multiples of*

$$\rho(x) = \left(1 + |x|^2\right)^{-\frac{1}{1-q}}$$

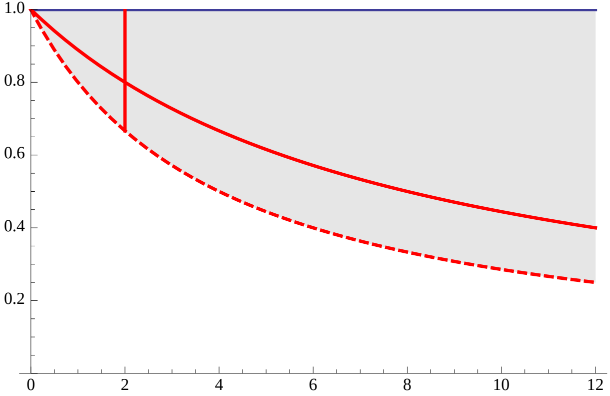
*and the optimal constant is*

$$\mathcal{C}_{N,2,q} = \frac{1}{2}\, c_{N,2,q}^{\frac{2\,q}{N\,(1-q)}}$$

By rearrangement inequalities it is enough to prove (1) for symmetric non-increasing $\rho$'s, and so $\int_{\mathbb{R}^N} x\,\rho\,dx = 0$. Therefore

$$I_2[\rho] = 2\int_{\mathbb{R}^N} \rho\,dx \int_{\mathbb{R}^N} |x|^2 \rho\,dx$$

and the optimal function is optimal for Carlson's inequality




*$N = 4$, region of the parameters $(\lambda, q)$ for which $\mathcal{C}_{N,\lambda,q} > 0$. The dashed, red curve is the threshold case $q = N/(N+\lambda)$*

# The threshold case $q = N/(N+\lambda)$ and below

**Proposition**

*If* $0 < q \le N/(N+\lambda)$, *then* $\mathcal{C}_{N,\lambda,q} = 0 = \lim_{q \to N/(N+\lambda)_+} \mathcal{C}_{N,\lambda,q}$

Let $\rho$, $\sigma \ge 0$ such that $\int_{\mathbb{R}^N} \sigma\, dx = 1$, smooth (+ compact support)

$$\rho_\varepsilon(x) := \rho(x) + M\, \varepsilon^{-N}\, \sigma(x/\varepsilon)$$

Then $\int_{\mathbb{R}^N} \rho_\varepsilon\, dx = \int_{\mathbb{R}^N} \rho\, dx + M$ and, by simple estimates,

$$\int_{\mathbb{R}^N} \rho_\varepsilon^q\, dx \to \int_{\mathbb{R}^N} \rho^q\, dx \quad \text{as} \quad \varepsilon \to 0_+$$

and

$$I_\lambda[\rho_\varepsilon] \to I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx \quad \text{as} \quad \varepsilon \to 0_+$$

If $0 < q < N/(N+\lambda)$, *i.e.*, $\alpha > 1$, take $\rho_\varepsilon$ as a trial function,

$$\mathcal{C}_{N,\lambda,q} \le \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx}{\left(\int_{\mathbb{R}^N} \rho\, dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q\, dx\right)^{(2-\alpha)/q}} =: \mathcal{Q}[\rho, M]$$

*The threshold case:* If $\alpha = 1$, *i.e.*, $q = N/(N+\lambda)$, by taking the limit as $M \to +\infty$, we obtain

$$\mathcal{C}_{N,\lambda,q} \le \frac{2\int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx}{\left(\int_{\mathbb{R}^N} \rho^q\,dx\right)^{(2-\alpha)/q}}$$

For any $R > 1$, we take

$$\rho_R(x) := |x|^{-(N+\lambda)}\,\mathbb{1}_{1\le|x|\le R}(x)$$

Then

$$\int_{\mathbb{R}^N} |x|^\lambda\,\rho_R\,dx = \int_{\mathbb{R}^N} \rho_R^q\,dx = \left|\mathbb{S}^{N-1}\right|\,\log R$$

and, as a consequence,

$$\frac{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_R\,dx}{\left(\int_{\mathbb{R}^N} \rho_R^{N/(N+\lambda)}\,dx\right)^{(N+\lambda)/N}} = \left(\left|\mathbb{S}^{N-1}\right|\,\log R\right)^{-\lambda/N} \to 0 \quad \text{as} \quad R \to \infty$$

This proves that $\mathcal{C}_{N,\lambda,q} = 0$ for $q = N/(N+\lambda)$

# A relaxed inequality

$$I_\lambda[\rho]+2M\int_{\mathbb{R}^N}|x|^\lambda\,\rho\,dx\;\ge\mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx+M\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q} \qquad (2)$$

**Proposition**

*If $q > N/(N+\lambda)$, the relaxed inequality (2) holds with the same optimal constant $\mathcal{C}_{N,\lambda,q}$ as (1) and admits an optimizer $(\rho, M)$*

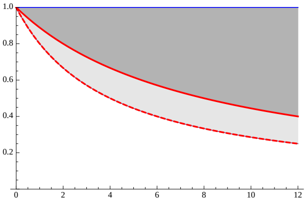
Heuristically, this is the extension of the reverse HLS inequality (1)

$$I_\lambda[\rho]\;\ge\mathcal{C}_{N,\lambda,q}\left(\int_{\mathbb{R}^N}\rho\,dx\right)^\alpha\left(\int_{\mathbb{R}^N}\rho^q\,dx\right)^{(2-\alpha)/q}$$

to measures of the form $\rho + M\,\delta$

# Existence of minimizers and relaxation

# Existence of a minimizer: first case




The $\alpha < 0$ case: dark grey region

**Proposition**

*If $\lambda > 0$ and $\frac{2N}{2N+\lambda} < q < 1$, there is a minimizer $\rho$ for $\mathcal{C}_{N,\lambda,q}$*

The limit case $\alpha = 0$, $q = \frac{2N}{2N+\lambda}$ is the *conformally invariant* case: see (Dou, Zhu 2015) and (Ngô, Nguyen 2017)

A minimizing sequence $\rho_j$ can be taken radially symmetric non-increasing by rearrangement, and such that

$$\int_{\mathbb{R}^N} \rho_j(x)\,dx = \int_{\mathbb{R}^N} \rho_j(x)^q\,dx = 1 \quad \text{for all } j \in \mathbb{N}$$

Since $\rho_j(x) \le C\,\min\left\{|x|^{-N},\,|x|^{-N/q}\right\}$ by Helly's selection theorem we may assume that $\rho_j \to \rho$ a.e., so that

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \ge I_\lambda[\rho] \quad \text{and} \quad 1 \ge \int_{\mathbb{R}^N} \rho(x)\,dx$$

by Fatou's lemma. Pick $p \in (N/(N+\lambda), q)$ and apply (1) with the same $\lambda$ and $\alpha = \alpha(p)$:

$$I_\lambda[\rho_j] \ge \mathcal{C}_{N,\lambda,p} \left(\int_{\mathbb{R}^N} \rho_j^p\,dx\right)^{(2-\alpha(p))/p}$$

Hence the $\rho_j$ are uniformly bounded in $\mathrm{L}^p(\mathbb{R}^N)$: $\rho_j(x) \le C'\,|x|^{-N/p}$,

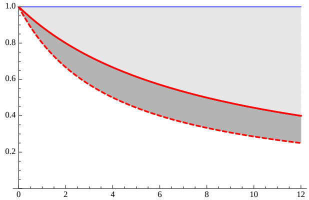
$$\int_{\mathbb{R}^N} \rho_j^q\,dx \to \int_{\mathbb{R}^N} \rho^q\,dx = 1$$

by dominated convergence

# Existence of a minimizer: second case

If $N/(N+\lambda) < q < 2N/(2N+\lambda)$ we consider the *relaxed inequality*

$$I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \rho \, dx \ \geq \mathcal{C}_{N,\lambda,q} \left( \int_{\mathbb{R}^N} \rho \, dx + M \right)^\alpha \left( \int_{\mathbb{R}^N} \rho^q \, dx \right)^{(2-\alpha)/q}$$




The $0 < \alpha < 1$ case: dark grey region

**Proposition**

*If $q > N/(N+\lambda)$, the relaxed inequality holds with the same optimal constant $\mathcal{C}_{N,\lambda,q}$ as (1) and admits an optimizer $(\rho, M)$*

Let $(\rho_j, M_j)$ be a minimizing sequence with $\rho_j$ radially symmetric non-increasing by rearrangement, such that

$$\int_{\mathbb{R}^N} \rho_j\,dx + M_j = \int_{\mathbb{R}^N} \rho_j^q = 1$$

Local estimates + Helly's selection theorem: $\rho_j \to \rho$ almost everywhere and $M_j \to M := L + \lim_{j\to\infty} M_j$, so that $\int_{\mathbb{R}^N} \rho\,dx + M = 1$, and $\int_{\mathbb{R}^N} \rho(x)^q\,dx = 1$
We cannot invoke Fatou's lemma because $\alpha \in (0,1)$: let $d\mu_j := \rho_j\,dx$

$$\mu_j\left(\mathbb{R}^N \setminus B_R(0)\right) = \int_{\{|x|\ge R\}} \rho_j\,dx \le C \int_{\{|x|\ge R\}} \frac{dx}{|x|^{N/q}} = C'\,R^{-N\,(1-q)/q}$$

$\mu_j$ are tight: up to a subsequence, $\mu_j \to \mu$ weak * and $d\mu = \rho\,dx + L\,\delta$

$$\liminf_{j\to\infty} I_\lambda[\rho_j] \ge I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx\,,$$

$$\liminf_{j\to\infty} \int_{\mathbb{R}^N} |x|^\lambda\,\rho_j\,dx \ge \int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx$$

Conclusion: $\liminf_{j\to\infty} \mathcal{Q}[\rho_j, M_j] \ge \mathcal{Q}[\rho, M]$

# Optimizers are positive

$$\mathcal{Q}[\rho, M] := \frac{I_\lambda[\rho] + 2M \int_{\mathbb{R}^N} |x|^\lambda \, \rho \, dx}{\left(\int_{\mathbb{R}^N} \rho \, dx + M\right)^\alpha \left(\int_{\mathbb{R}^N} \rho^q \, dx\right)^{(2-\alpha)/q}}$$

**Lemma**

*Let $\lambda > 0$ and $N/(N+\lambda) < q < 1$. If $\rho \geq 0$ is an optimal function for some $M > 0$, then $\rho$ is radial (up to a translation), monotone non-increasing and positive a.e. on $\mathbb{R}^N$*

If $\rho$ vanishes on a set $E \subset \mathbb{R}^N$ of finite, positive measure, then

$$\mathcal{Q}\left[\rho, M + \varepsilon \, \mathbb{1}_E\right] = \mathcal{Q}[\rho, M] \left(1 - \frac{2-\alpha}{q} \, \frac{|E|}{\int_{\mathbb{R}^N} \rho(x)^q \, dx} \, \varepsilon^q + o(\varepsilon^q)\right)$$

as $\varepsilon \to 0_+$, a contradiction if $(\rho, M)$ is a minimizer of $\mathcal{Q}$

# Euler–Lagrange equation

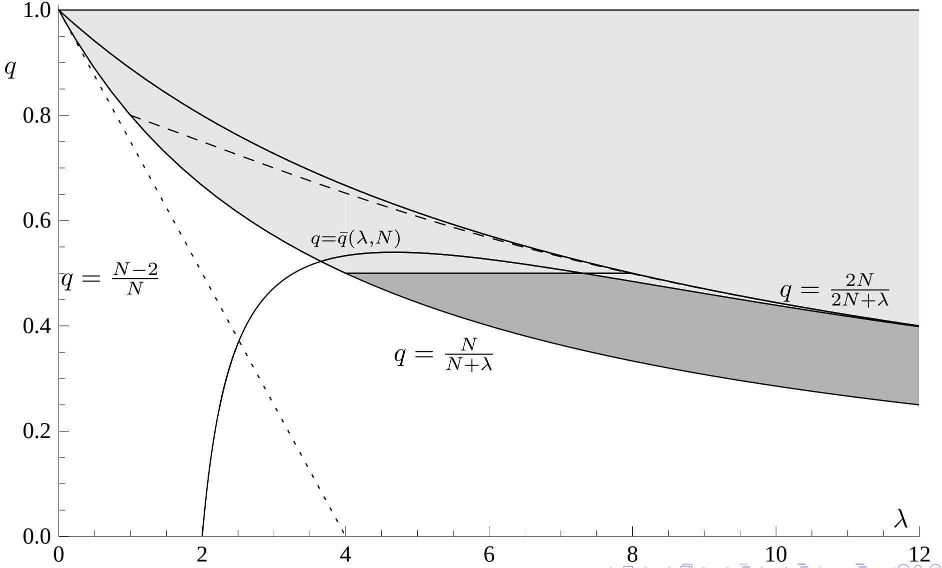
Euler–Lagrange equation for a minimizer $(\rho_*, M_*)$

$$\frac{2\int_{\mathbb{R}^N}|x-y|^\lambda\,\rho_*(y)\,dy+M_*|x|^\lambda}{I_\lambda[\rho_*]+2M_*\int_{\mathbb{R}^N}|y|^\lambda\,\rho_*\,dy}-\frac{\alpha}{\int_{\mathbb{R}^N}\rho_*\,dy+M_*}-\frac{(2-\alpha)\,\rho_*(x)^{-1+q}}{\int_{\mathbb{R}^N}\rho_*(y)^q\,dy}=0$$

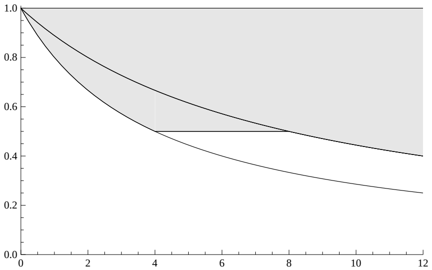
We can reformulate the question of the optimizers of (1) as: when is it true that $M_*=0$ ? We already know that $M_*=0$ if

$$\frac{2N}{2N+\lambda}<q<1$$

# Regions of no concentration and regularity of measure valued minimizers


1.0
0.8
0.6
0.4
0.2
0.0
$q$
$\lambda$
0
2
4
6
8
10
12
$q=\bar{q}(\lambda,N)$
$q=\frac{N-2}{N}$
$q=\frac{2N}{2N+\lambda}$
$q=\frac{N}{N+\lambda}$

# No concentration 1




**Proposition**

*Let* $N \geq 1$, $\lambda > 0$ *and* $\dfrac{N}{N+\lambda} < q < \dfrac{2N}{2N+\lambda}$

*If* $N \geq 3$ *and* $\lambda > 2N/(N-2)$, *assume further that* $q \geq \dfrac{N-2}{N}$

*If* $(\rho_*, M_*)$ *is a minimizer, then* $M_* = 0$

# Two ingredients of the proof

- Based on the Brézis–Lieb lemma

**Lemma**

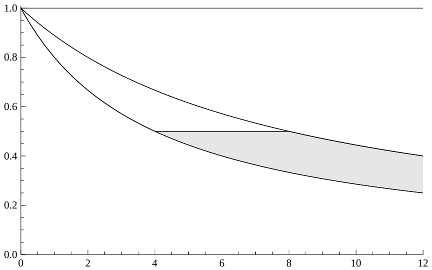
*Let $0 < q < p$, let $f \in \mathrm{L}^p \cap \mathrm{L}^q(\mathbb{R}^N)$ be a symmetric non-increasing function and let $g \in \mathrm{L}^q(\mathbb{R}^N)$. Then, for any $\tau > 0$, as $\varepsilon \to 0_+$,*

$$\int_{\mathbb{R}^N} \left| f(x) + \varepsilon^{-N/p}\, \tau\, g(x/\varepsilon) \right|^q dx = \int_{\mathbb{R}^N} f^q\, dx + \varepsilon^{N(1-q/p)}\, \tau^q \int_{\mathbb{R}^N} |g|^q\, dx + o\left(\varepsilon^{N(1-q/p)}\, \tau^q\right)$$

- $$I_\lambda\left[\rho_* + \varepsilon^{-N} \tau\, \sigma(\cdot/\varepsilon)\right] + 2\,(M_* - \tau) \int_{\mathbb{R}^N} |x|^\lambda \left(\rho_*(x) + \varepsilon^{-N} \tau\, \sigma(x/\varepsilon)\right) dx$$

$$= I_\lambda[\rho_*] + 2\, M_* \int_{\mathbb{R}^N} |x|^\lambda\, \rho_\star\, dx + \underbrace{\left\{ \begin{array}{l} 2\,\tau \iint_{\mathbb{R}^N \times \mathbb{R}^N} \rho_*(x) \left(|x-y|^\lambda - |x|^\lambda\right) \frac{\sigma\left(\frac{y}{\varepsilon}\right)}{\varepsilon^N}\, dx\, dy \\ + \varepsilon^\lambda\, \tau^2\, I_\lambda[\sigma] + 2\,(M_* - \tau)\, \tau\, \varepsilon^\lambda \int_{\mathbb{R}^N} |x|^\lambda\, \sigma\, dx \end{array} \right.}_{= O(\varepsilon^\beta\, \tau) \quad \text{with} \quad \beta := \min\{2,\lambda\}}$$

# Regularity and concentration




## Proposition

*If* $N \geq 3$, $\lambda > 2N/(N-2)$ *and*

$$\frac{N}{N+\lambda} < q < \min\left\{\frac{N-2}{N}, \frac{2N}{2N+\lambda}\right\},$$

*and* $(\rho_*, M_*) \in \mathrm{L}^{N\,(1-q)/2}(\mathbb{R}^N) \times [0, +\infty)$ *is a minimizer, then* $M_* = 0$

# Regularity

## Proposition

*Let $N \geq 1$, $\lambda > 0$ and $N/(N+\lambda) < q < 2N/(2N+\lambda)$*
*Let $(\rho_*, M_*)$ be a minimizer*

1. *If* $\int_{\mathbb{R}^N} \rho_*\,dx > \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$, *then* $M_* = 0$ *and* $\rho_*$, *bounded and*

$$\rho_*(0) = \left(\frac{(2-\alpha) I_\lambda[\rho_*] \int_{\mathbb{R}^N} \rho_*\,dx}{\left(\int_{\mathbb{R}^N} \rho_*^q\,dx\right)\left(2\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx \int_{\mathbb{R}^N} \rho_*\,dx - \alpha I_\lambda[\rho_*]\right)}\right)^{\frac{1}{1-q}}$$

2. *If* $\int_{\mathbb{R}^N} \rho_*\,dx = \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$, *then* $M_* = 0$ *and* $\rho_*$ *is unbounded*
3. *If* $\int_{\mathbb{R}^N} \rho_*\,dx < \frac{\alpha}{2}\,\frac{I_\lambda[\rho_*]}{\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx}$, *then* $\rho_*$ *is unbounded and*

$$M_* = \frac{\alpha I_\lambda[\rho_*] - 2\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx \;\int_{\mathbb{R}^N} \rho_*\,dx}{2\,(1-\alpha)\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx} > 0$$

# An ingredient of the proof

**Lemma**

*For constants* $A,\, B>0$ *and* $0<\alpha<1$*, define*

$$f(M)=\frac{A+M}{(B+M)^\alpha}\quad \textit{for}\quad M\ge 0$$

*Then* $f$ *attains its minimum on* $[0,\infty)$ *at* $M=0$ *if* $\alpha\,A\le B$ *and at* $M=(\alpha\,A-B)/(1-\alpha)>0$ *if* $\alpha\,A>B$

# No concentration 2

For any $\lambda \geq 1$ we deduce from

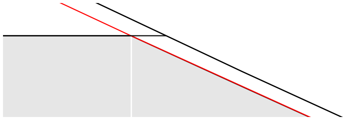
$$|x-y|^\lambda \leq \big(|x|+|y|\big)^\lambda \leq 2^{\lambda-1}\left(|x|^\lambda+|y|^\lambda\right)$$

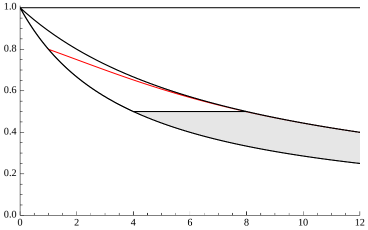
that

$$I_\lambda[\rho] < 2^\lambda \int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx \int_{\mathbb{R}^N} \rho(x)\,dx$$

For all $\alpha \leq 2^{-\lambda+1}$, we infer that $M_* = 0$ if

$$q \geq \frac{2N\left(1-2^{-\lambda}\right)}{2N\left(1-2^{-\lambda}\right)+\lambda}$$



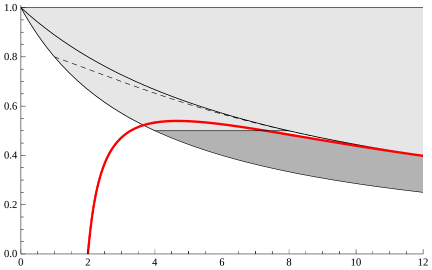


# No concentration 3

Layer cake representation (superlevel sets are balls)

$$I_\lambda[\rho] \le 2\, A_{N,\lambda} \int_{\mathbb{R}^N} |x|^\lambda\, \rho\, dx \int_{\mathbb{R}^N} \rho(x)\, dx$$

$$A_{N,\lambda} := \sup_{0 \le R, S < \infty} \frac{\iint_{B_R \times B_S} |x-y|^\lambda\, dx\, dy}{|B_R| \int_{B_S} |x|^\lambda\, dx + |B_S| \int_{B_R} |y|^\lambda\, dy}$$



### Proposition

*Assume that* $N \geq 3$ *and* $\lambda > 2N/(N-2)$ *and observe that*

$$\frac{N}{N+\lambda} < \bar{q}(\lambda, N) \leq \frac{2N\left(1-2^{-\lambda}\right)}{2N\left(1-2^{-\lambda}\right)+\lambda} < \frac{2\,N}{2\,N+\lambda}$$

*for* $\lambda > 2$ *large enough. If*

$$\max\left\{\bar{q}(\lambda, N), \frac{N}{N+\lambda}\right\} < q < \frac{N-2}{N}$$

*and if* $(\rho_*, M_*)$ *is a minimizer, then* $M_* = 0$ *and* $\rho_* \in \mathrm{L}^\infty(\mathbb{R}^N)$

# More on regularity

## Lemma

*Assume that $\rho_*$ is an unbounded minimizer*

- *if $\lambda < 2$, there is a constant $c > 0$ such that*

$$\rho_*(x) \geq c\,|x|^{-\lambda/(1-q)} \quad \textit{as} \quad x \to 0$$

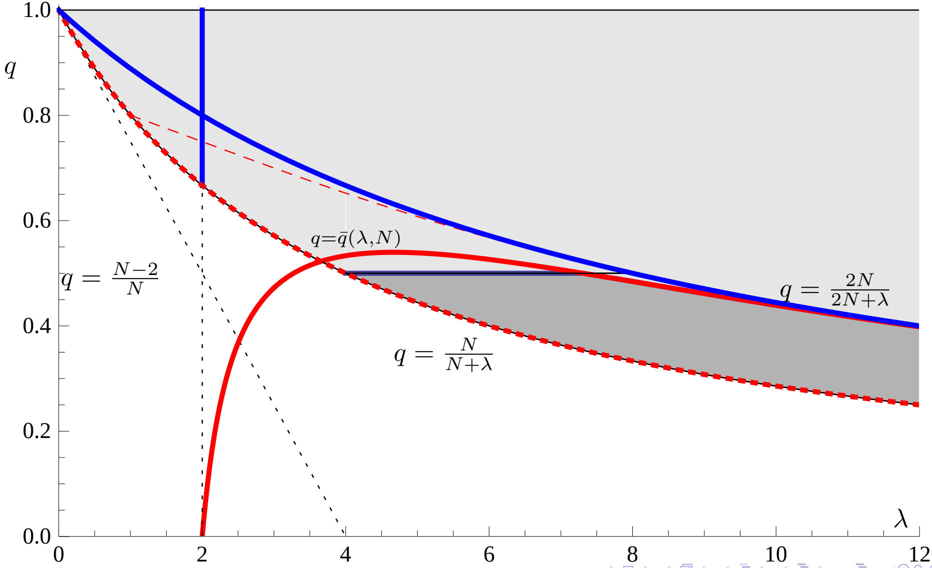
- *if $\lambda \geq 2$, there is a constant $C > 0$ such that*

$$\rho_*(x) = C\,|x|^{-2/(1-q)}\,\big(1 + o(1)\big) \quad \textit{as} \quad x \to 0$$

## Corollary

$$q \neq \frac{2N}{2N+\lambda}\,, \quad \frac{N}{N+\lambda} < q < 1 \quad \textit{and} \quad q \geq \frac{N-2}{N} \; \textit{if}\; N \geq 3$$

*If $\rho_*$ is a minimizer for $\mathcal{C}_{N,\lambda,q}$, then $\rho_* \in \mathrm{L}^\infty(\mathbb{R}^N)$*


1.0
0.8
0.6
0.4
0.2
0.0
q
λ
0
2
4
6
8
10
12
$q=\bar{q}(\lambda,N)$
$q = \frac{N-2}{N}$
$q = \frac{2N}{2N+\lambda}$
$q = \frac{N}{N+\lambda}$

# Free energy point of view

# A toy model

Assume that $u$ solves the *fast diffusion with external drift* $V$ given by

$$\frac{\partial u}{\partial t} = \Delta u^q + \nabla \cdot \big( u \, \nabla V \big)$$

To fix ideas: $V(x) = 1 + \frac{1}{2}\,|x|^2 + \frac{1}{\lambda}\,|x|^\lambda$. *Free energy* functional

$$\mathcal{F}[u] := \int_{\mathbb{R}^N} V\, u\, dx - \frac{1}{1-q} \int_{\mathbb{R}^N} u^q\, dx$$

- Under the mass constraint $M = \int_{\mathbb{R}^N} u\, dx$, smooth minimizers are

$$u_\mu(x) = \big( \mu + V(x) \big)^{-\frac{1}{1-q}}$$

- The equation can be seen as a gradient flow

$$\frac{d}{dt} \mathcal{F}[u(t,\cdot)] = -\int_{\mathbb{R}^N} u \left| \tfrac{q}{1-q} \nabla u^{q-1} - \nabla V \right|^2 dx$$

# A toy model (continued)

If $\lambda = 2$, the so-called *Barenblatt profile* $u_\mu$ has finite mass if and only if

$$q > q_c := \frac{N-2}{N}$$

- For $\lambda > 2$, the integrability condition is $q > 1 - \lambda/N$ but $q = q_c$ is a threshold for the regularity: the mass of $u_\mu = (\mu + V)^{1/(1-q)}$ is

$$M(\mu) := \int_{\mathbb{R}^N} u_\mu \, dx \le M_\star = \int_{\mathbb{R}^N} \left(\tfrac{1}{2}\,|x|^2 + \tfrac{1}{\lambda}\,|x|^\lambda\right)^{-\frac{1}{1-q}} dx$$

- If one tries to minimize the free energy under the mass contraint $\int_{\mathbb{R}^N} u \, dx = M$ for an arbitrary $M > M_\star$, the limit of a minimizing sequence is the measure

$$\left(M - M_\star\right) \delta + u_{-1}$$

# A model for nonlinear springs: heuristics

$$V=\rho*W_\lambda\,,\quad W_\lambda(x):=\tfrac{1}{\lambda}\,|x|^\lambda$$

is motivated by the study of the nonnegative solutions of the evolution equation

$$\frac{\partial\rho}{\partial t}=\Delta\rho^q+\nabla\cdot(\rho\,\nabla W_\lambda*\rho)$$

Optimal functions for (1) are energy minimizers (eventually measure valued) for the *free energy* functional

$$\mathcal{F}[\rho]:=\frac{1}{2}\int_{\mathbb{R}^N}\rho\,(W_\lambda*\rho)\,dx-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx=\frac{1}{2\lambda}\,I_\lambda[\rho]-\frac{1}{1-q}\int_{\mathbb{R}^N}\rho^q\,dx$$

under a *mass* constraint $M=\int_{\mathbb{R}^N}\rho\,dx$ while smooth solutions obey to

$$\frac{d}{dt}\mathcal{F}[\rho(t,\cdot)]=-\int_{\mathbb{R}^N}\rho\left|\tfrac{q}{1-q}\nabla\rho^{q-1}-\nabla W_\lambda*\rho\right|^2\,dx$$

# Free energy or minimization of the quotient

$$\mathcal{F}[\rho] = -\frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q \, dx + \frac{1}{2\,\lambda} I_\lambda[\rho]$$

- If $0 < q \le N/(N+\lambda)$, then $\mathcal{C}_{N,\lambda,q} = 0$: take test functions $\rho_n \in \mathrm{L}^1_+ \cap \mathrm{L}^q(\mathbb{R}^N)$ such that $\|\rho_n\|_{\mathrm{L}^1(\mathbb{R}^N)} = I_\lambda[\rho_n] = 1$ and $\int_{\mathbb{R}^N} \rho_n^q \, dx = n \in \mathbb{N}$

$$\lim_{n\to+\infty} \mathcal{F}[\rho_n] = -\infty$$

- If $N/(N+\lambda) < q < 1$, $\rho_\ell(x) := \ell^{-N}\,\rho(x/\ell)/\|\rho\|_{\mathrm{L}^1(\mathbb{R}^N)}$

$$\mathcal{F}[\rho_\ell] = -\,\ell^{(1-q)\,N}\,\mathsf{A} + \ell^\lambda\,\mathsf{B}$$

has a minimum at $\ell = \ell_\star$ and

$$\mathcal{F}[\rho] \ge \mathcal{F}[\rho_{\ell_\star}] = -\,\kappa_\star\,(\mathsf{Q}_{q,\lambda}[\rho])^{-\frac{N\,(1-q)}{\lambda - N\,(1-q)}}$$

**Proposition**

*$\mathcal{F}$ is bounded from below if and only if $\mathcal{C}_{N,\lambda,q} > 0$*

*Relaxed free energy*

$$\mathcal{F}^{\mathrm{rel}}[\rho, M] := -\frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q\,dx + \frac{1}{2\,\lambda}\,I_\lambda[\rho] + \frac{M}{\lambda}\int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx$$

## Corollary

*Let* $q \in (0,1)$ *and* $N/(N+\lambda) < q < 1$

$$\inf\left\{\mathcal{F}^{\mathrm{rel}}[\rho, M]\,:\,0 \le \rho \in \mathrm{L}^1 \cap \mathrm{L}^q(\mathbb{R}^N)\,,\ M \ge 0\,,\ \int_{\mathbb{R}^N} \rho\,dx + M = 1\right\}$$

*is achieved by a minimizer of* (2) *such that* $\int_{\mathbb{R}^N} \rho_*\,dx + M_* = 1$ *and*

$$I_\lambda[\rho_*] + 2\,M_*\int_{\mathbb{R}^N} |x|^\lambda\,\rho_*\,dx = 2\,N\int_{\mathbb{R}^N} \rho_*^q\,dx$$

# Uniqueness

## Proposition

*Let $N/(N+\lambda) < q < 1$ and assume either that $(N-1)/N < q < 1$ and $\lambda \geq 1$, or $2 \leq \lambda \leq 4$. Then the minimizer of*

$$\mathcal{F}^{\rm rel}[\rho, M] := \frac{1}{2\,\lambda}\,I_\lambda[\rho] + \frac{M}{\lambda}\int_{\mathbb{R}^N} |x|^\lambda\,\rho\,dx - \frac{1}{1-q}\int_{\mathbb{R}^N} \rho^q\,dx$$

*is unique up to translation, dilation and multiplication by a positive constant*

- If $(N-1)/N < q < 1$ and $\lambda \geq 1$, the lower semi-continuous extension of $\mathcal{F}$ to probability measures is strictly geodesically convex in the Wasserstein-$p$ metric for $p \in (1,2)$

- By strict rearrangement inequalities a minimizer $(\rho, M)$ such that $M \in [0,1)$ of the relaxed free energy $\mathcal{F}^{\mathrm{rel}}$ is (up to a translation) such that $\rho$ is radially symmetric and $\int_{\mathbb{R}^N} x\,\rho\,dx = 0$

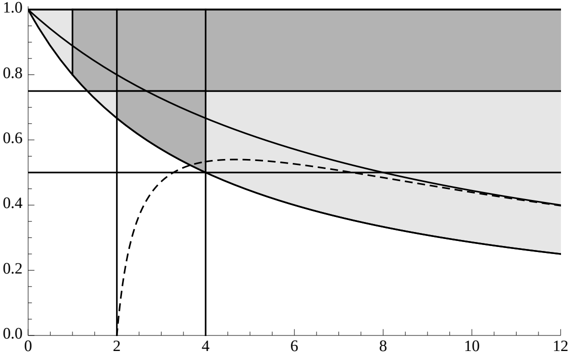
Let $(\rho, M)$ and $(\rho', M')$ be two minimizers and

$$[0,1] \ni t \mapsto f(t) := \mathcal{F}^{\mathrm{rel}}\big[(1-t)\,\rho + t\,\rho', (1-t)\,M + t\,M'\big]$$

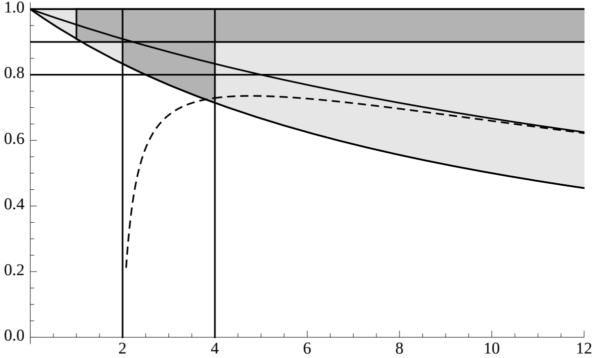
$$f''(t) = \frac{1}{\lambda}\,I_\lambda[\rho'-\rho] + \frac{2}{\lambda}\,(M'-M)\int_{\mathbb{R}^N} |x|^\lambda\,(\rho'-\rho)\,dx \\ + q\int_{\mathbb{R}^N}\big((1-t)\,\rho + t\,\rho'\big)^{q-2}(\rho'-\rho)^2\,dx$$

(Lopes, 2017) $I_\lambda[h] \geq 0$ *if* $2 \leq \lambda \leq 4$, *for all* $h$ *such that* $\int_{\mathbb{R}^N}\big(1+|x|^\lambda\big)\,|h|\,dx < \infty$ *with* $\int_{\mathbb{R}^N} h\,dx = 0$ *and* $\int_{\mathbb{R}^N} x\,h\,dx = 0$

# $N = 4$



# $N = 10$



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Thank you for your attention !