Multi-bubbling phenomena

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Plan

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I — Multi-bubbling for the exponential nonlinearity in the slightly supercritical case

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An ODE approach

Exponential nonlinearity

[Gelfand57, Joseph-Lundgren73]

$$-\Delta u = \lambda\, e^{u} \quad |x| < 1\,,\ x \in \mathbb{R}^n$$
$$u = 0 \quad \text{if} \quad |x| = 1$$

Bifurcation diagrams in $L^\infty(\Omega)$ (bounded solutions are radial):
1. If $n = 2$, the branch has an asymptote at $\lambda = \lambda^* = 0$, the equation has exactly two solutions for any $\lambda \in (0, \lambda_1^+)$ and no solution if $\lambda > \lambda_1^+$.
2. If $2 < n < 10$, the branch oscillates around an asymptote at $\lambda = \lambda^* > 0$, the equation has at least one solution for any $\lambda \in (0, \lambda_1^+)$, $\lambda_1^+ > \lambda^*$, and no solution if $\lambda > \lambda_1^+$.
3. If $n \geq 10$, the branch has an asymptote at $\lambda = \lambda^* > 0$, the equation has (exactly) one solution iff $\lambda \in (0, \lambda^*)$.

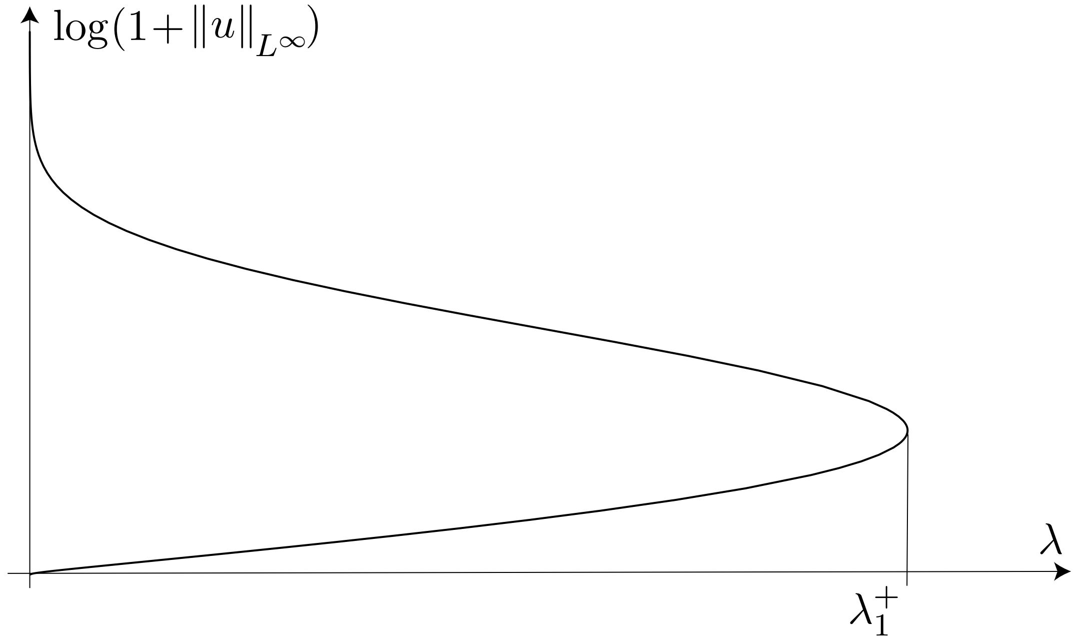

Figure 1: *Critical case:* $p = n$

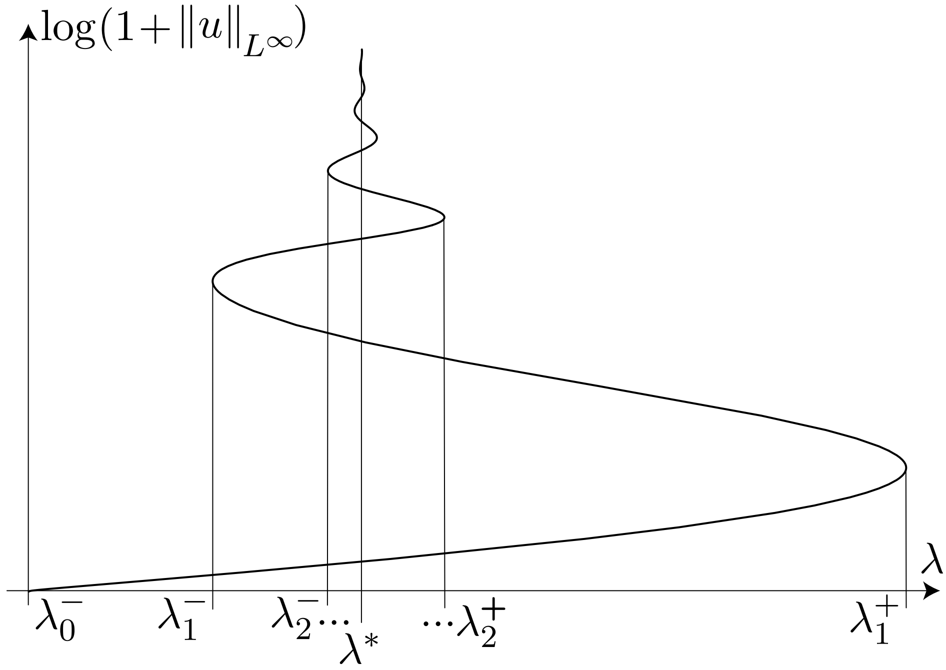

Figure 2: *Supercritical case:* $p < n < p\,\frac{p+3}{p-1}$

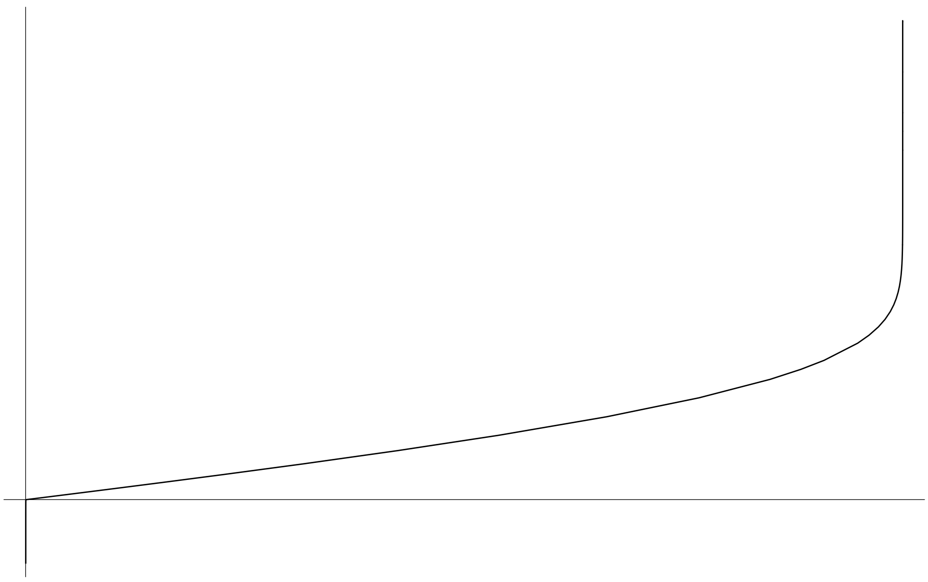

Figure 3: *Supercritical case:* $n \geq p\,\frac{p+3}{p-1}$

For $\lambda^* = 2(n-2)$, there exists a unique radial singular solution u^* such that

$$e^{u^*(x)} = \frac{1}{|x|^2}\,,$$

[Cabre,Cabre-Martel98,Cabre-Martel99,Mignot-Puel98]. Radial solutions solve

$$\begin{cases} u'' + \frac{n-1}{r}\,u' + \lambda\,e^u = 0\,, & r \in (0,1) \\[2ex] u'(0) = 0\,, \quad u(1) = 0 & \end{cases}$$

It is natural to consider n as a real parameter. With $n = 2+\varepsilon$

$$|x|^{N-2-\varepsilon}\text{div}(|x|^{-(N-2-\varepsilon)}\nabla u) + \lambda\,e^u = 0$$

In analogy with the situation observed in the Brezis-Nirenberg problem [Brezis-Nirenberg83], $n = 2$ appears as the *critical* case and $n = 2+\varepsilon$, $\varepsilon > 0$ as the *supercritical* case.

"Criticality" in the Brezis-Nirenberg problem

$$-\Delta u = u^p + \lambda\, u \quad |x| < 1\,,\ x \in \mathbb{R}^n$$
$$u = 0 \quad \text{if} \quad |x| = 1$$

1. For $n \geq 3$, the *critical* exponent is $(n+2)/(n-2)$. In terms of the parameter λ, the "first branch" is monotone decreasing.

2. the "first branch" is oscillating in the *supercritical* regime $p > (n+2)/(n-2)$, around an asymptotic value $\lambda = \lambda^*$.

"First branch": the branch of positive radial bounded solutions which bifurcates from the trivial solution at the first eigenvalue of $-\Delta$.

In the supercritical regime, there exists a radial singular solution if and only if $\lambda = \lambda^*$ [Merle-Peletier91].

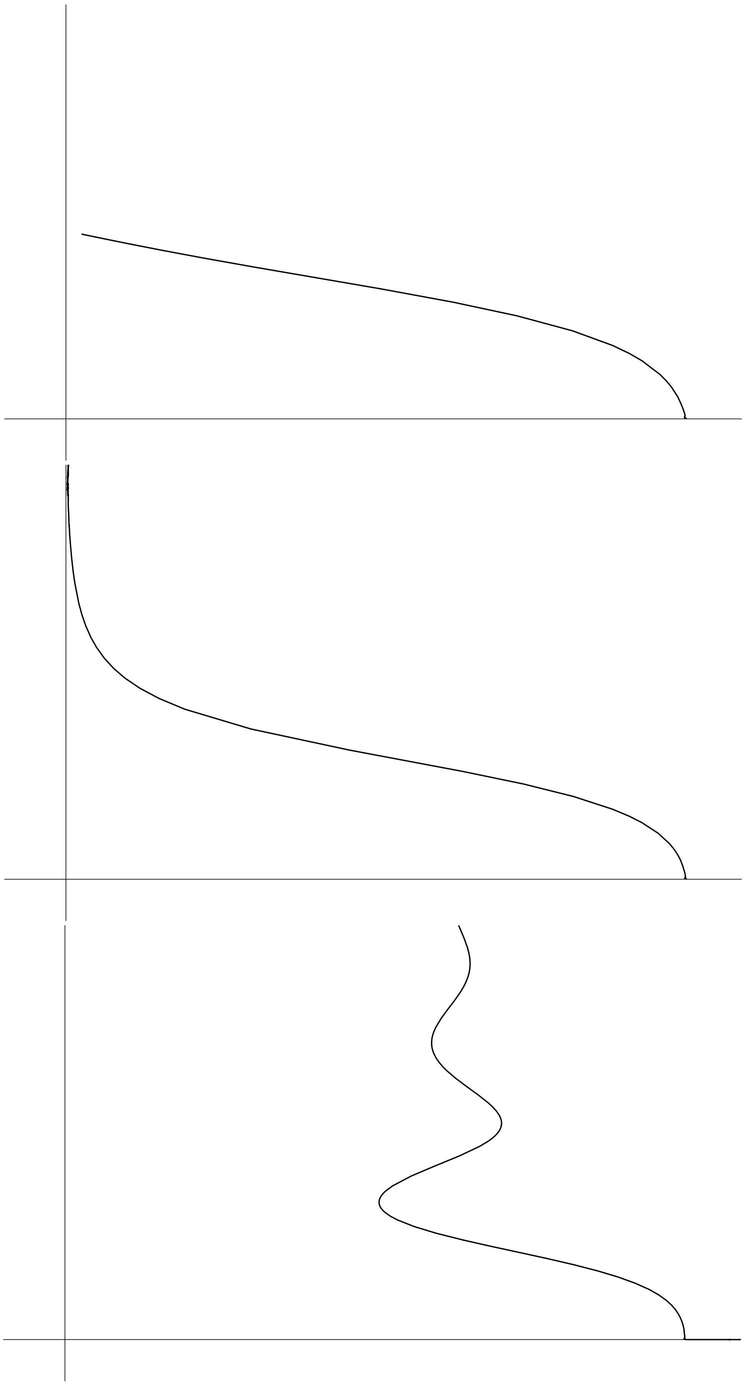

Figure 4: *Positive solutions for Brezis Nirenberg. Top: subcritical case. Middle: critical case. Bottom: supercritical case*

Back to the exponential nonlinearity : a more general equation

$$\begin{cases} \Delta_p u + \lambda\, e^u = 0 & \text{in} \quad \Omega \\ \\ u > 0\,, \quad u = 0 & \text{on} \quad \partial\Omega \end{cases} \tag{1}$$

with $1 < p < n$, where Ω is the unit ball in $\mathbb{R}^n$. Here we use the standard notation $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$. Written in radial coordinates, the equation is

$$\begin{cases} \Delta_{p,n} u + \lambda\, e^u = 0\,, \quad r \in (0,1) \\ \\ u(0) > 0\,, \quad \frac{du}{dr}(0) = 0\,, \quad u(1) = 0 \end{cases}$$

$$\Delta_{p,n} u := \frac{1}{r^{n-1}} \frac{d}{dr} \left(r^{n-1} \left| \frac{du}{dr} \right|^{p-2} \frac{du}{dr} \right)$$

Small parameter in the slightly supercritical regime: $\varepsilon = n - p > 0$. The properties of the bifurcation diagram for $p > 1$ are very similar to the ones of the case $p = 2$ [Jacobsen-Schmitt02]

1. If $n = p$, the branch has an asymptote at $\lambda = 0$ and the equation has exactly two solutions for any $\lambda \in (0, \lambda_1^+)$.

2. Let $p_\pm(n) = \frac{1}{2}[\, n - 3 \pm \sqrt{(n-1)(n-9)}\,]$. If $p < n < \frac{p(p+3)}{(p-1)}$, i.e. $1 < p < n$ if $1 < n < 9$, $\quad 1 < p < p_-(n)$ or $p_+(n) < p < n$ if $n \geq 9$ the branch oscillates around an asymptote at $\lambda^* := p^{p-1}(n-p)$ if $3 \leq n < 10$. There is a unique radial singular solution $u^* := -p \log r$

3. The branch is monotone with an asymptote at $\lambda = \lambda^* > 0$ if $n \geq p(p+3)/(p-1)$, i.e. $p \in [p_-(n), p_+(n)]$.

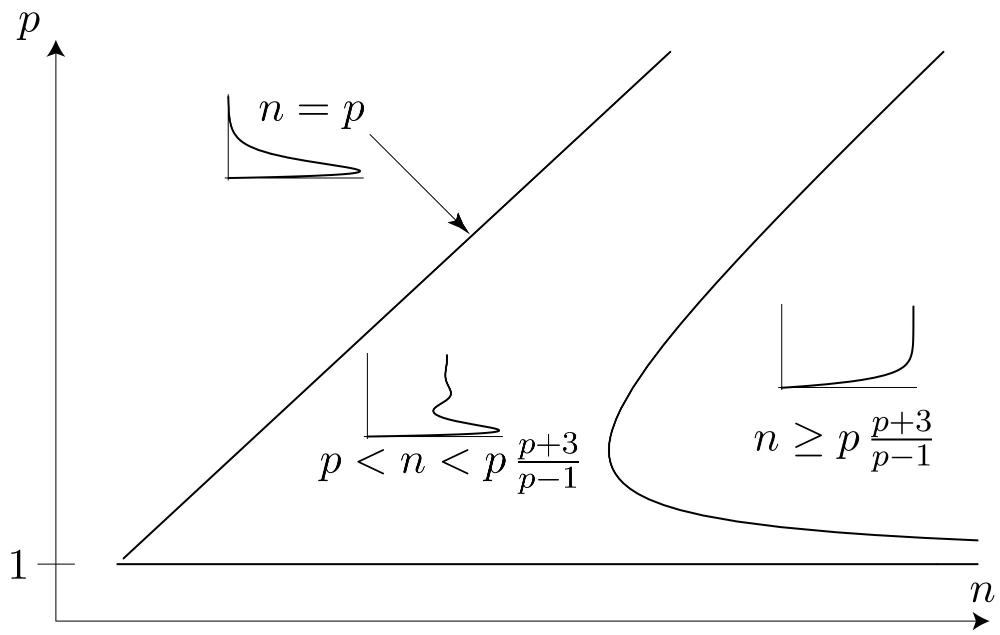

Figure 5: *Types of bifurcation diagrams in terms of n and p.*

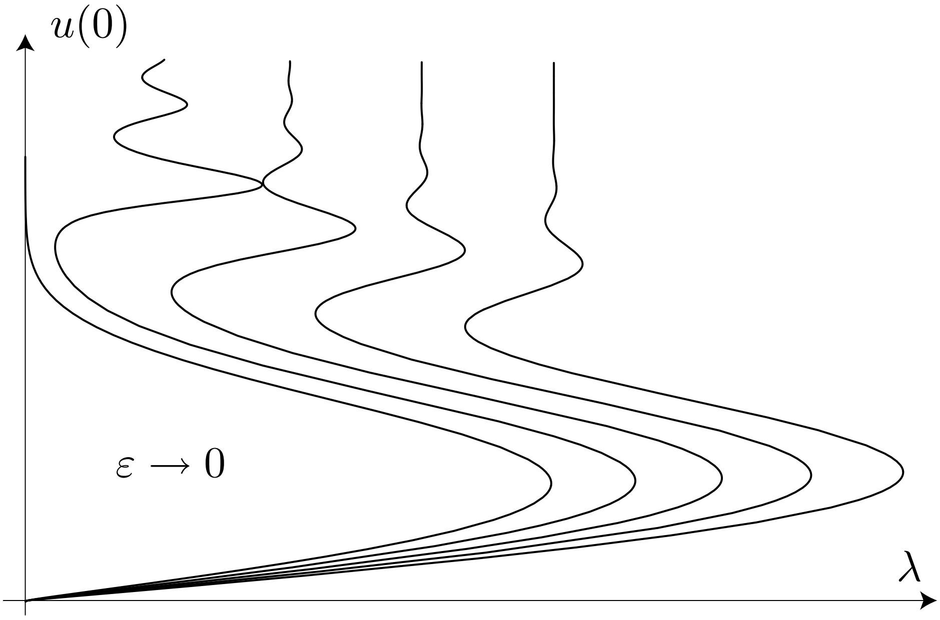

Figure 6: *The critical limit* $\varepsilon = n - p \searrow 0$.

Theorem 1 *Let $k \in \mathbb{N}$, $k \geq 1$. There exists $\bar{\lambda}_k^+ > 0$ such that for any $\lambda \in (0, \bar{\lambda}_k^+)$, Equation (1) has a solution u^ε which can be written as:*

$$\lambda\,|x|^p\,e^{u^\varepsilon(x)} = \left[\sum_{j=1}^{k} w_j^k(\log|x| + \mu_j(\varepsilon))\right](1 + o(1)) \quad \text{as} \quad \varepsilon \to 0$$

uniformly on Ω. Moreover

$$\lim_{i \to +\infty} (\mu_{j+1} - \mu_j) = +\infty \quad \forall\, j = 1, 2, \ldots k-1$$

Here the functions w_j^k are smooth, even, positive and such that $w_j^k(s) \searrow 0$ as $s \to \pm\infty$.

Precision: true only for sequences $(\varepsilon_i)_{i\in\mathbb{N}}$ with $\varepsilon_i \searrow 0$.
Conjecture: w_j^k depends neither on k nor on $(\varepsilon_i)_{i\in\mathbb{N}}$.

The generalized Emden-Fowler change of variables

See [Damascelli-Pacella-Ramaswamy99] and [Brock01] for some recent result on the symmetry properties of the solutions.

Emden-Fowler change of variables: For $r = e^s$, $s \in (-\infty, 0]$, define $v(s) := u(r)$. Then (1) is equivalent to

$$\begin{cases} (p-1)\,|v'|^{p-2}\,v'' + (n-p)\,|v'|^{p-2}\,v' + \lambda\,e^{v+ps} = 0\,, \quad s \in (-\infty, 0) \\ \lim_{s\to-\infty} v(s) > 0\,, \quad \lim_{s\to-\infty} e^{-s}\,v'(s) = 0\,, \quad v(0) = 0 \end{cases}$$

where $v' = \frac{dv}{ds}$. The equation for v can be reduced to an *autonomous* ODE system as follows. Let

$$x(s) = \lambda\,e^{v(s)+ps} \quad \text{and} \quad y(s) = |v'(s)|^{p-2}\,v'(s)\,.$$

$$\begin{cases} x' = x\left(|y|^{p^*-2}\, y + p\right), & x(0) = \lambda \\ y' = (p-n)\, y - x\,, & \lim_{s \to -\infty} e^{-s}\, |y(s)|^{p^*-2}\, y(s) = 0 \end{cases}$$

where $p^* = (1 - 1/p)^{-1}$: $y = |v'|^{p-2}\, v' \iff v' = |y|^{p^*-2}\, y$. Two fixed points: $P^- = (0,0)$ and $P^+ = p^{p-1}(n-p,-1)$.

Lemma 2 *Let $\lambda^* = p^{p-1}(n-p)$, $p < n < p\,(p+3)/(p-1)$. There exists two sequences $(\lambda_k^-)_{k\geq 1}$ and $(\lambda_k^+)_{k\geq 1}$ such that:*
(i) $(\lambda_k^-)_{k\geq 1}$ *is increasing and* $\lim_{k\to+\infty} \lambda_k^- = \lambda^*$.
(ii) $(\lambda_k^+)_{k\geq 1}$ *is decreasing and* $\lim_{k\to+\infty} \lambda_k^+ = \lambda^*$.
(iii) *(1) has no solutions if $\lambda > \lambda_1^+$, $2k-1$ solutions if $\lambda = \lambda_k^+$ or $\lambda \in (\lambda_{k-1}^-, \lambda_k^-)$, and $2k$ solutions if $\lambda = \lambda_k^-$ or $\lambda \in (\lambda_{k+1}^+, \lambda_k^+)$.*
(iv) *(1) has infinitely many solutions if and only if $\lambda = \lambda^*$.*

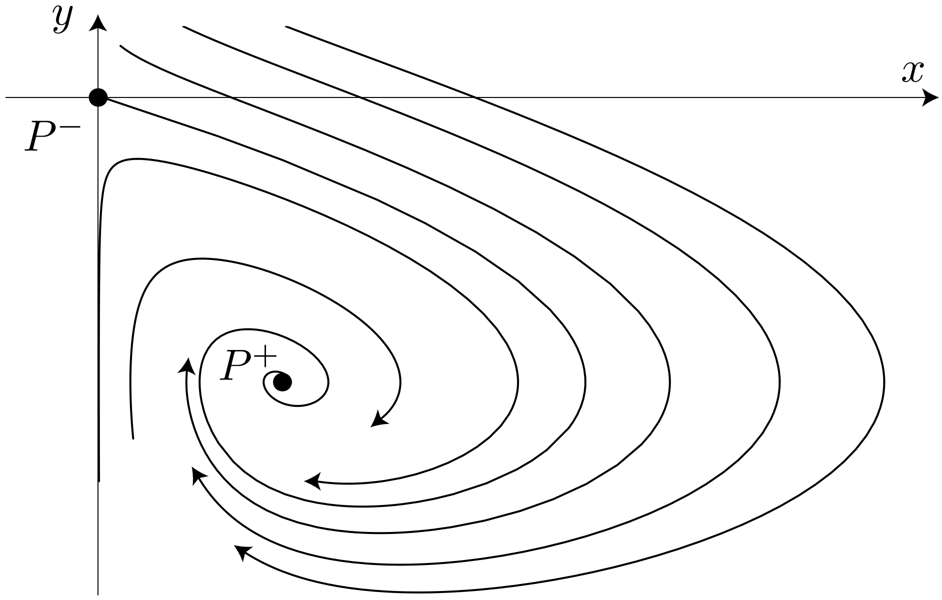

Figure 7: *Phase portrait in the supercritical case* $p < n < p\,\frac{p+3}{p-1}$ *(here* $n = 2$*,* $p = 1.5$*).*

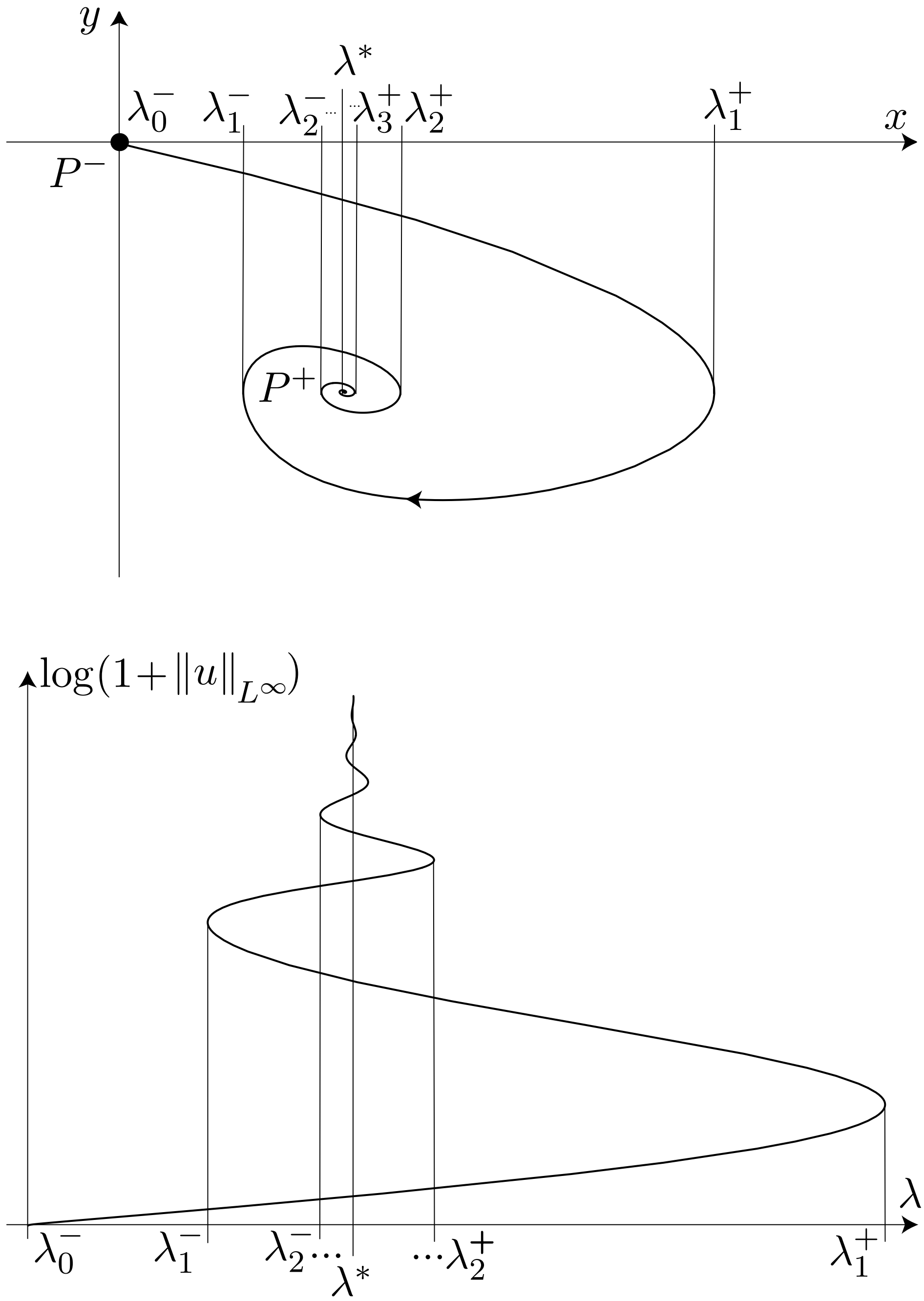

Figure 8: *Parametrization of the solutions in the supercritical case ($n = 2$, $p = 1.5$). Left: $(\bar{x}, \bar{y})$ in the phase space. Right: the bifurcation diagram.*

The critical case: $p = n$ The system becomes Hamiltonian:

$$x' = x\,(|y|^{p^*-2}y + p)\,, \quad y' = -x\,,$$

which is explicitely solvable in the case $p = 2$ [Bratu14]:
$u(r) = 2\log(a^2+1) - 2\log(a^2+r^2)$ is a solution of (1) for any $a > 0$ such that $\lambda = 8\,a^2\,(a^2+1)^{-2}$.

Lemma 3 *Assume that* $p = n$ *and let* $\lambda_1^+ := \sup_{s\in\mathbb{R}} \bar{x}(s)$. *Then Equation (1) has no solutions if* $\lambda > \lambda_1^+$, *one and only one solution if* $\lambda = \lambda_1^+$ *and two and only two solutions if* $\lambda \in (0, \lambda_1^+)$.

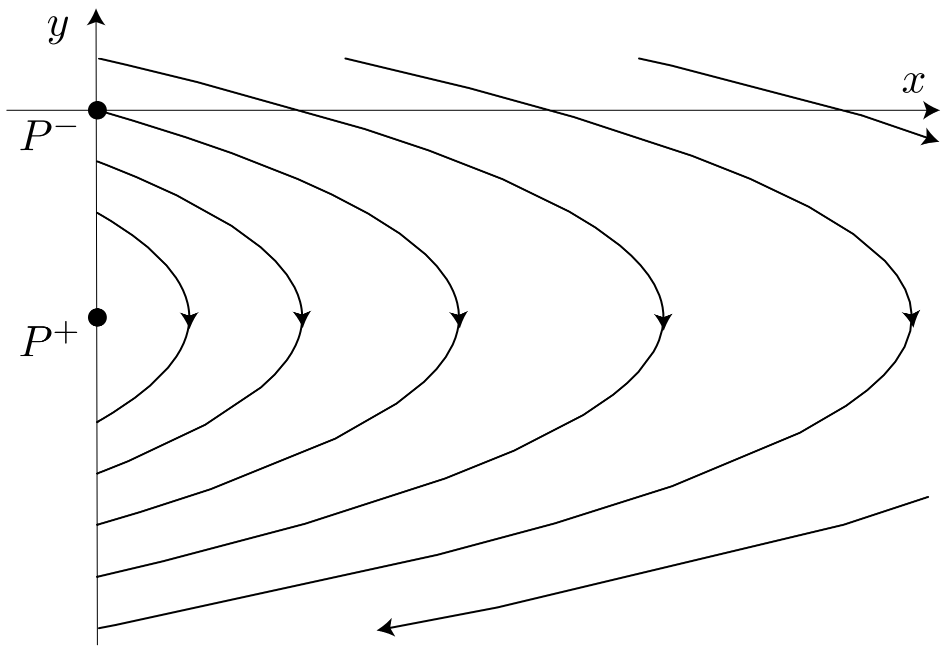

Figure 9: *Phase portrait in the critical case $n = p$ (here $n = 2$).*

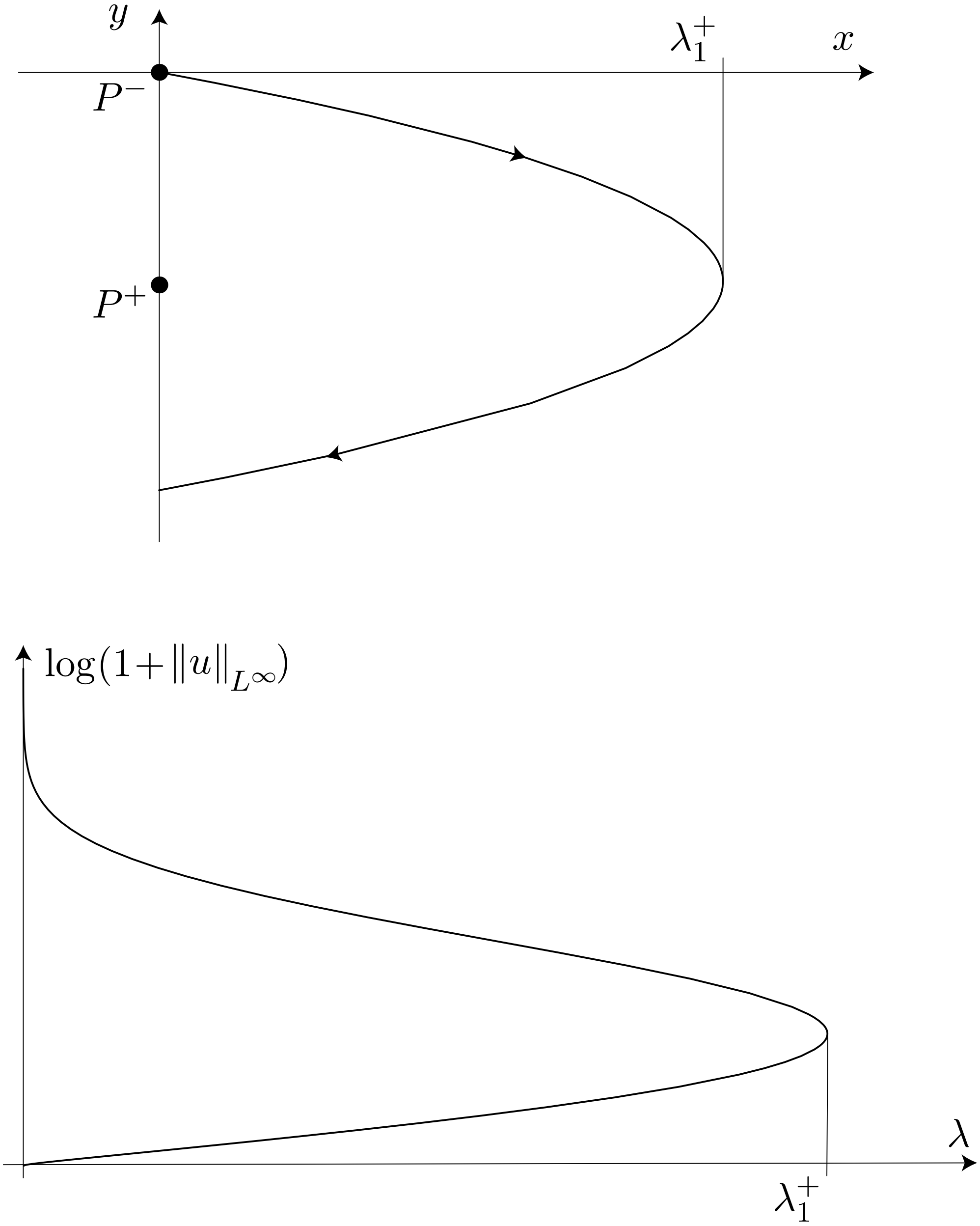

Figure 10: *Parametrization of the solutions in the critical case ($n = p = 2$). Top: ($\bar{x}, \bar{y}$) in the phase space. Bottom: the bifurcation diagram.*

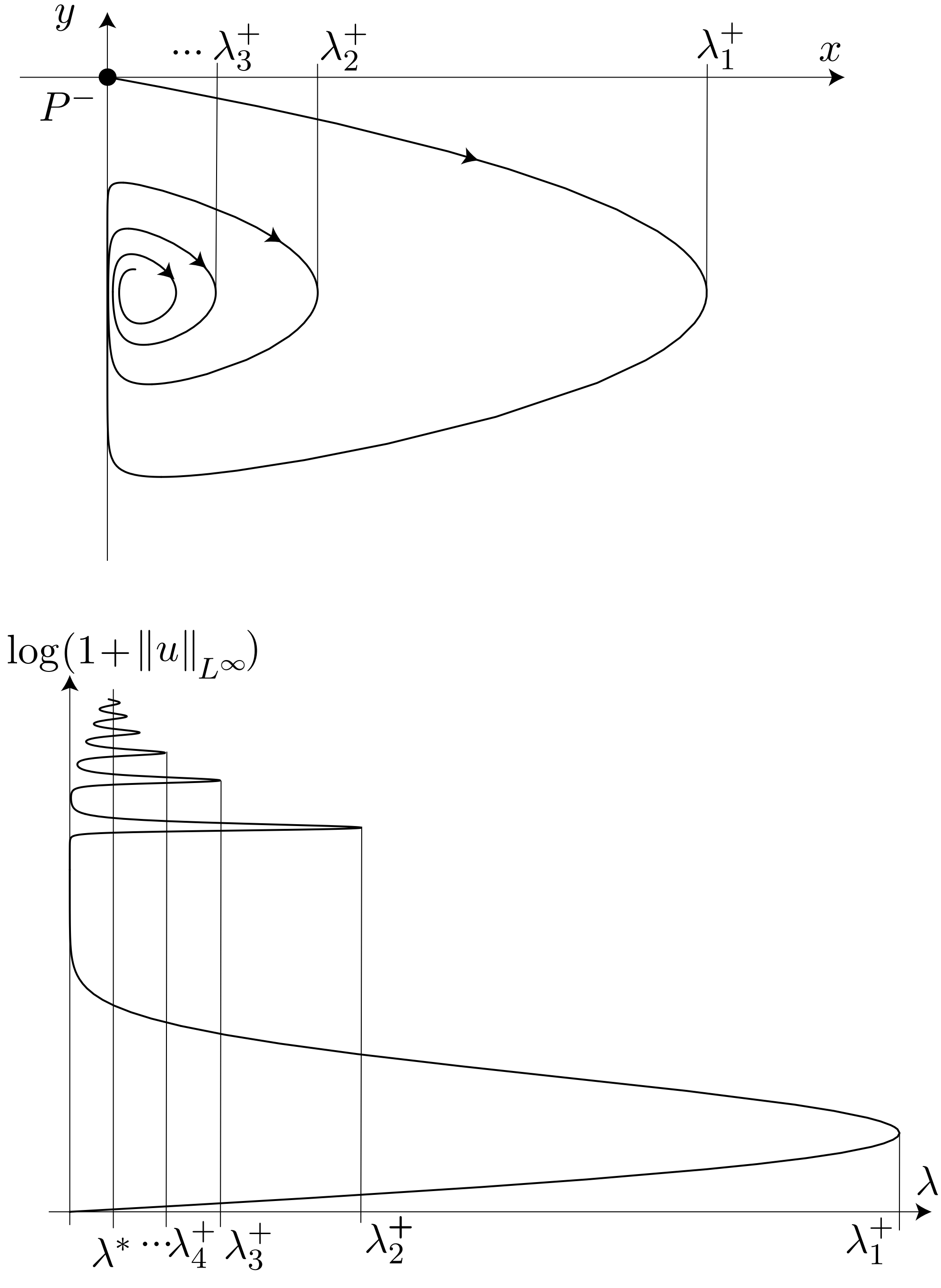

Figure 11: *Top: the solution* $(\bar{x}^\varepsilon, \bar{y}^\varepsilon)$ *in the slightly supercritical regime* $\varepsilon > 0$, $\varepsilon \to 0$. *Bottom: the corresponding bifurcation diagram. Here* $n = 2$, $\varepsilon = 0.05$.

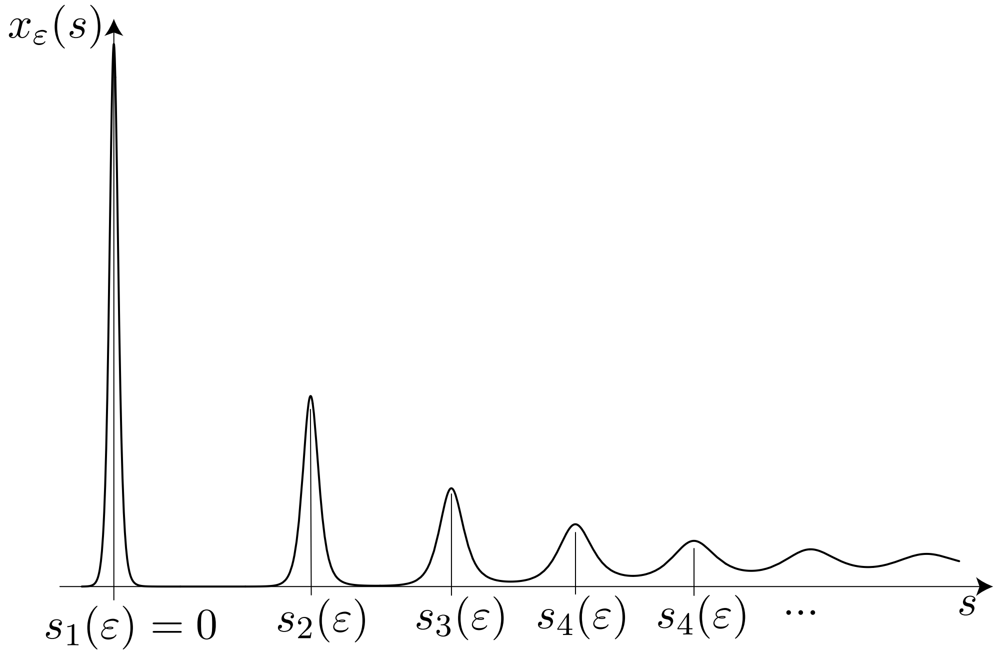

Figure 12: *Bubbles in the logarithmic scale, after the Emden-Fowler transformation.*

Description of the critical limit

Lemma 4 *For any $k \geq 1$,*

$$\bar{\lambda}_k^+ > 0 \qquad (P_k)$$

and $\bar{\lambda}_1^+ = \lambda_1^{0,+}$. Moreover $(\bar{\lambda}_k^+)_{k\in\mathbb{N}}$ is a strictly decreasing sequence.

Corollary 5 *For any $k \geq 1$, as $\varepsilon \to 0$,*

$$\bar{x}^\varepsilon(s) \to \sum_{j=1}^{k} x_j(s - s_j(\varepsilon))$$

uniformly on any interval $(-\infty, a(\varepsilon)) \in \mathbb{R}$ such that $s_k(\varepsilon) < a(\varepsilon) < s_{k+1}(\varepsilon)$ with $\liminf_{\varepsilon\to 0}(s_{k+1}(\varepsilon) - a(\varepsilon)) = \liminf_{\varepsilon\to 0}(a(\varepsilon) - s_k(\varepsilon)) = +\infty$.

Let $\lambda \in (0, \bar{\lambda}_k^+)$ and define $s_k^\pm(\lambda) \in \mathbb{R}$ as the two solutions of $x_k(s_k^\pm(\lambda)) = \lambda$, $\pm s_k^\pm(\lambda) > 0$. A careful rewriting of the Emden-Fowler change of variables then allows to see the solution of (1) as a superposition of bubbles.

Lemma 6 *Let $\lambda \in (0, \bar{\lambda}_k^+]$ for some $k \geq 1$. Then there exist two solutions $u^\pm$ of (1) which take the form*

$$\lambda\, r^p \;\; e^{u^\pm(r)} = \left[\sum_{j=1}^{k} x_j(\log r + s_k(\varepsilon) - s_j(\varepsilon) + s_k^\pm(\lambda)) \right] (1 + o(1))$$

as $\varepsilon \to 0$.

This actually amounts to saying that there is a *k-bubble solution.*

Multi-Bubbling

In the new coordinates $V = \log x$, $U = -y$, the system becomes

$$\begin{cases} U' = e^V - \varepsilon U \,, & U(-\infty) = 0 \\ V' = U_*^{p^*-1} - U^{p^*-1} \,, & V(0) = \log \lambda_1^{\varepsilon,+} \end{cases}$$

Define an *energy* and an *angle* respectively by:

$$E = e^V - e^{V_*} - e^{V_*}(V - V_*) + \frac{1}{p^*}\left(U^{p^*} - U_*^{p^*}\right) - U_*^{p^*-1}(U - U_*) \,.$$

$$\cos\theta = \frac{U - U_*}{\sqrt{|U - U_*|^2 + |V - V_*|^2}} \quad \text{and} \quad \sin\theta = \frac{V - V_*}{\sqrt{|U - U_*|^2 + |V - V_*|^2}} \,.$$

Lemma 7 *There exists a constant $\nu > 0$ such that*

$$0 \geq \frac{dE}{ds} \geq -\varepsilon\, \nu\, E \quad \forall\, s \in \mathbb{R}\ .$$

Given $C > 0$, there exists a constant $\omega > 0$ such that, if

$$V(s) \geq \log(\varepsilon\, U_*) - C \quad \forall\, s \in [s_1, s_2] \subset \mathbb{R}^+\,,$$

then for $\varepsilon > 0$ small enough and any $s \in [s_1, s_2]$,

$$\frac{d\theta}{ds} \geq \varepsilon\, \omega\ .$$

Corollary 8 *Under the above assumptions,*

$$E(s) \geq E(s_0)\, e^{-\frac{\nu}{\omega}\,[\theta(s) - \theta(s_0)]}\ .$$

II — The Brezis-Nirenberg problem:

multi-bubbling in a ball

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A phase plane analysis

References / History

$p = \frac{N+2}{N-2}$, $\varepsilon \geq 0$, $N \geq 3$, B is the unit ball in $\mathbb{R}^N$

$$\begin{cases} -\Delta u = u^{p+\varepsilon} + \lambda\, u\,, \quad u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

- <1950: Lane, Emden, Fowler, Chandrasekhar (astrophysics)
- Sobolev, Rellich, Nash, Gagliardo, Nirenberg, Pohozaev
- 1976: Aubin, Talenti
- 1983: Brezis, Nirenberg: case $\varepsilon = 0$ is solvable for ($N \geq 4$) $0 < \lambda < \lambda_1 = \lambda_1(-\Delta)$. Uniqueness (Zhang, 1992).
- subcritical case ($0 > \varepsilon \to 0$): Brezis and Peletier, Rey, Han
- supercritical case: Symmetry (Gidas, Ni, Nirenberg, 1979).

Budd and Norbury (1987, case $\varepsilon > 0$): formal asymptotics, numerical computations. Merle and Peletier (1991): existence of a unique value $\lambda = \lambda_* > 0$ for which there exists a radial, singular, positive solution.

The Brezis-Nirenberg problem in a ball: radial solutions

$$\begin{cases} u'' + \frac{n-1}{r}\, u' + |u|^{p-1} u + \lambda\, u = 0 \,, \quad r \in (0,1) \\ u'(0) = 0 \,, \quad u(1) = 0 \end{cases} \tag{2}$$

The Emden-Fowler change of variables

With $r = e^s$, $u(r) = r^{-2/(p-1)} v(s)$, (2) is equivalent to

$$v'' + |v|^{p-1} v - \beta\, v = -\alpha\, v' - \lambda\, e^{2s}\, v \,, \quad s \in (-\infty, 0) \; + \; b.c.$$

$\alpha = n - 2 - 4/(p-1)$, $\beta = 2\,[(n-2)p - n]/(p-1)^2$, $p = \frac{n+2}{n-2} + \varepsilon$.
Let $x(s) = v(s)$, $y(s) = v'(s)$:

$$x' = y \,, \quad y' = -(|x|^{p-1} x - \beta\, x) - \lambda\, e^{2s}\, x - \alpha\, y$$

$x(0) = 0$, $y(0) = \gamma > 0$

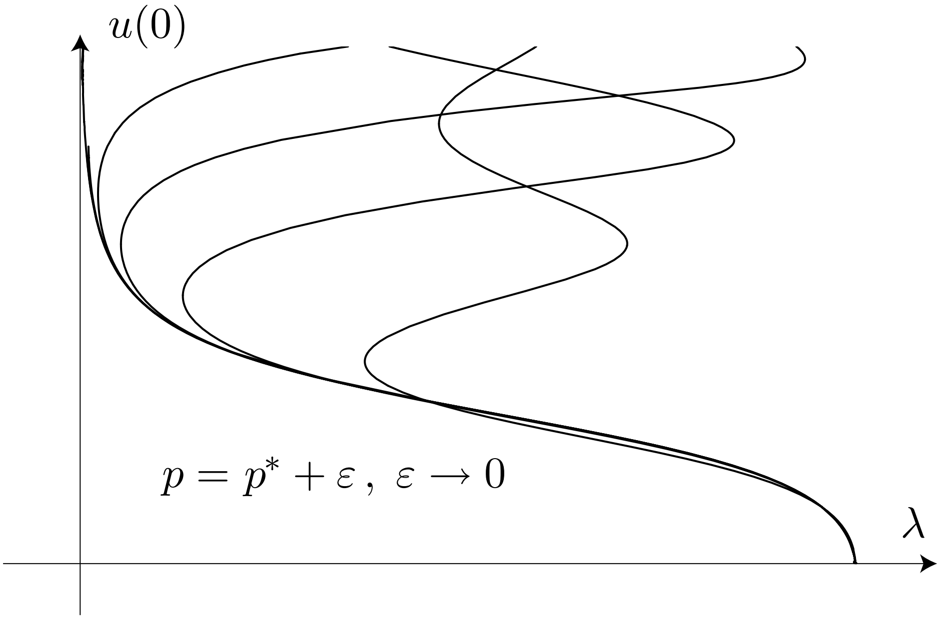

Figure 13: *Bifurcation curve of the positive solutions in the slightly supercritical case.*

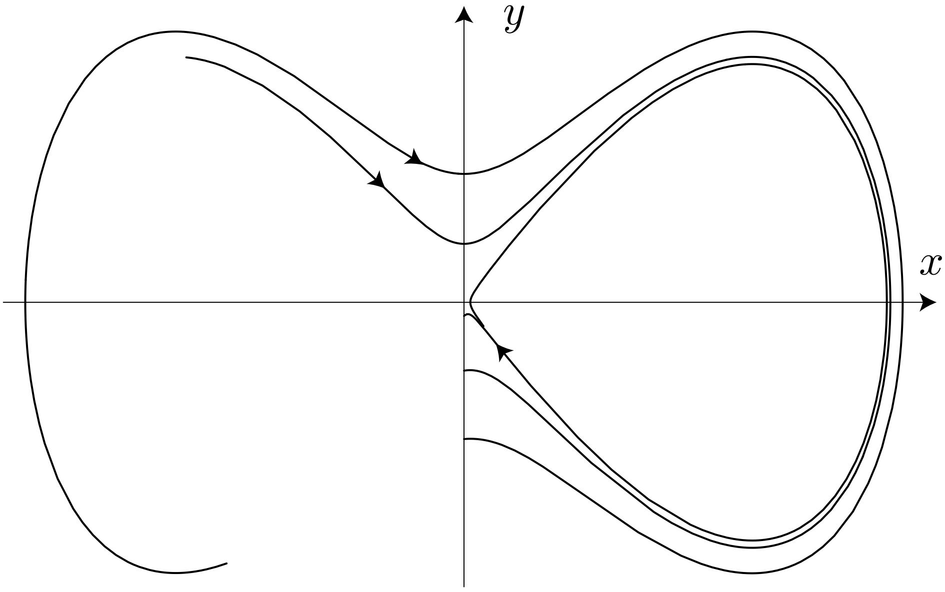

Figure 14: *Phase space*

The critical case [Brezis-Nirenberg83]
[Benguria-J.D.-Esteban00, J.D.-Esteban-Ramaswamy] Define the energy by

$$E(s) = \frac{1}{2}|y(s)|^2 + \frac{1}{p+1}|x(s)|^{p+1} - \frac{\beta}{2}|x(s)|^2$$

and consider its limiting value $\mathcal{E}_{\gamma,\lambda} = \lim_{s\to-\infty} E(s)$. Let $\lambda \in (\lambda^*, \lambda_1)$ be fixed:

1. If $\gamma \in (0, \gamma_*(\lambda))$, then $\mathcal{E}_{\gamma,\lambda} < 0$: positive singular
2. If $\gamma \in (\gamma_*(\lambda), +\infty)$, then $\mathcal{E}_{\gamma,\lambda} > 0$: singular and oscillating
3. If $\gamma = \gamma_*(\lambda)$, then $\mathcal{E}_{\gamma,\lambda} = 0$: (unique) positive bounded

Theorem 9 *Let* $n \geq 3$, $p^* = \frac{n+2}{n-2} < p = p^* + \varepsilon$, $\kappa \in \mathbb{N}$, $\kappa \geq 1$. *If* $\lambda \in (\lambda^*, \lambda_1)$, *then there exists a solution*

$$x(s) = x_0^\lambda(s) + \sum_{i=1}^{k} x^*(s + s_i(\varepsilon)) + o(1)$$

for some $k \geq \kappa$, *where* x_0^λ *is the bounded solution in the critical case* ($p = p^*$, $\varepsilon = 0$) *with* $\gamma = \gamma^*(\lambda)$ *and* x^* *is the unique positive solution of the asymptotic problem*

$$x'' + |x|^{p^*-1}x - \frac{1}{4}(n-2)^2 x = 0$$

such that $x'(0) = 0$ *and with zero energy.*

III – The Brezis-Nirenberg problem: Lyapunov-Schmidt reduction

Bubble-tower radial solutions in the slightly supercritical Brezis-Nirenberg problem

We consider the Brezis-Nirenberg problem

$$\begin{cases} \Delta u + u^{p+\varepsilon} + \lambda\, u = 0 & \text{in } B \\ u > 0 \quad \text{in } B\,, \quad u = 0 & \text{on } \partial B \end{cases} \tag{3}$$

in dimension $N \geq 4$, in the supercritical case: $p = \frac{N+2}{N-2}$, $\varepsilon > 0$. If $\varepsilon \to 0$ and if, simultaneously, $\lambda \to 0$ at the appropriate rate, then there are radial solutions which behave like a superposition of *bubbles*: $M_j \to +\infty$ and $M_j = o(M_{j+1})$ for all j and

$$(N(N-2))^{(N-2)/4} \sum_{j=1}^{k} \left(1 + M_j^{\frac{4}{N-2}}\, |y|^2\right)^{-(N-2)/2} M_j\,(1 \;+\; o(1)\,)$$

1. Parametrization of the solutions

Let B be the unit ball in $\mathbb{R}^N$, $N \geq 4$, and consider for $p = \frac{N+2}{N-2}$ and $\varepsilon \geq 0$ the positive solutions of

$$\begin{cases} \Delta u + u^{p+\varepsilon} + \lambda\, u = 0 & \text{in } B \\ u > 0 \quad \text{in } B\,, \quad u = 0 & \text{on } \partial B \end{cases}$$

Denote by $\rho = \rho(a) > 0$ the first zero of v given by

$$\begin{cases} v'' + \frac{N-1}{r}\, v' + v^{p+\varepsilon} + v = 0 \quad \text{in } [0, +\infty) \\ v(0) = a > 0\,, \quad v'(0) = 0 \end{cases}$$

To any solution u of (1) corresponds a function v on $[0, \sqrt{\lambda})$ s.t.

$$v(|x|) = \lambda^{-1/(p+\varepsilon-1)}\, u(x/\sqrt{\lambda}) \Longleftrightarrow u(x) = \rho^{2/(p+\varepsilon-1)}\, v(\rho\, |x|)$$

with $\lambda = \rho^2(a)$. The bifurcation diagram $(\lambda, \|u\|_{L^\infty})$ is therefore fully parametrized by $a \mapsto (\rho^2,\, a\, \rho^{2/(p+\varepsilon-1)})$ with $\rho = \rho^2(a)$.

2. Heuristics

Consider a family of (radial, noincreasing) solutions u_ε of (1) for $\lambda = \lambda_\varepsilon \to 0$. The problem at $\lambda = 0$, $\varepsilon = 0$ has no solution:

$$M_\varepsilon = \gamma^{-1} \max u_\varepsilon = \gamma^{-1} u_\varepsilon(0) \to +\infty$$

for some fixed constant $\gamma > 0$. Let $u_\varepsilon(z) = M_\varepsilon\, v_\varepsilon \left(M_\varepsilon^{(p+\varepsilon-1)/2}\, z \right)$

$$\Delta v_\varepsilon + v_\varepsilon^{p+\varepsilon} + M_\varepsilon^{-(p+\varepsilon-1)} \lambda_\varepsilon v_\varepsilon = 0, \quad |z| < M_\varepsilon^{(p+\varepsilon-1)/2}.$$

Locally over compacts around the origin, $v_\varepsilon \to w$ s.t.

$$\Delta w + w^p = 0$$

with $w(0) = \gamma := (N(N-2))^{\frac{N-2}{4}}$: $w(z) = \gamma \left(\frac{1}{1+|z|^2} \right)^{\frac{N-2}{2}}$.

Guess: $u_\varepsilon(y) = \gamma \left(1 + M_\varepsilon^{\frac{4}{N-2}} |y|^2 \right)^{-\frac{N-2}{2}} M_\varepsilon \,(\, 1 + o(1)\,)$ as $\varepsilon \to 0$.

Theorem 10 [k-bubble solution] *Assume $N \geq 5$. Then, given an integer $k \geq 1$, there exists a number $\mu_k > 0$ s.t. if $\mu > \mu_k$ and*

$$\lambda = \mu\, \varepsilon^{\frac{N-4}{N-2}}\,,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \ldots, k$ which depend on k, N and μ and two solutions $u_\varepsilon^\pm$ of Problem (1) of the form

$$u_\varepsilon^\pm(y) = \gamma \sum_{j=1}^{k} \left(1 + \left[\alpha_j^\pm\, \varepsilon^{\frac{1}{2}-j} \right]^{\frac{4}{N-2}} |y|^2 \right)^{-(N-2)/2} \alpha_j^\pm\, \varepsilon^{\frac{1}{2}-j}\,(1 + o(1))\,,$$

where $\gamma = (N(N-2))^{\frac{N-2}{4}}$ and $o(1) \to 0$ uniformly on B as $\varepsilon \to 0$.

Bifurcation curve: $\lambda = \varepsilon^{\frac{N-4}{N-2}} f_k\left(c_k^{-1} \varepsilon^{k-\frac{1}{2}}\, m \right)$ for $m \sim \varepsilon^{\frac{1}{2}-k}$.

After lengthy computations... The numbers $\alpha_j^{\pm}$ can be expressed by the formulae

$$\alpha_j^{\pm} = b_3^{1-j} \frac{(k-j)!}{(k-1)!} s_k^{\pm}(\mu), \quad j = 1, \ldots, k \ ,$$

where $b_3 = \frac{(N-2)\sqrt{\pi}\,\Gamma(\frac{N}{2})}{2^{N+2}\,\Gamma(\frac{N+1}{2})}$ and $s_k^{\pm}(\mu)$ are the two solutions of

$$\mu = f_k(s) := k b_1 s^{\frac{4}{N-2}} + b_2 s^{-2\frac{N-4}{N-2}}$$

with $b_1 = \left(\frac{N-2}{4}\right)^3 \frac{N-4}{N-1}$ and $b_2 = (N-2)\frac{\Gamma\,(N-1)}{\Gamma\left(\frac{N-4}{2}\right)\Gamma\left(\frac{N}{2}\right)}$.

Remind that $\mu > \mu_k$ where μ_k is the minimum value of the function $f_k(s)$:

$$\mu_k = (N-2)\left[\frac{b_1\, k}{N-4}\right]^{\frac{N-4}{N-2}} \left[\frac{b_2}{2}\right]^{\frac{2}{N-2}}$$

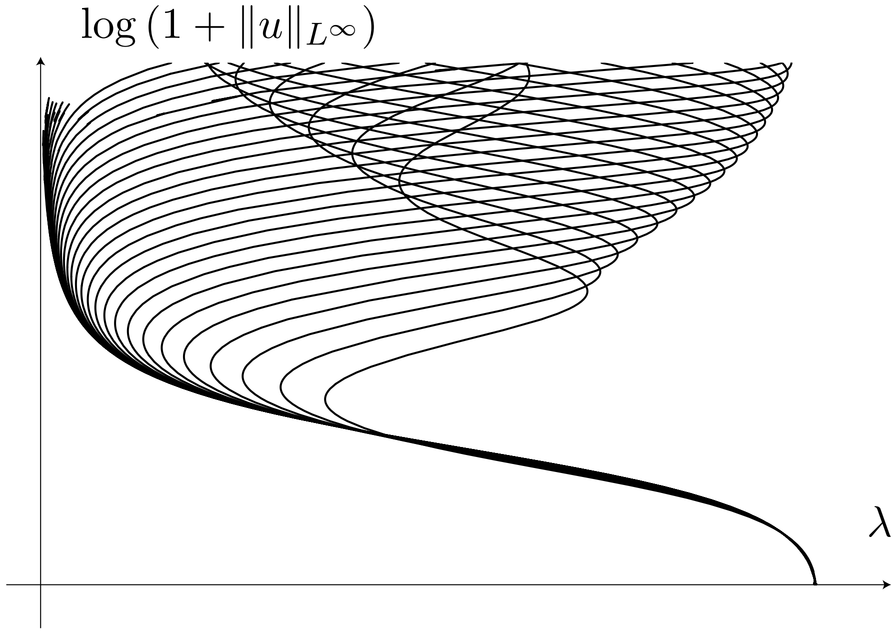

Figure 15: *Slightly supercritical case*

3. The asymptotic expansion

The solution of

$$v'' - v + e^{\varepsilon x}\, v^{p+\varepsilon} + \left(\frac{p-1}{2}\right)^2 \lambda\, e^{-(p-1)x}\, v = 0 \quad \text{on } (0,\infty)$$

with $v(0) = v(\infty) = 0$, $v > 0$ is given as a critical point of

$$E_\varepsilon(w) = I_\varepsilon(w) - \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \lambda \int_0^\infty e^{-(p-1)x} |w|^2\, dx$$

$$I_\varepsilon(w) = \frac{1}{2}\int_0^\infty |w'|^2\, dx + \frac{1}{2}\int_0^\infty |w|^2\, dx - \frac{1}{p+\varepsilon+1}\int_0^\infty e^{\varepsilon x}|w|^{p+\varepsilon+1}\, dx$$

$U(x) = \left(\frac{4\,N}{N-2}\right)^{\frac{N-2}{4}} e^{-x}\left(1 + e^{-\frac{4}{N-2}\,x}\right)^{-\frac{N-2}{2}}$ is the solution of

$$U'' - U + U^p = 0$$

Ansatz: $v(x) = V(x) + \phi$, $V(x) = \sum_{i=1}^{k} (U(x-\xi_i) - U(\xi_i)\, e^{-x})$.

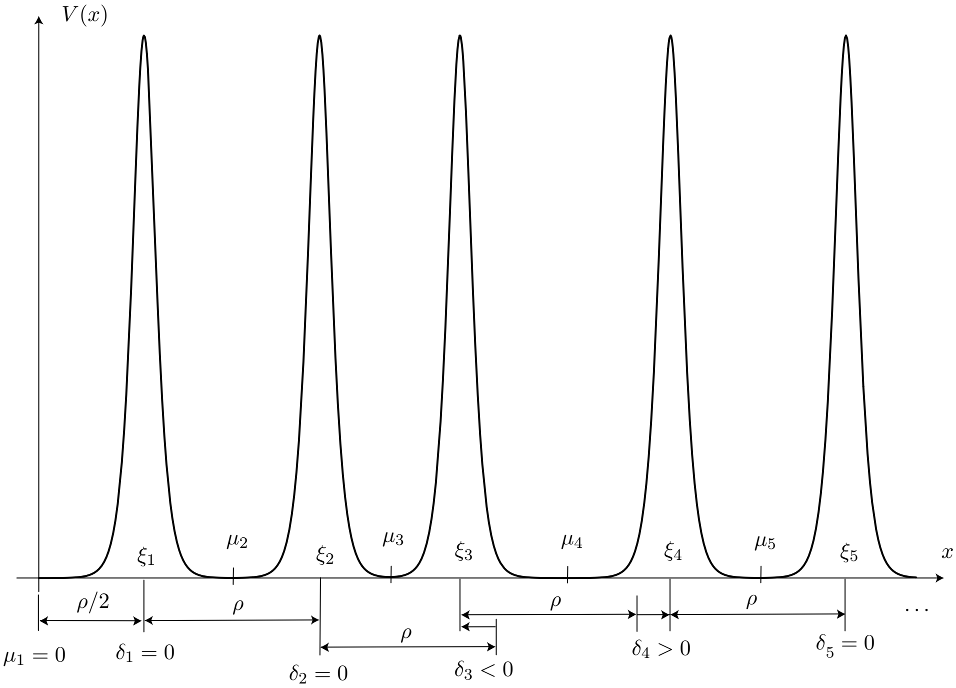

Figure 16: *The ansatz*

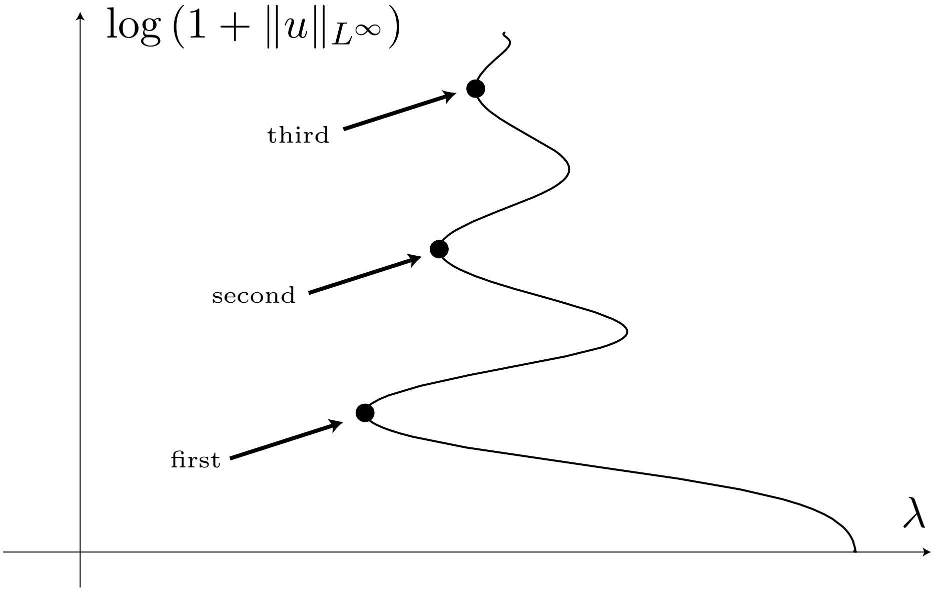

Figure 17: *Turning points*

Further choices: $\xi_1 = -\frac{1}{2}\log\varepsilon + \log\Lambda_1$,
$\xi_{i+1} - \xi_i = -\log\varepsilon - \log\Lambda_{i+1}$, $i = 1, \ldots, k-1$.

Lemma 11 *Let $N \geq 5$ and $\lambda = \mu\,\varepsilon^{\frac{N-4}{N-2}}$. Then*

$$E_\varepsilon(V) \;=\; k\,a_0 + \varepsilon\,\Psi_k(\Lambda) + \frac{k^2}{2}\,a_3\,\varepsilon\,\log\varepsilon + a_5\,\varepsilon + \varepsilon\,\theta_\varepsilon(\Lambda)$$

$$\Psi_k(\Lambda) = a_1\,\Lambda_1^{-2} - k\,a_3\log\Lambda_1 - a_4\,\mu\,\Lambda_1^{-(p-1)} + \sum_{i=2}^{k}\left[(k-i+1)\,a_3\log\Lambda_i - a_2\,\Lambda_i\right],$$

and $\lim_{\varepsilon\to 0}\theta_\varepsilon(\Lambda) = 0$ *uniformly and in the C^1-sense.*

$\Psi_k(\Lambda) = \varphi_k^\mu(\Lambda_1) + \sum_{i=2}^{k}\varphi_i(\Lambda_i)$
$\varphi_k^\mu(s)' = f_k(s) - \mu = 0$ has 2 solutions: $\Psi_k(\Lambda)$ has 2 critical points.

4. The finite dimensional reduction

Let $\mathcal{J}_\varepsilon(\xi) = E_\varepsilon(V + \phi)$ where ϕ is the solution of

$$\mathcal{L}_\varepsilon \phi = h + \sum_{i=1}^{k} c_i \, Z_i \qquad (4)$$

such that $\phi(0) = \phi(\infty) = 0$ and $\int_0^\infty Z_i \, \phi \, dx = 0$,

$$\mathcal{L}_\varepsilon \phi = -\phi'' + \phi - (p+\varepsilon) e^{\varepsilon x} V^{p+\varepsilon-1} \phi - \lambda \left(\frac{p-1}{2}\right)^2 e^{-(p-1)x} \phi$$

and $Z_i(x) = U_i'(x) - U_i'(0) e^{-x}$, $i = 1, \ldots, k$. If $h = N_\varepsilon(\phi) + R_\varepsilon$, $N_\varepsilon(\phi) = e^{\varepsilon x} \left[(V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon) V^{p+\varepsilon-1} \phi \right]$ and $R_\varepsilon = e^{\varepsilon x} [V^{p+\varepsilon} - V^p] + V^p [e^{\varepsilon x} - 1] + [V^p - \sum_{i=1}^k V_i^p] + \lambda \left(\frac{p-1}{2}\right)^2 e^{-(p-1)x} \, V$,

$$\nabla_\xi \mathcal{J}_\varepsilon(\xi) = 0$$

Under technical conditions, one finds a solution to (4) if h is small w.r.t. $\|h\|_* = \sup_{x>0} \left(\sum_{i=1}^k e^{-\sigma|x-\xi_i|}\right)^{-1} |h(x)|$, σ small enough.

Let us consider for a number M large but fixed, the conditions:

$$\left\{ \begin{array}{c} \xi_1 > \frac{1}{2} \log(M\varepsilon)^{-1}, \quad \log(M\varepsilon)^{-1} < \min_{1\leq i<k}(\xi_{i+1} - \xi_i) \,, \\ \\ \xi_k < k \log(M\varepsilon)^{-1}, \quad \lambda < M\,\varepsilon^{\frac{3-p}{2}} \,. \end{array} \right. \tag{5}$$

For σ chosen small enough:

$$\|N_\varepsilon(\phi)\|_* \leq C\,\|\phi\|_*^{\min\{p,2\}} \quad \text{and} \quad \|R^\varepsilon\|_* \leq C\,\varepsilon^{\frac{3-p}{2}} \,.$$

Lemma 12 *Assume that (5) holds. Then there is a $C > 0$ s.t., for $\varepsilon > 0$ small enough, there exists a unique solution ϕ with*

$$\|\phi\|_* \leq C\varepsilon \quad \text{and} \quad \|D_\xi \phi\|_* \leq C\varepsilon \; .$$

Lemma 13 *Assume that (5) holds. The following expansion holds*

$$\mathcal{J}_\varepsilon(\xi) = E_\varepsilon(V) + o(\varepsilon) \; ,$$

where the term $o(\varepsilon)$ is uniform in the C^1-sense.

5. The case $N = 4$

Theorem 14 *Let $N = 4$. Given a number $k \geq 1$, if*

$$\mu > \mu_k = k \frac{\pi}{2^5} e^2 \quad \textit{and} \quad \lambda e^{-2/\lambda} = \mu \varepsilon \ ,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \ldots, k$, which depend on k and μ, and two solutions $u_\varepsilon^\pm$:

$$u_\varepsilon^\pm(y) = \gamma \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^2 |y|^2} \right) M_j \ (1 + o(1)) \ ,$$

uniformly on B as $\varepsilon \to 0$, with $M_j^\pm = \alpha_j^\pm \varepsilon^{\frac{1}{2}-j} |\log \varepsilon|^{-\frac{1}{2}}$.

The proof is similar to the case $N \geq 5$. For $N = 4$, the order of the height of each bubble is corrected with a logarithmic term.

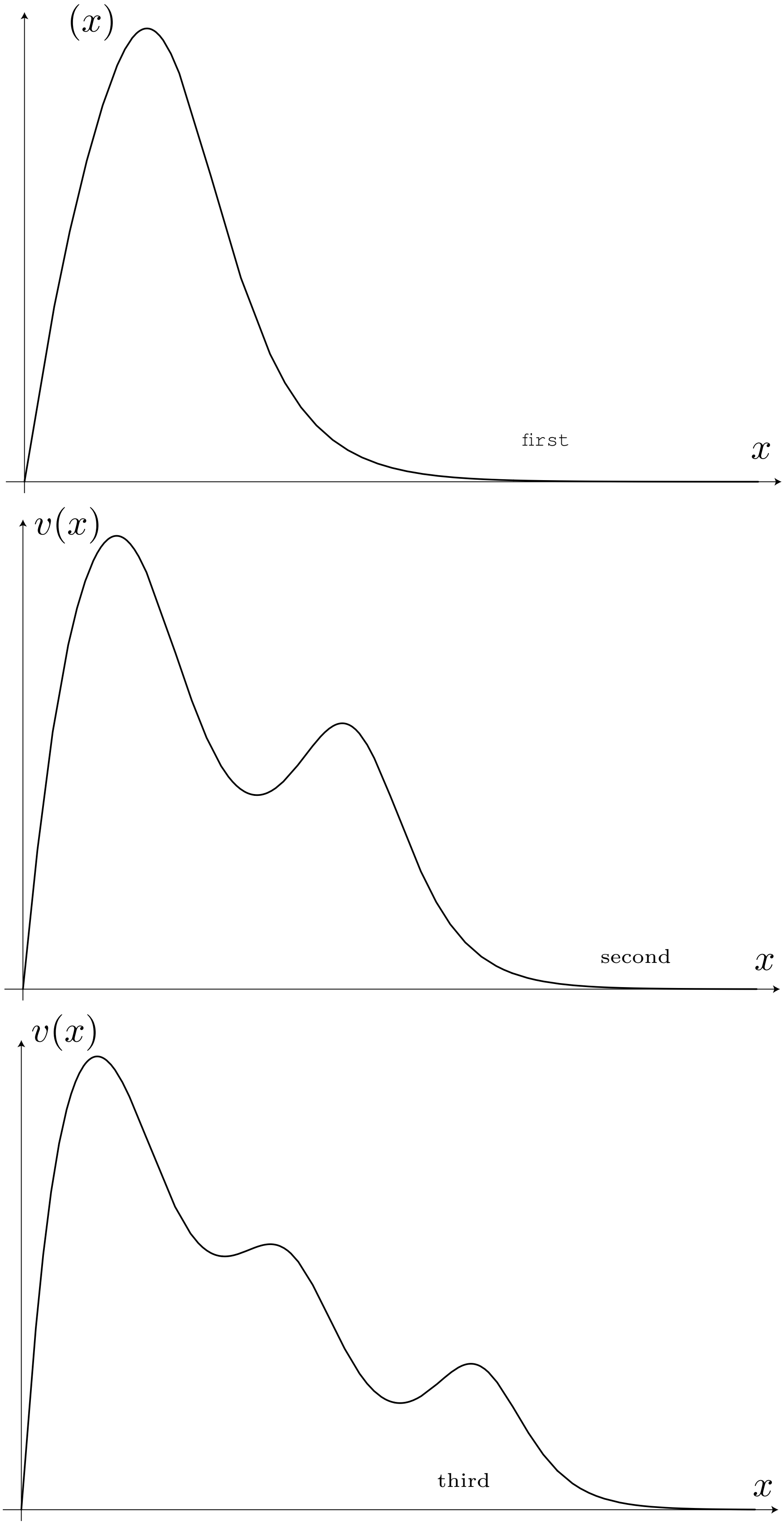

Figure 18: Functions corresponding to the first three *turning* points *to the right* in the previous bifurcation diagram, with $\varepsilon = 0.2$

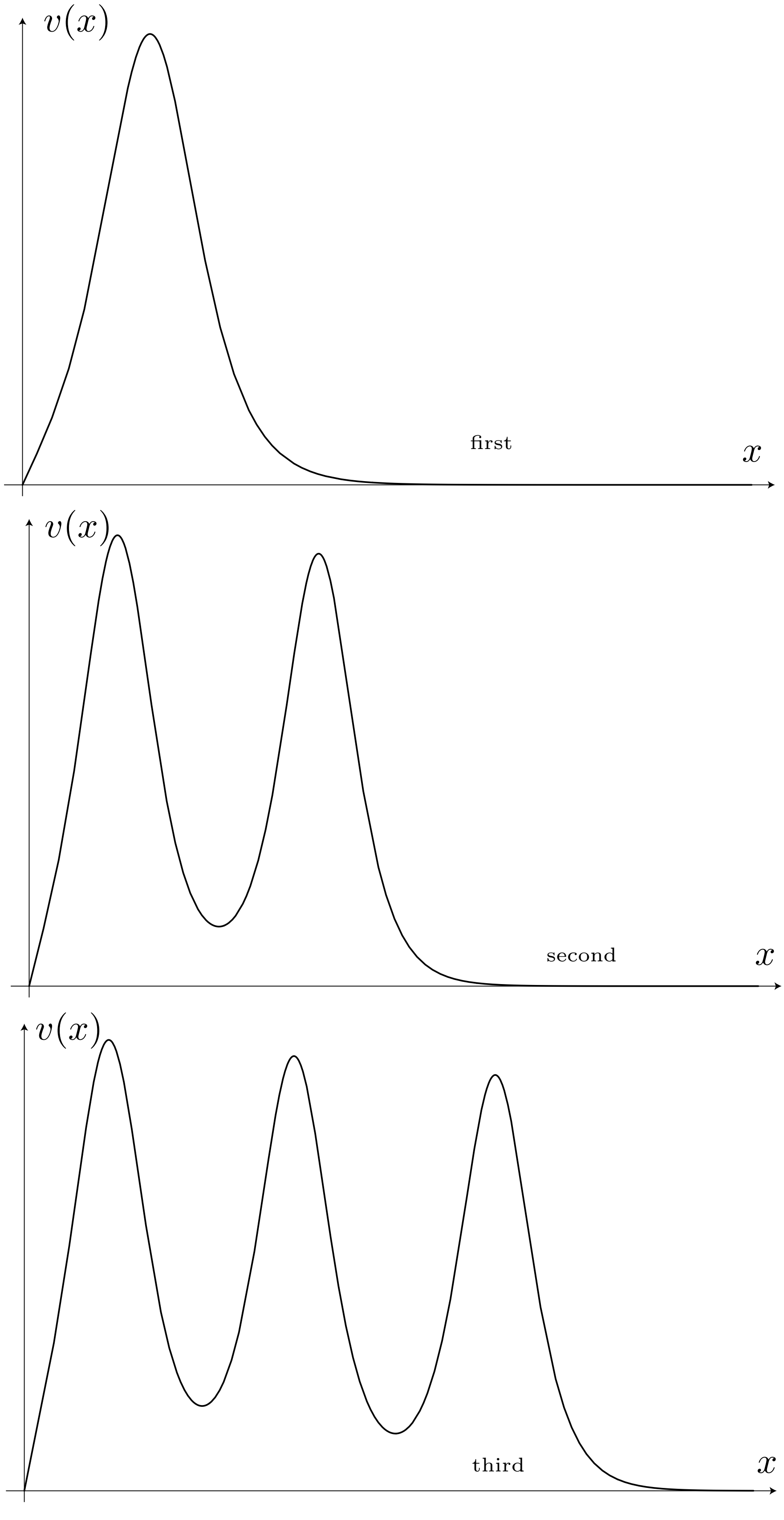

Figure 19: Functions corresponding to the first three *turning* points *to the right*, with $\varepsilon = 0.01$

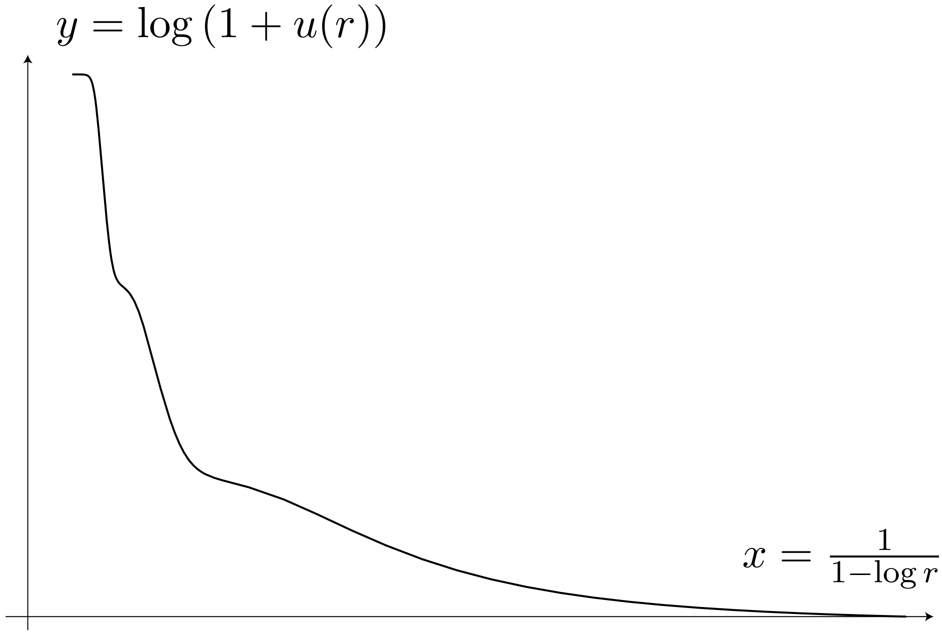

Figure 20: *A 3-bubble solution, with* $\varepsilon = 0.01$.

IV – The Brezis-Nirenberg problem: the general case

Case $N \geq 4$: [Ge,Jing,Pacard04]. From now on: N=3

$\Omega \subset \mathbb{R}^3$, bounded domain with smooth boundary:

$$\begin{cases} \Delta u + \lambda u + u^q = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

If $1 < q < 5$ and $0 < \lambda < \lambda_1$ subcritical solutions are critical points of :

$$Q_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{2}{q+1}}}, \quad u \in H_0^1(\Omega) \setminus \{0\}$$

and

$$S_\lambda = \inf_{u \in H_0^1(\Omega) \setminus \{0\}} Q_\lambda(u). \tag{2}$$

S_λ is achieved thanks to compactness of Sobolev embedding.

Critical case:

Let $q = 5 = \frac{N+2}{N-2}$, $N = 3$, and define $\lambda^* = \inf\{\lambda > 0 \ / S_\lambda < S_0\}$.

$$S_0 = \inf_{u \in C_0^1(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{\left(\int_{\mathbb{R}^3} |u|^6\right)^{\frac{1}{3}}}$$

Brezis and Nirenberg : $0 < \lambda^* < \lambda_1$, S_λ is achieved for $\lambda^* < \lambda < \lambda_1$ and (1) is solvable in this range. When Ω is a ball : $\lambda^* = \frac{\lambda_1}{4}$.

Theorem 15 **(a)** *Assume that $\lambda^* < \lambda < \lambda_1$. Then there exists a number $q_1 > 5$ such that Problem (1) is solvable for any $q \in (5, q_1)$.*

(b) *Assume that Ω is a ball and that $\frac{\lambda_1}{4} < \lambda < \lambda_1$. Then, given $k \geq 1$ there exists a number $q_k > 5$ such that Problem (1) has at least k radial solutions for any $q \in (5, q_k)$.*

Blowing-up solution for (1) near the critical exponent: sequence of solutions u_n of (1) for $\lambda = \lambda_n$ bounded, and $q = q_n \to 5$.

$$M_n = \alpha^{-1} \max_{\Omega} u_n = \alpha^{-1} u_n(x_n) \to +\infty$$

with $\alpha > 0$ to be chosen, we see then that the scaled function

$$u_n(y) = M_n \, v_n(x_n + M_n^{(q_n-1)/2} \, y)$$

satisfies

$$\Delta v_n + v_n^{q_n} + M_n^{-(q_n-1)} \lambda_n v_n = 0$$

in the expanding domain $\Omega_n = M_n^{(q_n-1)/2}(\Omega - x_n)$.

If x_n stays away from the boundary of Ω: locally over compacts around the origin, v_n converges up to subsequences to $w > 0$

$$\Delta w + w^5 = 0 \quad \text{in } \mathbb{R}^3$$

$w(0) = \max w = \alpha = 3^{1/4}$. Explicit form:

$$w(z) = 3^{1/4} \left(\frac{1}{1 + |z|^2} \right)^{1/2}$$

(extremal of the Sobolev constant S_0). In the original variable, "near x_n"

$$u_n(x) \sim 3^{1/4} \left(\frac{1}{1 + M_n^4 \, |x - x_n|^2} \right)^{1/2} M_n \, (\, 1 + o(1) \,)$$

The convergence holds only local over compacts. We say that the solution $u_n(x)$ is a *single bubble* if the equivalent holds with $o(1) \to 0$ uniformly in Ω.

$N = 3$. Let $\lambda < \lambda_1$ and consider Green's function $G_\lambda(x, y)$

$$-\Delta_y G_\lambda - \lambda G_\lambda = \delta_x \quad y \in \Omega, \quad G_\lambda(x, y) = 0 \quad y \in \partial\Omega\,.$$

Robin's function: $g_\lambda(x) = H_\lambda(x, x)$, where

$$H_\lambda(x, y) = \frac{1}{4\pi|y - x|} - G_\lambda(x, y)$$

$g_\lambda(x)$ is a smooth function which goes to $+\infty$ as x approaches $\partial\Omega$. Its minimum value is not necessarity positive but it is decreasing in λ. It is strictly positive when λ is close to 0 and approaches $-\infty$ as $\lambda \uparrow \lambda_1$. [Druet] : the number λ^* can be characterized as

$$\lambda^* = \sup\{\lambda > 0 \ / \ \min_{\Omega} g_\lambda > 0\}$$

As $\lambda \downarrow \lambda^*$, u_λ constitute a single-bubble with blowing-up near the set where g_{λ_*} attains its minimum value zero.

Role of *non-trivial critical values* of g_λ for existence, not only in the critical case $q = 5$ and in the sub-critical $q = 5 - \varepsilon$.

Let $\mathcal{D}$ be an open subset of Ω with smooth boundary. We recall that *g_λ links non-trivially in $\mathcal{D}$ at critical level $\mathcal{G}_\lambda$ relative to B and B_0* if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: if we set

$$\Gamma = \{\Phi \in C(B, \mathcal{D}) \ / \ \Phi|_{B_0} = Id\}$$

$$\text{then } \sup_{y \in B_0} g_\lambda(y) < \mathcal{G}_\lambda \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} g_\lambda(\Phi(y)),$$

and for all $y \in \partial\mathcal{D}$ such that $g_\lambda(y) = \mathcal{G}_\lambda$, $\exists \tau_y$ tangent to $\partial\mathcal{D}$ at y such that $\nabla g_\lambda(y) \cdot \tau_y \neq 0$.

Theorem 16 **(a)** *Super-critical case: Assume that* $\mathcal{G}_\lambda < 0$, $q = 5+\varepsilon$. *Then Problem* (1) *is solvable for all sufficiently small* $\varepsilon > 0$;

$$u_\varepsilon(y) = 3^{\frac{1}{4}} \left(\frac{1}{1 + M_\varepsilon^4 \, |y - \zeta_\varepsilon|^2} \right)^{\frac{1}{2}} M_\varepsilon \, (\, 1 + o(1) \,)$$

where $o(1) \to 0$ *uniformly in* $\overline{\Omega}$ *as* $\varepsilon \to 0$, $M_\varepsilon = \frac{2^{\frac{3}{2}}}{3^{\frac{1}{8}}\pi} (-\mathcal{G}_\lambda)^{1/2} \varepsilon^{-\frac{1}{2}}$ *and* $\zeta_\varepsilon \in \mathcal{D}$ *is such that* $g_\lambda(\zeta_\varepsilon) \to \mathcal{G}_\lambda, \quad \nabla g_\lambda(\zeta_\varepsilon) \to 0, \quad$ *as* $\varepsilon \to 0$.

(b) *Sub-critical case: Assume that* $\mathcal{G}_\lambda > 0$, $q = 5 - \varepsilon$. *Then Problem* (1) *has a solution* u_ε *of* (1) *exactly as in part (a) but with* $M_\varepsilon = \frac{2^{\frac{3}{2}}}{3^{\frac{1}{8}}\pi} (\mathcal{G}_\lambda)^{1/2} \varepsilon^{-\frac{1}{2}}$

$\Omega \subset \mathbb{R}^3$ is symmetric with respect to the coordinate planes if for all $(y_1, y_2, y_3) \in \Omega$ we have that

$$(-y_1, y_2, y_3),\ (y_1, -y_2, y_3),\ (y_1, y_2, -y_3)\ \in \Omega.$$

Theorem 17 *If Ω is symmetric, $g_\lambda(0) < 0$ and $q = 5 + \varepsilon$, then, given $k \geq 1$, there exists for all sufficiently small $\varepsilon > 0$ a solution u_ε*

$$u_\varepsilon(x) = 3^{\frac{1}{4}} \sum_{j=1}^{k} \left(\frac{1}{1 + M_{j\varepsilon}^4 |x|^2} \right)^{\frac{1}{2}} M_{j\varepsilon} \left(1 + o(1)\right)$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$ and, for $j = 1, \ldots, k$,

$$M_{j\varepsilon} = (-g_\lambda(0))^{1/2} \left(\frac{2^5}{3^{\frac{1}{4}} k\pi} \right) \left(\frac{2^{\frac{15}{2}}}{\pi} \right)^{j-1} \frac{(k-j)!}{(k-1)!} \varepsilon^{\frac{1}{2}-j},$$

Sketch of Proofs

- a careful analysis of Robin's function
- the Emden-Fowler change of coordinates around a critical point of Robin's function
- an energy expansion: 20 pages of computations, but at the end the constants are explicit !
- build a fixed point in the appropriate weighted norm, using the exponential decay of the bubbles in the new variables, and get the appropriate continuity estimates
- finite dimensional reduction