Multi-bulles pour des problèmes légèrement surcritiques

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I. Bubble-tower radial solutions in the slightly supercritical Brezis-Nirenberg problem

We consider the Brezis-Nirenberg problem

\[
\begin{aligned}
\Delta u + u^{p+\varepsilon} + \lambda u &= 0 \quad \text{in } B \\
u &> 0 \quad \text{in } B , \quad u = 0 \quad \text{on } \partial B
\end{aligned}
\]  \hspace{1cm} \text{(1)}

in dimension \(N \geq 4\), in the supercritical case: \(p = \frac{N+2}{N-2}, \varepsilon > 0\).

If \(\varepsilon \to 0\) and if, simultaneously, \(\lambda \to 0\) at the appropriate rate, then there are radial solutions which behave like a superposition of bubbles: \(M_j \to +\infty\) and \(M_j = o(M_{j+1})\) for all \(j\) and

\[
(N(N - 2))^{(N-2)/4} \sum_{j=1}^{k} \left(1 + M_j^{\frac{4}{N-2}} |y|^2\right)^{-\frac{(N-2)}{2}} M_j (1 + o(1))
\]
1. Parametrization of the solutions

Let $B$ be the unit ball in $\mathbb{R}^N$, $N \geq 4$, and consider for $p = \frac{N+2}{N-2}$ and $\varepsilon \geq 0$ the positive solutions of

$$
\begin{cases}
\Delta u + u^{p+\varepsilon} + \lambda u = 0 & \text{in } B \\
u > 0 & \text{in } B, \\
 u = 0 & \text{on } \partial B
\end{cases}
$$

Denote by $\rho = \rho(a) > 0$ the first zero of $v$ given by

$$
\begin{cases}
v'' + \frac{N-1}{r} v' + v^{p+\varepsilon} + v = 0 & \text{in } [0, +\infty) \\
v(0) = a > 0, \\
v'(0) = 0
\end{cases}
$$

To any solution $u$ of (1) corresponds a function $v$ on $[0, \sqrt{\lambda})$ s.t.

$$v(|x|) = \lambda^{-1/(p+\varepsilon-1)} u(x/\sqrt{\lambda}) \iff u(x) = \rho^{2/(p+\varepsilon-1)} v(\rho |x|)
$$

with $\lambda = \rho^2(a)$. The bifurcation diagram $(\lambda, \|u\|_{L^\infty})$ is therefore fully parametrized by $a \mapsto (\rho^2, a \rho^{2/(p+\varepsilon-1)})$ with $\rho = \rho^2(a)$. 
2. References, heuristics and main result

\[ p = \frac{N+2}{N-2}, \quad \varepsilon \geq 0, \quad N \geq 4, \quad B \text{ is the unit ball in } \mathbb{R}^N \]

\[
\begin{aligned}
-\Delta u &= u^{p+\varepsilon} + \lambda u, \quad u > 0 \quad \text{in } B \\
u &= 0 \quad \text{on } \partial B
\end{aligned}
\]

- <1950: Lane, Emden, Fowler, Chandrasekhar (astrophysics)
- Sobolev, Rellich, Nash, Gagliardo, Nirenberg, Pohozaev
- 1976: Aubin, Talenti
- 1983: Brezis, Nirenberg: case \( \varepsilon = 0 \) is solvable for \( 0 < \lambda \leq \lambda_1 = \lambda_1(-\Delta) \). Uniqueness (Zhang, 1992).
- subcritical case \( (0 > \varepsilon \rightarrow 0) \): Brezis and Peletier, Rey, Han
- supercritical case: Symmetry (Gidas, Ni, Nirenberg, 1979).
Consider a family of (radial, noincreasing) solutions \( u_\varepsilon \) of (1) for \( \lambda = \lambda_\varepsilon \to 0 \). The problem at \( \lambda = 0, \varepsilon = 0 \) has no solution:

\[
M_\varepsilon = \gamma^{-1} \max u_\varepsilon = \gamma^{-1} u_\varepsilon(0) \to +\infty
\]

for some fixed constant \( \gamma > 0 \). Let \( v_\varepsilon(z) = M_\varepsilon u_\varepsilon \left( M_\varepsilon^{(p+\varepsilon-1)/2} z \right) \)

\[
\Delta v_\varepsilon + v_\varepsilon^{p+\varepsilon} + M_\varepsilon^{-(p+\varepsilon-1)} \lambda_\varepsilon v_\varepsilon = 0, \quad |z| < M_\varepsilon^{(p+\varepsilon-1)/2}.
\]

Locally over compacts around the origin, \( v_\varepsilon \to w \) s.t.

\[
\Delta w + w^p = 0
\]

with \( w(0) = \gamma := (N(N-2))^{N-2}/4 \): \( w(z) = \gamma \left( \frac{1}{1 + |z|^2} \right)^{\frac{N-2}{2}} \).

Guess: \( u_\varepsilon(y) = \gamma \left( 1 + M_\varepsilon^{\frac{4}{N-2}} |y|^2 \right)^{-\frac{N-2}{2}} M_\varepsilon \left( 1 + o(1) \right) \) as \( \varepsilon \to 0 \).
Theorem 1 [ $k$-bubble solution ] Assume $N \geq 5$. Then, given an integer $k \geq 1$, there exists a number $\mu_k > 0$ s.t. if $\mu > \mu_k$ and

$$\lambda = \mu \varepsilon^{\frac{N-4}{N-2}},$$

then there are constants $0 < \alpha_j^- < \alpha_j^+, \ j = 1, \ldots, k$ which depend on $k$, $N$ and $\mu$ and two solutions $u_{\varepsilon}^\pm$ of Problem (1) of the form

$$u_{\varepsilon}^\pm(y) = \gamma \sum_{j=1}^k \left(1 + \left[\alpha_j^\pm \varepsilon^{\frac{1}{2}-j}\right]^{\frac{4}{N-2}} |y|^2 \right)^{-(N-2)/2} \alpha_j^\pm \varepsilon^{\frac{1}{2}-j} (1 + o(1)),$$

where $\gamma = (N(N-2))^{\frac{N-2}{4}}$ and $o(1) \to 0$ uniformly on $B$ as $\varepsilon \to 0$.

Bifurcation curve: $\lambda = \varepsilon^{\frac{N-4}{N-2}} f_k \left(c_k^{-1} \varepsilon^{k-\frac{1}{2}} m \right)$ for $m \sim \varepsilon^{\frac{1}{2}-k}$. 
The numbers $\alpha_j^\pm$ can be expressed by the formulae

$$\alpha_j^\pm = b_3^{1-j} \frac{(k-j)!}{(k-1)!} s_k^\pm(\mu), \quad j = 1, \ldots, k,$$

where $b_3 = \frac{(N-2) \sqrt{\pi} \Gamma \left(\frac{N}{2}\right)}{2^{N+2} \Gamma\left(\frac{N+1}{2}\right)}$ and $s_k^\pm(\mu)$ are the two solutions of

$$\mu = f_k(s) := kb_1 s^{\frac{4}{N-2}} + b_2 s^{-2\frac{N-4}{N-2}}$$

with $b_1 = \left(\frac{N-2}{4}\right)^3 \frac{N-4}{N-1}$ and $b_2 = (N-2) \frac{\Gamma(N-1)}{\Gamma\left(\frac{N-4}{2}\right) \Gamma\left(\frac{N}{2}\right)}$.

Remind that $\mu > \mu_k$ be the minimum value of the function $f_k(s)$:

$$\mu_k = (N-2) \left[ \frac{b_1 k}{N-4} \right]^{\frac{N-4}{N-2}} \left[ \frac{b_2}{2} \right]^{\frac{2}{N-2}}$$
3. The asymptotic expansion

The solution of

\[ v'' - v + e^{\varepsilon x} v^{p+\varepsilon} + \left(\frac{p-1}{2}\right)^2 \lambda e^{-(p-1)x} v = 0 \quad \text{on} \ (0, \infty) \]

with \( v(0) = v(\infty) = 0, \ v > 0 \) is given as a critical point of

\[ E_\varepsilon(w) = I_\varepsilon(w) - \frac{1}{2} \left(\frac{p-1}{2}\right)^2 \lambda \int_0^\infty e^{-(p-1)x} |w|^2 \, dx \]

\[ I_\varepsilon(w) = \frac{1}{2} \int_0^\infty |w'|^2 \, dx + \frac{1}{2} \int_0^\infty |w|^2 \, dx - \frac{1}{p + \varepsilon + 1} \int_0^\infty e^{\varepsilon x} |w|^{p+\varepsilon+1} \, dx \]

\[ U(x) = \left(\frac{4N}{N-2}\right)^{\frac{N-2}{4}} e^{-x} \left(1 + e^{-\frac{4}{N-2} x}\right)^{-\frac{N-2}{2}} \]

is the solution of

\[ U'' - U + U^p = 0 \]

Ansatz: \( v(x) = V(x) + \phi, \ V(x) = \sum_{i=1}^k (U(x - \xi_i) - U(\xi_i) e^{-x}). \)
Further choices:

\[ \xi_1 = -\frac{1}{2} \log \varepsilon + \log \Lambda_1 , \]

\[ \xi_{i+1} - \xi_i = - \log \varepsilon - \log \Lambda_{i+1} , \quad i = 1, \ldots, k - 1 . \]

**Lemma 1** Let \( N \geq 5 \) and \( \lambda = \mu \varepsilon^{\frac{N-4}{N-2}} \). Then

\[
E_\varepsilon(V) = k a_0 + \varepsilon \Psi_k(\Lambda) + \frac{k^2}{2} a_3 \varepsilon \log \varepsilon + a_5 \varepsilon + \varepsilon \theta_\varepsilon(\Lambda)
\]

\[
\Psi_k(\Lambda) = a_1 \Lambda_1^{-2} - k a_3 \log \Lambda_1 - a_4 \mu \Lambda_1^{-(p-1)}
\]

\[
+ \sum_{i=2}^{k} [(k-i+1) a_3 \log \Lambda_i - a_2 \Lambda_i] ,
\]

and \( \lim_{\varepsilon \to 0} \theta_\varepsilon(\Lambda) = 0 \) uniformly and in the \( C^1 \)-sense.
Constants are explicit:

\[
\begin{align*}
a_0 &= \frac{1}{2} \int_{-\infty}^{\infty} (|U'|^2 + U^2) \, dx - \frac{1}{p+1} \int_{-\infty}^{\infty} U^{p+1} \, dx \\
a_1 &= \left( \frac{4N}{N-2} \right)^{(N-2)/2} \\
a_2 &= \left( \frac{N}{N-2} \right)^{(N-2)/4} \int_{-\infty}^{\infty} e^x U^p \, dx \\
a_3 &= \frac{1}{p+1} \int_{-\infty}^{\infty} U^{p+1} \, dx \\
a_4 &= \frac{1}{2} \left( \frac{p-1}{2} \right)^2 \int_{-\infty}^{\infty} e^{-(p-1)x} U^2 \, dx \\
a_5 &= \frac{1}{p+1} \int_{-\infty}^{\infty} U^{p+1} \log U \, dx + \frac{1}{(p+1)^2} \int_{-\infty}^{\infty} U^{p+1} \, dx
\end{align*}
\]

\[\Psi_k(\Lambda) = \varphi^\mu_k(\Lambda_1) + \sum_{i=2}^{k} \varphi_i(\Lambda_i)\]

\[\varphi^\mu_k(s) = a_1 s^{-2} - k a_3 \log s - a_4 \mu s^{-(p-1)}\]

\[\varphi_i(s) = (k - i + 1) a_3 \log s - a_2 s\]

\[\varphi^\mu_k(s)' = f_k(s) - \mu = 0\] has 2 solutions: \(\Psi_k(\Lambda)\) has 2 critical points.
4. The finite dimensional reduction

Let $I_{\varepsilon}(\xi) = E_{\varepsilon}(V + \phi)$ where $\phi$ is the solution of

$$
\mathcal{L}_{\varepsilon}\phi = h + \sum_{i=1}^{k} c_i Z_i \tag{2}
$$

such that $\phi(0) = \phi(\infty) = 0$ and $\int_{0}^{\infty} Z_i \phi \, dx = 0$,

$$
\mathcal{L}_{\varepsilon}\phi = -\phi'' + \phi - (p + \varepsilon) e^{\varepsilon x} V^{p+\varepsilon-1} \phi - \lambda \left( \frac{p-1}{2} \right)^2 e^{-(p-1)x} \phi
$$

and $Z_i(x) = U_i'(x) - U_i'(0) e^{-x}$, $i = 1, \ldots, k$. If $h = N_{\varepsilon}(\phi) + R_\varepsilon$, $N_{\varepsilon}(\phi) = e^{\varepsilon x} \left[ (V + \phi)^{p+\varepsilon} - V^{p+\varepsilon} - (p + \varepsilon) V^{p+\varepsilon-1} \phi \right]$ and $R_\varepsilon = e^{\varepsilon x} [V^{p+\varepsilon} - V^p] + V^p [e^{\varepsilon x} - 1] + [V^p - \sum_{i=1}^{k} V_i^p] + \lambda \left( \frac{p-1}{2} \right)^2 e^{-(p-1)x} V$,

$$
\nabla_{\xi} I_{\varepsilon}(\xi) = 0
$$
Under technical conditions, one finds a solution to (2) if $h$ is small w.r.t. $\|h\|_* = \sup_{x > 0} \left( \sum_{i=1}^{k} e^{-\sigma|x - \xi_i|} \right)^{-1} |h(x)|$, $\sigma$ small enough. Let us consider for a number $M$ large but fixed, the conditions:

$$
\left\{
\begin{array}{ll}
\xi_1 > \frac{1}{2} \log(M\varepsilon)^{-1}, & \log(M\varepsilon)^{-1} < \min_{1 \leq i < k} (\xi_{i+1} - \xi_i), \\
\xi_k < k \log(M\varepsilon)^{-1}, & \lambda < M \varepsilon^{\frac{3-p}{2}}.
\end{array}
\right.
$$

(3)

For $\sigma$ chosen small enough:

$$
\|N_{\varepsilon}(\phi)\|_* \leq C \|\phi\|^\min\{p,2\} \quad \text{and} \quad \|R_{\varepsilon}\|_* \leq C \varepsilon^{\frac{3-p}{2}}.
$$
Lemma 2 Assume that (3) holds. Then there is a $C > 0$ s.t., for $\varepsilon > 0$ small enough, there exists a unique solution $\phi$ with
\[
\|\phi\|_* \leq C\varepsilon \quad \text{and} \quad \|D_{\xi}\phi\|_* \leq C\varepsilon .
\]

Lemma 3 Assume that (3) holds. The following expansion holds
\[
J_\varepsilon(\xi) = E_\varepsilon(V) + o(\varepsilon) ,
\]
where the term $o(\varepsilon)$ is uniform in the $C^1$-sense.
5. The case $N = 4$

**Theorem 2** Let $N = 4$. Given a number $k \geq 1$, if

$$\mu > \mu_k = k \frac{\pi}{2} e^2 \quad \text{and} \quad \lambda e^{-2/\lambda} = \mu \epsilon,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+, j = 1, \ldots, k,$ which depend on $k$ and $\mu$, and two solutions $u_{\epsilon}^\pm$:

$$u_{\epsilon}^\pm(y) = \gamma \sum_{j=1}^{k} \left( \frac{1}{1 + M_j^2 |y|^2} \right) M_j (1 + o(1)),$$

uniformly on $B$ as $\epsilon \to 0$, with $M_j^\pm = \alpha_j^\pm \epsilon^{1/2-j} |\log \epsilon|^{-1/2}$.

The proof is similar to the case $N \geq 5$. For $N = 4$, the order of the height of each bubble is corrected with a logarithmic term.
II. Bubbling solutions in the slightly supercritical Brezis-Nirenberg problem

\( \Omega \subset \mathbb{R}^3 \), bounded domain with smooth boundary:

\[
\begin{aligned}
\Delta u + \lambda u + u^q &= 0 \quad \text{in } \Omega \\
u > 0 & \quad \text{in } \Omega \\
u &= 0 \quad \text{on } \partial \Omega.
\end{aligned}
\]  

(1)

If \( 1 < q < 5 \) and \( 0 < \lambda < \lambda_1 \) subcritical solutions are ascritical points of:

\[
Q_\lambda(u) = \frac{\int_{\Omega} |\nabla u|^2 - \lambda \int_{\Omega} |u|^2}{\left( \int_{\Omega} |u|^{q+1} \right)^{\frac{2}{q+1}}}, \quad u \in H^1_0(\Omega) \setminus \{0\}
\]

and

\[
S_\lambda = \inf_{u \in H^1_0(\Omega) \setminus \{0\}} Q_\lambda(u).
\]  

(2)
$S_\lambda$ is achieved thanks to compactness of Sobolev embedding.

$$S_0 = \inf_{u \in C^1_0(\mathbb{R}^3) \setminus \{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2}{\left( \int_{\mathbb{R}^3} |u|^6 \right)^{\frac{1}{3}}}$$

Let $q = 5$ and $\lambda^* = \inf\{\lambda > 0 \mid S_\lambda < S_0\}$.

Brezis and Nirenberg: $0 < \lambda^* < \lambda_1$, $S_\lambda$ is achieved for $\lambda^* < \lambda < \lambda_1$ and (1) is solvable in this range. When $\Omega$ is a ball: $\lambda^* = \frac{\lambda_1}{4}$, no solution for $\lambda \leq \lambda^*$.

**Theorem 3 (a)** Assume that $\lambda^* < \lambda < \lambda_1$, where $\lambda^*$. Then there exists a number $q_1 > 5$ such that Problem (1) is solvable for any $q \in (5, q_1)$.

**(b)** Assume that $\Omega$ is a ball and that $\frac{\lambda_1}{4} < \lambda < \lambda_1$. Then, given $k \geq 1$ there exists a number $q_k > 5$ such that Problem (1) has at least $k$ radial solutions for any $q \in (5, q_k)$. 

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Blowing-up solution for (1) near the critical exponent: sequence of solutions $u_n$ of (1) for $\lambda = \lambda_n$ bounded, and $q = q_n \to 5$.

$$M_n = \alpha^{-1} \max_{\Omega} u_n = \alpha^{-1} u_n(x_n) \to +\infty$$

with $\alpha > 0$ to be chosen, we see then that the scaled function

$$v_n(y) = M_n u_n(x_n + M_n^{(q_n-1)/2} y)$$

satisfies

$$\Delta v_n + v_n^{q_n} + M_n^{-(q_n-1)} \lambda_n v_n = 0$$

in the expanding domain $\Omega_n = M_n^{(q_n-1)/2} (\Omega - x_n)$. 

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If $x_n$ stays away from the boundary of $\Omega$: locally over compacts around the origin, $v_n$ converges up to subsequences to $w > 0$

$$\Delta w + w^5 = 0 \quad \text{in } \mathbb{R}^3$$

$w(0) = \max w = \alpha = 3^{1/4}$. Explicit form:

$$w(z) = 3^{1/4} \left( \frac{1}{1 + |z|^2} \right)^{1/2}$$

(extremal of the Sobolev constant $S_0$). In the original variable, “near $x_n$”

$$u_n(x) \sim 3^{1/4} \left( \frac{1}{1 + M_n^4 |x - x_n|^2} \right)^{1/2} M_n \left( 1 + o(1) \right)$$

The convergence holds only local over compacts. We say that the solution $u_n(x)$ is a *single bubble* if the equivalent holds with $o(1) \to 0$ uniformly in $\Omega$. 

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Let $\lambda < \lambda_1$ and consider Green’s function $G_\lambda(x, y)$

$$-\Delta_y G_\lambda - \lambda G_\lambda = \delta_x \quad y \in \Omega, \quad G_\lambda(x, y) = 0 \quad y \in \partial \Omega.$$  
Robin’s function: $g_\lambda(x) = H_\lambda(x, x)$, where

$$H_\lambda(x, y) = \frac{1}{4\pi|y - x|} - G_\lambda(x, y)$$

$g_\lambda(x)$ is a smooth function which goes to $+\infty$ as $x$ approaches $\partial \Omega$. Its minimum value is not necessarily positive but it is decreasing in $\lambda$. It is strictly positive when $\lambda$ is close to 0 and approaches $-\infty$ as $\lambda \uparrow \lambda_1$. [Druet]: the number $\lambda^*$ can be characterized as

$$\lambda^* = \sup\{\lambda > 0 \mid \min_{\Omega} g_\lambda > 0\}$$

As $\lambda \downarrow \lambda^*$, $u_\lambda$ constitute a single-bubble with blowing-up near the set where $g_{\lambda^*}$ attains its minimum value zero.
Role of **Non-trivial critical values** of $g_\lambda$ for existence, not only in the critical case $q = 5$ and in the sub-critical $q = 5 - \varepsilon$. Apparently new even in the case of the ball is established: duality between the sub and super-critical cases.

Let $\mathcal{D}$ be an open subset of $\Omega$ with smooth boundary. We recall that $g_\lambda$ **links non-trivially in $\mathcal{D}$ at critical level $\mathcal{G}_\lambda$ relative to $B$ and $B_0$** if $B$ and $B_0$ are closed subsets of $\bar{\mathcal{D}}$ with $B$ connected and $B_0 \subset B$ such that the following conditions hold: if we set

$$
\Gamma = \{ \Phi \in C(B, \mathcal{D}) \mid \Phi|_{B_0} = Id \}
$$

then

$$
\sup_{y \in B_0} g_\lambda(y) < \mathcal{G}_\lambda \equiv \inf_{\Phi \in \Gamma} \sup_{y \in B} g_\lambda(\Phi(y)),
$$

and for all $y \in \partial \mathcal{D}$ such that $g_\lambda(y) = \mathcal{G}_\lambda$, $\exists \tau_y$ tangent to $\partial \mathcal{D}$ at $y$ such that $\nabla g_\lambda(y) \cdot \tau_y \neq 0$. 
Theorem 4 Under the above assumptions,
(a) Assume that \( g_\lambda < 0 \) \( q = 5 + \varepsilon \). Then Problem (1) is solvable for all sufficiently small \( \varepsilon > 0 \);

\[ u_\varepsilon(y) = 3^4 \left( \frac{1}{1 + M_\varepsilon^4 |y - \zeta_\varepsilon|^2} \right)^{1/2} M_\varepsilon (1 + o(1)) \]

where \( o(1) \rightarrow 0 \) uniformly in \( \overline{\Omega} \) as \( \varepsilon \rightarrow 0 \), \( M_\varepsilon = \frac{2^\frac{3}{2}}{3^\frac{8}{\pi}} (-g_\lambda)^{1/2} \varepsilon^{-\frac{1}{2}} \)

and \( \zeta_\varepsilon \in \mathcal{D} \) is such that \( g_\lambda(\zeta_\varepsilon) \rightarrow g_\lambda, \nabla g_\lambda(\zeta_\varepsilon) \rightarrow 0 \), as \( \varepsilon \rightarrow 0 \).

(b) Assume that \( g_\lambda > 0 \), \( q = 5 - \varepsilon \). Then Problem (1) has a solution \( u_\varepsilon \) of (1) exactly as in part (a) but with

\[ M_\varepsilon = \frac{2^\frac{3}{1}}{3^\frac{8}{\pi}} (g_\lambda)^{1/2} \varepsilon^{-\frac{1}{2}} \]
Ω ⊂ \mathbb{R}^3 is symmetric with respect to the coordinate planes if for all \((y_1, y_2, y_3) \in \Omega\) we have that
\[-(y_1, y_2, y_3), (y_1, -y_2, y_3), (y_1, y_2, -y_3) \in \Omega.\]

**Theorem 5** If \(\Omega\) is symmetric, \(g_\lambda(0) < 0\) and \(q = 5 + \varepsilon\), then, given \(k \geq 1\), there exists for all sufficiently small \(\varepsilon > 0\) a solution \(u_\varepsilon\)

\[
u_\varepsilon(x) = 3^{\frac{1}{4}} \sum_{j=1}^{k} \left( \frac{1}{1 + M_{j\varepsilon}^4 |x|^2} \right)^{\frac{1}{2}} M_{j\varepsilon} (1 + o(1))
\]

where \(o(1) \to 0\) uniformly in \(\Omega\) and, for \(j = 1, \ldots, k\),

\[
M_{j\varepsilon} = (-g_\lambda(0))^{1/2} \left( \frac{2^5}{3^{\frac{1}{4}} k \pi} \right) \left( \frac{2^{\frac{15}{2}}}{\pi} \right)^{-1} \frac{(k - j)!}{(k - 1)!} \varepsilon^{\frac{1}{2} - j},
\]