Multi-bulles pour des problèmes légr̀ement surcritiques

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I. Bubble-tower radial solutions in the slightly supercritical Brezis-Nirenberg problem

We consider the Brezis-Nirenberg problem

$$\begin{cases} \Delta u + u^{p+\varepsilon} + \lambda\, u = 0 & \text{in } B \\ u > 0 \quad \text{in } B\,, \quad u = 0 & \text{on } \partial B \end{cases} \tag{1}$$

in dimension $N \geq 4$, in the supercritical case: $p = \frac{N+2}{N-2}$, $\varepsilon > 0$. If $\varepsilon \to 0$ and if, simultaneously, $\lambda \to 0$ at the appropriate rate, then there are radial solutions which behave like a superposition of *bubbles*: $M_j \to +\infty$ and $M_j = o(M_{j+1})$ for all j and

$$(N(N-2))^{(N-2)/4} \sum_{j=1}^{k} \left(1 + M_j^{\frac{4}{N-2}}\, |y|^2\right)^{-(N-2)/2} M_j\,(\,1 \;+\; o(1)\,)$$

1. Parametrization of the solutions

Let B be the unit ball in $I\!R^N$, $N \geq 4$, and consider for $p = \frac{N+2}{N-2}$ and $\varepsilon \geq 0$ the positive solutions of

$$\begin{cases} \Delta u + u^{p+\varepsilon} + \lambda\, u = 0 & \text{in } B \\ u > 0 \quad \text{in } B\,, \quad u = 0 & \text{on } \partial B \end{cases}$$

Denote by $\rho = \rho(a) > 0$ the first zero of v given by

$$\begin{cases} v'' + \frac{N-1}{r}\, v' + v^{p+\varepsilon} + v = 0 \quad \text{in } [0, +\infty) \\ v(0) = a > 0\,, \quad v'(0) = 0 \end{cases}$$

To any solution u of (1) corresponds a function v on $[0, \sqrt{\lambda})$ s.t.

$$v(|x|) = \lambda^{-1/(p+\varepsilon-1)}\, u(x/\sqrt{\lambda}) \Longleftrightarrow u(x) = \rho^{2/(p+\varepsilon-1)}\, v(\rho\, |x|)$$

with $\lambda = \rho^2(a)$. The bifurcation diagram $(\lambda, \|u\|_{L^\infty})$ is therefore fully parametrized by $a \mapsto (\rho^2,\, a\, \rho^{2/(p+\varepsilon-1)})$ with $\rho = \rho^2(a)$.

2. References, heuristics and main result

$p = \frac{N+2}{N-2}$, $\varepsilon \geq 0$, $N \geq 4$, B is the unit ball in $\mathbb{R}^N$

$$\begin{cases} -\Delta u = u^{p+\varepsilon} + \lambda\, u\,, \quad u > 0 & \text{in } B \\ u = 0 & \text{on } \partial B \end{cases}$$

- <1950: Lane, Emden, Fowler, Chandrasekhar (astrophysics)
- Sobolev, Rellich, Nash, Gagliardo, Nirenberg, Pohozaev
- 1976: Aubin, Talenti
- 1983: Brezis, Nirenberg: case $\varepsilon = 0$ is solvable for $0 < \lambda < \lambda_1 = \lambda_1(-\Delta)$. Uniqueness (Zhang, 1992).
- subcritical case ($0 > \varepsilon \to 0$): Brezis and Peletier, Rey, Han
- supercritical case: Symmetry (Gidas, Ni, Nirenberg, 1979). Budd and Norbury (1987, case $\varepsilon > 0$): formal asymptotics, numerical computations. Merle and Peletier (1991): existence of a unique value $\lambda = \lambda_* > 0$ for which there exists a radial, singular, positive solution. Branch of solutions: Flores (thesis, 2001).

Consider a family of (radial, noincreasing) solutions u_ε of (1) for $\lambda = \lambda_\varepsilon \to 0$. The problem at $\lambda = 0$, $\varepsilon = 0$ has no solution:

$$M_\varepsilon = \gamma^{-1} \max u_\varepsilon = \gamma^{-1} u_\varepsilon(0) \to +\infty$$

for some fixed constant $\gamma > 0$. Let $v_\varepsilon(z) = M_\varepsilon \, u_\varepsilon \left(M_\varepsilon^{(p+\varepsilon-1)/2} \, z \right)$

$$\Delta v_\varepsilon + v_\varepsilon^{p+\varepsilon} + M_\varepsilon^{-(p+\varepsilon-1)} \lambda_\varepsilon v_\varepsilon = 0, \quad |z| < M_\varepsilon^{(p+\varepsilon-1)/2}.$$

Locally over compacts around the origin, $v_\varepsilon \to w$ s.t.

$$\Delta w + w^p = 0$$

with $w(0) = \gamma := (N(N-2))^{\frac{N-2}{4}}$:$w(z) = \gamma \left(\frac{1}{1+|z|^2} \right)^{\frac{N-2}{2}}$.

Guess: $u_\varepsilon(y) = \gamma \left(1 + M_\varepsilon^{\frac{4}{N-2}} |y|^2 \right)^{-\frac{N-2}{2}} M_\varepsilon \, (\, 1 + o(1) \,)$ as $\varepsilon \to 0$.

Theorem 1 [k-bubble solution] *Assume $N \geq 5$. Then, given an integer $k \geq 1$, there exists a number $\mu_k > 0$ s.t. if $\mu > \mu_k$ and*

$$\lambda = \mu\, \varepsilon^{\frac{N-4}{N-2}}\,,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \ldots, k$ which depend on k, N and μ and two solutions $u_\varepsilon^\pm$ of Problem (1) of the form

$$u_\varepsilon^\pm(y) = \gamma \sum_{j=1}^{k} \left(1 + \left[\alpha_j^\pm\, \varepsilon^{\frac{1}{2}-j} \right]^{\frac{4}{N-2}} |y|^2 \right)^{-(N-2)/2} \alpha_j^\pm\, \varepsilon^{\frac{1}{2}-j}\, (1 + o(1)\,)\,,$$

where $\gamma = (N(N-2))^{\frac{N-2}{4}}$ and $o(1) \to 0$ uniformly on B as $\varepsilon \to 0$.

Bifurcation curve: $\lambda = \varepsilon^{\frac{N-4}{N-2}} f_k\left(c_k^{-1} \varepsilon^{k-\frac{1}{2}}\, m \right)$ for $m \sim \varepsilon^{\frac{1}{2}-k}$.

The numbers $\alpha_j^{\pm}$ can be expressed by the formulae

$$\alpha_j^{\pm} = b_3^{1-j} \frac{(k-j)!}{(k-1)!} s_k^{\pm}(\mu), \quad j = 1, \ldots, k \ ,$$

where $b_3 = \frac{(N-2)\sqrt{\pi}\,\Gamma(\frac{N}{2})}{2^{N+2}\,\Gamma(\frac{N+1}{2})}$ and $s_k^{\pm}(\mu)$ are the two solutions of

$$\mu = f_k(s) := kb_1 s^{\frac{4}{N-2}} + b_2 s^{-2\frac{N-4}{N-2}}$$

with $b_1 = \left(\frac{N-2}{4}\right)^3 \frac{N-4}{N-1}$ and $b_2 = (N-2)\frac{\Gamma(N-1)}{\Gamma\left(\frac{N-4}{2}\right)\Gamma\left(\frac{N}{2}\right)}$.

Remind that $\mu > \mu_k$ be the minimum value of the function $f_k(s)$:

$$\mu_k = (N-2)\left[\frac{b_1\,k}{N-4}\right]^{\frac{N-4}{N-2}} \left[\frac{b_2}{2}\right]^{\frac{2}{N-2}}$$

3. The asymptotic expansion

The solution of

$$v'' - v + e^{\varepsilon x}\, v^{p+\varepsilon} + \left(\frac{p-1}{2}\right)^2 \lambda\, e^{-(p-1)x}\, v = 0 \quad \text{on } (0,\infty)$$

with $v(0) = v(\infty) = 0$, $v > 0$ is given as a critical point of

$$E_\varepsilon(w) = I_\varepsilon(w) - \frac{1}{2}\left(\frac{p-1}{2}\right)^2 \lambda \int_0^\infty e^{-(p-1)x}|w|^2\, dx$$

$$I_\varepsilon(w) = \frac{1}{2}\int_0^\infty |w'|^2\, dx + \frac{1}{2}\int_0^\infty |w|^2\, dx - \frac{1}{p+\varepsilon+1}\int_0^\infty e^{\varepsilon x}|w|^{p+\varepsilon+1}\, dx$$

$U(x) = \left(\frac{4\,N}{N-2}\right)^{\frac{N-2}{4}} e^{-x}\left(1 + e^{-\frac{4}{N-2}x}\right)^{-\frac{N-2}{2}}$ is the solution of

$$U'' - U + U^p = 0$$

Ansatz: $v(x) = V(x) + \phi$, $V(x) = \sum_{i=1}^k (U(x-\xi_i) - U(\xi_i)\, e^{-x})$.

Further choices:

$$\xi_1 = -\tfrac{1}{2}\log\varepsilon + \log\Lambda_1\ ,$$

$$\xi_{i+1} - \xi_i = -\log\varepsilon - \log\Lambda_{i+1}\ , \quad i = 1, \ldots, k-1\ .$$

Lemma 1 *Let $N \geq 5$ and $\lambda = \mu\,\varepsilon^{\frac{N-4}{N-2}}$. Then*

$$E_\varepsilon(V) \,=\, k\,a_0 + \varepsilon\,\Psi_k(\Lambda) + \frac{k^2}{2}a_3\,\varepsilon\,\log\varepsilon + a_5\,\varepsilon + \varepsilon\,\theta_\varepsilon(\Lambda)$$

$$\Psi_k(\Lambda) = a_1\,\Lambda_1^{-2} - k\,a_3\log\Lambda_1 - a_4\,\mu\,\Lambda_1^{-(p-1)} + \sum_{i=2}^{k}\left[(k-i+1)\,a_3\log\Lambda_i - a_2\,\Lambda_i\right]\ ,$$

and $\lim_{\varepsilon\to 0}\theta_\varepsilon(\Lambda) = 0$ *uniformly and in the C^1-sense.*

Constants are explicit:

$$\begin{cases} a_0 = \frac{1}{2}\int_{-\infty}^{\infty}\left(|U'|^2 + U^2\right)\,dx - \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\,dx \\ a_1 = \left(\frac{4N}{N-2}\right)^{(N-2)/2} \\ a_2 = \left(\frac{N}{N-2}\right)^{(N-2)/4}\int_{-\infty}^{\infty} e^x\,U^p\,dx \\ a_3 = \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\,dx \\ a_4 = \frac{1}{2}\left(\frac{p-1}{2}\right)^2\int_{-\infty}^{\infty} e^{-(p-1)\,x}\,U^2\,dx \\ a_5 = \frac{1}{p+1}\int_{-\infty}^{\infty} U^{p+1}\log U\,dx + \frac{1}{(p+1)^2}\int_{-\infty}^{\infty} U^{p+1}\,dx \end{cases}$$

$$\Psi_k(\Lambda) = \varphi_k^{\mu}(\Lambda_1) + \sum_{i=2}^{k}\varphi_i(\Lambda_i)$$

$\varphi_k^{\mu}(s) = a_1\,s^{-2} - k\,a_3\log s \;-\; a_4\,\mu\,s^{-(p-1)}$
$\varphi_i(s) = (k-i+1)\,a_3\log s - a_2\,s$
$\varphi_k^{\mu}(s)' = f_k(s) - \mu = 0$ has 2 solutions: $\Psi_k(\Lambda)$ has 2 critical points.

4. The finite dimensional reduction

Let $\mathcal{J}_\varepsilon(\xi) = E_\varepsilon(V+\phi)$ where ϕ is the solution of

$$\mathcal{L}_\varepsilon \phi = h + \sum_{i=1}^{k} c_i\, Z_i \tag{2}$$

such that $\phi(0)=\phi(\infty)=0$ and $\int_0^\infty Z_i\,\phi\,dx = 0$,

$$\mathcal{L}_\varepsilon \phi = -\phi'' + \phi - (p+\varepsilon)e^{\varepsilon x}V^{p+\varepsilon-1}\phi - \lambda\left(\frac{p-1}{2}\right)^2 e^{-(p-1)x}\phi$$

and $Z_i(x) = U_i'(x) - U_i'(0)e^{-x}$, $i=1,\dots,k$. If $h = N_\varepsilon(\phi)+R_\varepsilon$, $N_\varepsilon(\phi) = e^{\varepsilon x}\left[(V+\phi)_+^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi\right]$ and $R_\varepsilon = e^{\varepsilon x}[V^{p+\varepsilon} - V^p] + V^p[e^{\varepsilon x}-1] + [V^p - \sum_{i=1}^k V_i^p] + \lambda\left(\frac{p-1}{2}\right)^2 e^{-(p-1)x}\,V$,

$$\nabla_\xi \mathcal{J}_\varepsilon(\xi) = 0$$

Under technical conditions, one finds a solution to (2) if h is small w.r.t. $\|h\|_* = \sup_{x>0} \left(\sum_{i=1}^k e^{-\sigma|x-\xi_i|}\right)^{-1} |h(x)|$, σ small enough.

Let us consider for a number M large but fixed, the conditions:

$$\begin{cases} \xi_1 > \frac{1}{2}\log(M\varepsilon)^{-1}, \quad \log(M\varepsilon)^{-1} < \min_{1\le i<k}(\xi_{i+1} - \xi_i) \,, \\ \\ \xi_k < k\log(M\varepsilon)^{-1}, \quad \lambda < M\,\varepsilon^{\frac{3-p}{2}} \,. \end{cases} \tag{3}$$

For σ chosen small enough:

$$\|N_\varepsilon(\phi)\|_* \le C\,\|\phi\|_*^{\min\{p,2\}} \quad \text{and} \quad \|R^\varepsilon\|_* \le C\,\varepsilon^{\frac{3-p}{2}} \,.$$

Lemma 2 *Assume that (3) holds. Then there is a* $C > 0$ *s.t., for* $\varepsilon > 0$ *small enough, there exists a unique solution* ϕ *with*

$$\|\phi\|_* \le C\varepsilon \quad \text{and} \quad \|D_\xi \phi\|_* \le C\varepsilon \; .$$

Lemma 3 *Assume that (3) holds. The following expansion holds*

$$\mathcal{J}_\varepsilon(\xi) = E_\varepsilon(V) + o(\varepsilon) \; ,$$

where the term $o(\varepsilon)$ *is uniform in the* C^1*-sense.*

5. The case $N = 4$

Theorem 2 *Let $N = 4$. Given a number $k \geq 1$, if*

$$\mu > \mu_k = k\,\frac{\pi}{2^5}\,e^2 \quad \text{and} \quad \lambda\, e^{-2/\lambda} = \mu\,\varepsilon\ ,$$

then there are constants $0 < \alpha_j^- < \alpha_j^+$, $j = 1, \ldots, k$, which depend on k and μ, and two solutions $u_\varepsilon^\pm$:

$$u_\varepsilon^\pm(y) = \gamma \sum_{j=1}^{k} \left(\frac{1}{1 + M_j^2\,|y|^2} \right) M_j\ (1 + o(1))\ ,$$

uniformly on B as $\varepsilon \to 0$, with $M_j^\pm = \alpha_j^\pm \varepsilon^{\frac{1}{2}-j} |\log \varepsilon|^{-\frac{1}{2}}$.

The proof is similar to the case $N \geq 5$. For $N = 4$, the order of the height of each bubble is corrected with a logarithmic term.

II. Bubbling solutions in the slightly supercritical Brezis-Nirenberg problem

$\Omega \subset I\!R^3$, bounded domain with smooth boundary:

$$\begin{cases} \Delta u + \lambda u + u^q = 0 & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (1)$$

If $1 < q < 5$ and $0 < \lambda < \lambda_1$ subcritical solutions are ascritical points of :

$$Q_\lambda(u) = \frac{\int_\Omega |\nabla u|^2 - \lambda \int_\Omega |u|^2}{\left(\int_\Omega |u|^{q+1}\right)^{\frac{2}{q+1}}}, \quad u \in H_0^1(\Omega) \setminus \{0\}$$

and

$$S_\lambda = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} Q_\lambda(u). \qquad (2)$$

S_λ is achieved thanks to compactness of Sobolev embedding.

$$S_0 = \inf_{u \in C_0^1(\mathbf{R}^3)\setminus\{0\}} \frac{\int_{\mathbf{R}^3} |\nabla u|^2}{(\int_{\mathbf{R}^3} |u|^6)^{\frac{1}{3}}}$$

Let $q = 5$ and $\lambda^* = \inf\{\lambda > 0 \ / S_\lambda < S_0\}$.
Brezis and Nirenberg : $0 < \lambda^* < \lambda_1$, S_λ is achieved for $\lambda^* < \lambda < \lambda_1$ and (1) is solvable in this range. When Ω is a ball : $\lambda^* = \frac{\lambda_1}{4}$, no solution for $\lambda \leq \lambda^*$.

Theorem 3 (a) *Assume that* $\lambda^* < \lambda < \lambda_1$*, where* λ^**. Then there exists a number* $q_1 > 5$ *such that Problem* (1) *is solvable for any* $q \in (5, q_1)$*.*
(b) *Assume that* Ω *is a ball and that* $\frac{\lambda_1}{4} < \lambda < \lambda_1$*. Then, given* $k \geq 1$ *there exists a number* $q_k > 5$ *such that Problem* (1) *has at least* k *radial solutions for any* $q \in (5, q_k)$*.*

Blowing-up solution for (1) near the critical exponent: sequence of solutions u_n of (1) for $\lambda = \lambda_n$ bounded, and $q = q_n \to 5$.

$$M_n = \alpha^{-1} \max_{\Omega} u_n = \alpha^{-1} u_n(x_n) \to +\infty$$

with $\alpha > 0$ to be chosen, we see then that the scaled function

$$v_n(y) = M_n \, u_n(x_n + M_n^{(q_n-1)/2} \, y)$$

satisfies

$$\Delta v_n + v_n^{q_n} + M_n^{-(q_n-1)} \lambda_n v_n = 0$$

in the expanding domain $\Omega_n = M_n^{(q_n-1)/2}(\Omega - x_n)$.

If x_n stays away from the boundary of Ω: locally over compacts around the origin, v_n converges up to subsequences to $w > 0$

$$\Delta w + w^5 = 0 \quad \text{in } \mathbb{R}^3$$

$w(0) = \max w = \alpha = 3^{1/4}$. Explicit form:

$$w(z) = 3^{1/4} \left(\frac{1}{1 + |z|^2} \right)^{1/2}$$

(extremal of the Sobolev constant S_0). In the original variable, "near x_n"

$$u_n(x) \sim 3^{1/4} \left(\frac{1}{1 + M_n^4 \, |x - x_n|^2} \right)^{1/2} M_n \, (\, 1 + o(1) \,)$$

The convergence holds only local over compacts. We say that the solution $u_n(x)$ is a *single bubble* if the equivalent holds with $o(1) \to 0$ uniformly in Ω.

Let $\lambda < \lambda_1$ and consider Green's function $G_\lambda(x, y)$

$$-\Delta_y G_\lambda - \lambda G_\lambda = \delta_x \quad y \in \Omega, \quad G_\lambda(x, y) = 0 \quad y \in \partial\Omega \, .$$

Robin's function: $g_\lambda(x) = H_\lambda(x, x)$, where

$$H_\lambda(x, y) = \frac{1}{4\pi|y - x|} - G_\lambda(x, y)$$

$g_\lambda(x)$ is a smooth function which goes to $+\infty$ as x approaches $\partial\Omega$. Its minimum value is not necessarity positive but it is decreasing in λ. It is strictly positive when λ is close to 0 and approaches $-\infty$ as $\lambda \uparrow \lambda_1$. [Druet] : the number λ^* can be characterized as

$$\lambda^* = \sup\{\lambda > 0 \ / \ \min_{\Omega} g_\lambda > 0\}$$

As $\lambda \downarrow \lambda^*$, u_λ constitute a single-bubble with blowing-up near the set where g_{λ_*} attains its minimum value zero.

Role of *Non-trivial critical values* of g_λ for existence, not only in the critical case $q = 5$ and in the sub-critical $q = 5-\varepsilon$. Apparently new even in the case of the ball is established: duality between the sub and super-critical cases.

Let $\mathcal{D}$ be an open subset of Ω with smooth boundary. We recall that *g_λ links non-trivially in $\mathcal{D}$ at critical level $\mathcal{G}_\lambda$ relative to B and B_0* if B and B_0 are closed subsets of $\bar{\mathcal{D}}$ with B connected and $B_0 \subset B$ such that the following conditions hold: if we set

$$\Gamma = \{\Phi \in C(B, \mathcal{D}) \ / \ \Phi|_{B_0} = Id\}$$

$$\text{then} \sup_{y \in B_0} g_\lambda(y) < \mathcal{G}_\lambda \equiv \inf_{\Phi\in\Gamma} \sup_{y\in B} g_\lambda(\Phi(y)) \, ,$$

and for all $y \in \partial\mathcal{D}$ such that $g_\lambda(y) = \mathcal{G}_\lambda$, $\exists \tau_y$ tangent to $\partial\mathcal{D}$ at y such that $\nabla g_\lambda(y) \cdot \tau_y \neq 0$.

Theorem 4 *Under the above assumptions,*
(a) *Assume that* $\mathcal{G}_\lambda < 0 \quad q = 5 + \varepsilon$. *Then Problem* (1) *is solvable for all sufficiently small* $\varepsilon > 0$;

$$u_\varepsilon(y) = 3^{\frac{1}{4}} \left(\frac{1}{1 + M_\varepsilon^4 \, |y - \zeta_\varepsilon|^2} \right)^{\frac{1}{2}} M_\varepsilon \, (\, 1 + o(1)\,)$$

where $o(1) \to 0$ *uniformly in* $\overline{\Omega}$ *as* $\varepsilon \to 0$, $M_\varepsilon = \frac{2^{\frac{3}{2}}}{3^{\frac{1}{8}}\pi}(-\mathcal{G}_\lambda)^{1/2}\varepsilon^{-\frac{1}{2}}$
and $\zeta_\varepsilon \in \mathcal{D}$ *is such that* $g_\lambda(\zeta_\varepsilon) \to \mathcal{G}_\lambda, \quad \nabla g_\lambda(\zeta_\varepsilon) \to 0, \quad$ *as* $\varepsilon \to 0$.

(b) *Assume that* $\mathcal{G}_\lambda > 0\,, \quad q = 5 - \varepsilon$. *Then Problem* (1) *has a solution* u_ε *of* (1) *exactly as in part (a) but with*
$M_\varepsilon = \frac{2^{\frac{3}{2}}}{3^{\frac{1}{8}}\pi}(\mathcal{G}_\lambda)^{1/2}\varepsilon^{-\frac{1}{2}}$

$\Omega \subset \mathbb{R}^3$ is symmetric with respect to the coordinate planes if for all $(y_1, y_2, y_3) \in \Omega$ we have that

$$(-y_1, y_2, y_3),\ (y_1, -y_2, y_3),\ (y_1, y_2, -y_3)\ \in \Omega.$$

Theorem 5 *If Ω is symmetric, $g_\lambda(0) < 0$ and $q = 5 + \varepsilon$, then, given $k \geq 1$, there exists for all sufficiently small $\varepsilon > 0$ a solution u_ε*

$$u_\varepsilon(x) = 3^{\frac{1}{4}} \sum_{j=1}^{k} \left(\frac{1}{1 + M_{j\varepsilon}^4 |x|^2} \right)^{\frac{1}{2}} M_{j\varepsilon} \left(1 + o(1)\right)$$

where $o(1) \to 0$ uniformly in $\overline{\Omega}$ and, for $j = 1, \ldots, k$,

$$M_{j\varepsilon} = (-g_\lambda(0))^{1/2} \left(\frac{2^5}{3^{\frac{1}{4}} k\pi} \right) \left(\frac{2^{\frac{15}{2}}}{\pi} \right)^{j-1} \frac{(k-j)!}{(k-1)!} \varepsilon^{\frac{1}{2}-j},$$