# Stability in Gagliardo-Nirenberg-Sobolev inequalities

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# The stability result of G. Bianchi and H. Egnell

In Sobolev's inequality (with optimal contant $\mathsf{S}_d$),

$$\mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge 0$$

*is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions* when $d \ge 3$ ? A question raised in [Brezis, Lieb (1985)]

▷ [Bianchi, Egnell (1991)] There is a positive constant $\alpha$ such that

$$\mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \alpha \inf_{\varphi \in \mathscr{M}} \|\nabla f - \nabla \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants $\alpha$ and $\kappa$ and $f \mapsto \lambda(f)$ such that

$$\mathsf{S}_d \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \left(1 + \kappa \, \lambda(f)^\alpha\right) \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

*However, the question of **constructive** estimates is still widely open*

# From the carré du champ method to stability results

***Carré du champ method*** (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m , \quad \frac{d\mathscr{F}}{dt} = -\mathscr{I} , \quad \frac{d\mathscr{I}}{dt} \le -\Lambda\,\mathscr{I}$$

deduce that $\mathscr{I} - \Lambda\mathscr{F}$ is monotone non-increasing with limit 0

$$\boxed{\mathscr{I}[u] \ge \Lambda\,\mathscr{F}[u]}$$

▷ ***Improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star \ge \Lambda$ such that

$$\mathscr{I} - \Lambda\mathscr{F} \ge (\Lambda_\star - \Lambda)\,\mathscr{F}$$

▷ ***Improved entropy – entropy production inequality*** (weaker form)

$$\mathscr{I} \ge \Lambda\,\psi(\mathscr{F})$$

for some $\psi$ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathscr{I} - \Lambda\mathscr{F} \ge \Lambda\left(\psi(\mathscr{F}) - \mathscr{F}\right) \ge 0$$

# An application in the *Natural Sciences*

[Carlen, Figalli, 2013] *Stability for a GNS inequality and the log-HLS inequality, with application to the critical mass Keller-Segel equation*

▷ the log-HLS (Hardy-Littlewood-Sobolev) inequality

$$\mathscr{H}[u] := \int_{\mathbb{R}^2} u \log\left(\frac{u}{M}\right) dx + \frac{2}{M} \iint_{\mathbb{R}^2\times\mathbb{R}^2} u(x)\,u(y)\,\log|x-y|\,dx\,dy + M\,(1+\log\pi) \ge 0$$

with $M = \|u\|_1$ is linked with the $8\pi$ critical mass in the Keller-Segel model

▷ the GNS (Gagliardo-Nirenberg-Sobolev) inequality (special case)

$$\int_{\mathbb{R}^2} |\nabla f|^2\,dx \int_{\mathbb{R}^2} |f|^4\,dx \ge \pi \int_{\mathbb{R}^2} |f|^6\,dx$$

[Carlen, Carrillo, Loss, 2010] If $u$ solves $\frac{\partial u}{\partial t} = \Delta\sqrt{u}$ and $f = u^{1/4}$, then

$$-\frac{1}{8}\frac{d}{dt}\mathscr{H}[u] = \int_{\mathbb{R}^2} |\nabla f|^2\,dx - \pi\,\frac{\int_{\mathbb{R}^2}|f|^6\,dx}{\int_{\mathbb{R}^2}|f|^4\,dx}$$

- *Stability for log-HLS arises from the stability for GNS*

# Optimal transportation and gradient flows

- The fast diffusion flow seen as a gradient flow with respect to Wasserstein's distance
[McCann, 1997], [Otto, 2001]
[Cordero-Erausquin, Nazaret, Villani, 2004]
[Agueh, Ghoussoub, Kang, 2004]
[Carrillo, Lisini, Savaré, Slepčev, 2009]
[Zinsl, Matthes, 2015], [Zinsl, 2019]
[Iacobelli, Patacchini, Santambrogio, 2019]
[Ambrosio, Mondino, Savaré, 2019]
... already a long story (with apologizes for not quoting all relevant papers)

▷ the PDE point of view:
– *decay of the entropies*
– *regularization properties of the parabolic equations*
– *carré du champ method*
– *some spectral analysis*

# Outline

- Gagliardo-Nirenberg-Sobolev inequalities by variational methods

▷ A special family of Gagliardo-Nirenberg-Sobolev inequalities
▷ Concentration-compactness
▷ Stability results by variational methods

- The fast diffusion equation and the entropy methods

▷ Rényi entropy powers
▷ Spectral gaps and asymptotics
▷ Initial time layer

- Regularity and stability

▷ Uniform convergence in relative error
▷ The threshold time
▷ Improved entropy-entropy production inequality
▷ Some consequences

# Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\rm GNS}(p) \, \|f\|_{2p} \tag{GNS}$$

Up to translations, multiplications by a constant and scalings, there is a unique optimal function

$$\mathrm{g}(x) = \left(1 + |x|^2\right)^{-\frac{1}{p-1}}$$

# Gagliardo-Nirenberg-Sobolev inequalities

We consider the inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\rm GNS}(p)\,\|f\|_{2p}\qquad\text{(GNS)}$$

$\theta=\frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}$, $p\in(1,+\infty)$ if $d=1$ or $2$, $p\in(1,p^*]$ if $d\ge3$, $p^*=\frac{d}{d-2}$

## Theorem (del Pino, JD)

*Equality case in* (GNS) *is achieved if and only if*

$$f\in\mathfrak{M}:=\left\{g_{\lambda,\mu,y}\,:\,(\lambda,\mu,y)\in(0,+\infty)\times\mathbb{R}\times\mathbb{R}^d\right\}$$

*Aubin-Talenti functions:* $g_{\lambda,\mu,y}(x):=\mu\,\mathrm{g}((x-y)/\lambda)$, $\mathrm{g}(x)=\left(1+|x|^2\right)^{-\frac{1}{p-1}}$

# Related inequalities

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p}\qquad\text{(GNS)}$$

▷ *Sobolev's inequality*: $d\ge3$, $p=p^*=d/(d-2)$

$$\mathsf{S}_d\,\|\nabla f\|_2\ge\|f\|_{2p^*}$$

▷ *Euclidean Onofri inequality*

$$\int_{\mathbb{R}^2}e^{h-\overline{h}}\,\frac{dx}{\pi\,(1+|x|^2)^2}\le e^{\frac{1}{16\pi}\int_{\mathbb{R}^2}|\nabla h|^2\,dx}$$

$d=2$, $p\to+\infty$ with $f_p(x):=\mathrm{g}(x)\left(1+\frac{1}{2p}\,(h(x)-\overline{h})\right)$, $\overline{h}=\int_{\mathbb{R}^2}h(x)\,\frac{dx}{\pi\,(1+|x|^2)^2}$

▷ *Euclidean logarithmic Sobolev inequality in scale invariant form*

$$\frac{d}{2}\log\left(\frac{2}{\pi\,d\,e}\int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}x\right)\ge\int_{\mathbb{R}^d}|f|^2\log|f|^2\,\mathrm{d}x$$

or $\int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}x\ge\frac12\int_{\mathbb{R}^d}|f|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)\mathrm{d}x+\frac{d}{4}\log\left(2\pi\,e^2\right)\int_{\mathbb{R}^d}|f|^2\,\mathrm{d}x$

# Deficit, scale invariance

*Deficit functional*

$$\delta[f] := (p-1)^2 \, \|\nabla f\|_2^2 + 4\,\frac{d-p\,(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathcal{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

**Lemma**

(GNS) *is equivalent to* $\delta[f] \ge 0$ *if and only if*

$$\mathcal{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$$

*where* $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$ *and* $C(p,d)$ *is an explicit positive constant*

Take $f_\lambda(x) = \lambda^{\frac{d}{2p}}\,f(\lambda\,x)$ and optimize on $\lambda > 0$

$$\delta[f] \ge \delta[f] - \inf_{\lambda>0}\delta[f_\lambda] =: \delta_\star[f] \ge 0$$

A simplification: $\delta[f] = \delta[|f|]$ so we shall assume that $f \ge 0$ a.e.

# Relative entropy, relative Fisher information

▷ *Free energy* or *relative entropy functional*

$$\mathcal{E}[f|g] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1} - g^{p+1} - \tfrac{1+p}{2p}\,g^{1-p}\left(f^{2p} - g^{2p}\right)\right)dx$$

$$\text{If} \quad \int_{\mathbb{R}^d} f^{2p}\,(1,x,|x|^2)\,dx = \int_{\mathbb{R}^d} g^{2p}\,(1,x,|x|^2)\,dx\,, \quad g \in \mathfrak{M}$$

$$\text{then} \quad \mathcal{E}[f|g] = \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1} - g^{p+1}\right)dx \quad \text{and} \quad \delta_\star[f] \approx \mathcal{E}[f|g]^2$$

▷ *Relative Fisher information*

$$\mathcal{J}[f|g] := \frac{p+1}{p-1}\int_{\mathbb{R}^d}\left|(p-1)\,\nabla f + f^p\,\nabla g^{1-p}\right|^2 dx$$

▷ Nonlinear extension of the *Heisenberg uncertainty principle*

$$\left(\int_{\mathbb{R}^d} f^{p+1}\,dx\right)^2 \le \int_{\mathbb{R}^d} |\nabla f|^2\,dx \int_{\mathbb{R}^d} |x|^2 f^{2p}\,dx$$

# Some inequalities

**Lemma (Csiszár-Kullback inequality)**

*Let $d \ge 1$ and $p > 1$. There exists a constant $C_p > 0$ such that*

$$\left\| f^{2p} - \mathrm{g}^{2p} \right\|^2_{\mathrm{L}^1(\mathbb{R}^d)} \le C_p \, \mathscr{E}[f|\mathrm{g}] \quad \textit{if} \quad \|f\|_{2p} = \|\mathrm{g}\|_{2p}$$

**Lemma (Entropy - entropy production inequality)**

$$\frac{p+1}{p-1} \, \delta[f] = \mathscr{J}[f|g_f] - 4\,\mathscr{E}[f|g_f] \ge 0$$

**Lemma (A weak stability result)**

$$\delta[f] \ge \delta_\star[f] \approx \mathscr{E}[f|g]^2$$

# Concentration-compactness

$$I_M = \inf\left\{(p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} : f \in \mathcal{H}_p(\mathbb{R}^d),\quad \|f\|_{2p}^{2p} = M\right\}$$

$$I_1 = \mathcal{K}_{\mathrm{GNS}} \text{ and } I_M = I_1\,M^\gamma \text{ for any } M > 0$$

## Lemma

*If $d \geq 1$ and $p$ is an admissible exponent with $p < d/(d-2)$, then*

$$I_{M_1+M_2} < I_{M_1} + I_{M_2} \quad \forall\, M_1, M_2 > 0$$

## Lemma

*Let $d \geq 1$ and $p$ be an admissible exponent with $p < d/(d-2)$ if $d \geq 3$. If $(f_n)_n$ is minimizing and if $\limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^d} \int_{B(y)} |f_n|^{p+1}\,dx = 0$, then*

$$\lim_{n\to\infty} \|f_n\|_{2p} = 0$$

# Existence of a minimizer, further properties

**Proposition**

*Assume that $d \geq 1$ is an integer and let $p$ be an admissible exponent with $p < d/(d-2)$ if $d \geq 3$. Then there is an optimal function for* (GNS)

▷ *Pólya-Szegő principle* : there is a radial minimizer solving

$$-2\,(p-1)^2\,\Delta f + 4\,\big(d - p\,(d-2)\big)\,f^p - C\,f^{2\,p-1} = 0$$

If $f = \mathrm{g}$, then $C = 8\,p$

▷ *A rigidity result* : if $f$ is a minimizer and $\mathrm{P} = -\frac{p+1}{p-1}\,f^{1-p}$, then

$$\big(d - p\,(d-2)\big)\int_{\mathbb{R}^d} f^{p+1}\left|\Delta\mathrm{P} + (p+1)^2\,\frac{\int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}x}{\int_{\mathbb{R}^d} f^{p+1}\,\mathrm{d}x}\right|^2\mathrm{d}x + 2\,d\,p\int_{\mathbb{R}^d} f^{p+1}\left\|\mathrm{D}^2\mathrm{P} - \frac{1}{d}\,\Delta\mathrm{P}\,\mathrm{Id}\right\|^2\mathrm{d}x = 0$$

▷ *Weak stability result* : the minimizer is unique up to the invariances $f(x) = \mathrm{g}(x) = \big(1 + |x|^2\big)^{-1/(p-1)}$

# An abstract stability result

*Relative entropy*

$$\mathscr{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( f^{p+1} - g^{p+1} - \frac{1+p}{2p}\, g^{1-p} \left( f^{2p} - g^{2p} \right) \right) dx$$

*Deficit functional*

$$\delta[f] := a\, \|\nabla f\|_2^2 + b\, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}}\, \|f\|_{2p}^{2p\gamma} \ge 0$$

**Theorem**

*Let $d \ge 1$ and $p \in (1, p^*)$. There is a $\mathscr{C} > 0$ such that*

$$\delta[f] \ge \mathscr{C}\, \mathscr{F}[f]$$

*for any $f \in \mathscr{W} := \left\{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\, dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \right\}$ such that*

$$\int_{\mathbb{R}^d} f^{2p}(1,x)\, dx = \int_{\mathbb{R}^d} |g|^{2p}(1,x)\, dx$$

# A constructive result

The *relative entropy*

$$\mathscr{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( f^{p+1} - g^{p+1} - \frac{1+p}{2p}\, g^{1-p} \left( f^{2p} - g^{2p} \right) \right) dx$$

The *deficit functional*

$$\delta[f] := a\, \|\nabla f\|_2^2 + b\, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GN}\, \|f\|_{2p}^{2p\gamma} \ge 0$$

## Theorem

*Let $d \ge 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is a $\mathscr{C} > 0$ such that*

$$\delta[f] \ge \mathscr{C}\, \mathscr{F}[f]$$

*for any $f \in \mathscr{W} := \{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2 dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \}$ such that*

$$\int_{\mathbb{R}^d} f^{2p}\, dx = \int_{\mathbb{R}^d} |g|^{2p}\, dx\,, \quad \int_{\mathbb{R}^d} x\, f^{2p}\, dx = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} f^{2p}\, dx \le A \quad \text{and} \quad \mathscr{F}[f] \le G$$

# The fast diffusion equation and the entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy-entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)

# The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \ge 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{1}$$

with initial datum $u(t=0,x) = u_0(x) \ge 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \mathscr{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, \mathrm{d}x < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$, $\kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$ and $\mathscr{B}$ is the Barenblatt profile

$$\mathscr{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

# Entropy growth rate and Rényi entropy powers

With $p=\frac{1}{2m-1} \iff m=\frac{p+1}{2p}$, let us consider $f$ such that $u=f^{2p}$
$u^m=f^{p+1}$ and $u\,|\nabla^{m-1}u|^2=(p-1)^2\,|\nabla f|^2$

$$M=\|f\|_{2p}^{2p}\,,\quad \mathsf{E}[u]:=\int_{\mathbb{R}^d}u^m\,dx=\|f\|_{p+1}^{p+1}\quad\text{and}\quad \mathsf{I}[u]:=(p+1)^2\,\|\nabla f\|_2^2$$

By (GNS), if $u$ solves (1), then

$$\mathsf{E}'=\frac{p-1}{2p}\,\mathsf{I}=\frac{p-1}{2p}\,(p+1)^2\int_{\mathbb{R}^d}|\nabla f|^2\,dx$$
$$\geq\frac{p-1}{2p}\,(p+1)^2\left(\mathscr{C}_{\mathrm{GNS}(p)}\right)^{\frac{2}{\theta}}\,\|f\|_{2p}^{\frac{2}{\theta}}\,\|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}}\geq C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

with $C_0:=\frac{p-1}{2p}\,(p+1)^2\,\left(\mathscr{C}_{\mathrm{GNS}}(p)\right)^{\frac{2}{\theta}}\,M^{\frac{(d+2)\,m-d}{d\,(1-m)}}$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\geq\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\geq0$$

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx \quad \text{with} \quad \mathsf{P} = \frac{m}{m-1} \, u^{m-1} \text{ is the } \textit{pressure variable}$$

If $u$ solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m)\,\mathsf{I}$$

The *Rényi entropy power* $\mathsf{F} := \mathsf{E}^\sigma = \left(\int_{\mathbb{R}^d} u^m \, dx\right)^\sigma$ with $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$ applied to self-similar Barenblatt solutions has a linear growth in $t$

[Toscani, Savaré, 2014], [JD, Toscani, 2016]

# Nonlinear carré du champ method

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$

If $u$ is a smooth and rapidly decaying function on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$
$$= -2\int_{\mathbb{R}^d} u^m\,\Big\|\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id}\Big\|^2\,dx - 2\,(m-m_1)\int_{\mathbb{R}^d} u^m\,(\Delta \mathsf{P})^2\,dx$$

**Lemma**

*Let $d \ge 1$ and assume that $m \in (m_1, 1)$. If $u$ solves (1) with initial datum $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |x|^2\,u_0\,dx < +\infty$ and if, for any $t \ge 0$, $u(t,\cdot)$ is a smooth and rapidly decaying function on $\mathbb{R}^d$, then for any $t \ge 0$ we have*

$$-\frac{d}{dt}\log\left(\mathsf{I}^{\frac12}\,\mathsf{E}^{\frac{1-\theta}{\theta(p+1)}}\right) = \int_{\mathbb{R}^d} u^m\Big\|\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id}\Big\|^2 dx + (m-m_1)\int_{\mathbb{R}^d} u^m\Big|\Delta \mathsf{P} + \tfrac{\mathsf{I}}{\mathsf{E}}\Big|^2 dx$$

# Self-similar variables: entropy-entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\left(\tau,\frac{x}{\kappa\,R}\right)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau}+\nabla\cdot\left[v\left(\nabla u^{m-1}-2\,x\right)\right]=0 \tag{2}$$

*Generalized entropy (free energy)* and *Fisher information*

$$\mathscr{F}[v]:=-\frac{1}{m}\int_{\mathbb{R}^d}\left(v^m-\mathscr{B}^m-m\,\mathscr{B}^{m-1}\,(v-\mathscr{B})\right)\mathrm{d}x$$

$$\mathscr{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}+2\,x\right|^2\mathrm{d}x$$

are such that $\mathscr{I}[v]\geq 4\,\mathscr{F}[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathscr{F}[v(t,\cdot)]\leq\mathscr{F}[v_0]\,e^{-4\,t}$$

# Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \tag{H}$$

$$\mathscr{F}[v(t,\cdot)] \le C\, e^{-2\gamma(m)\,t} \quad \forall\, t \ge 0\,, \quad \gamma(m) := (1-m)\,\Lambda_{\alpha,d}$$

where $\Lambda_{\alpha,d} > 0$ is the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f \in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1} = 0$$

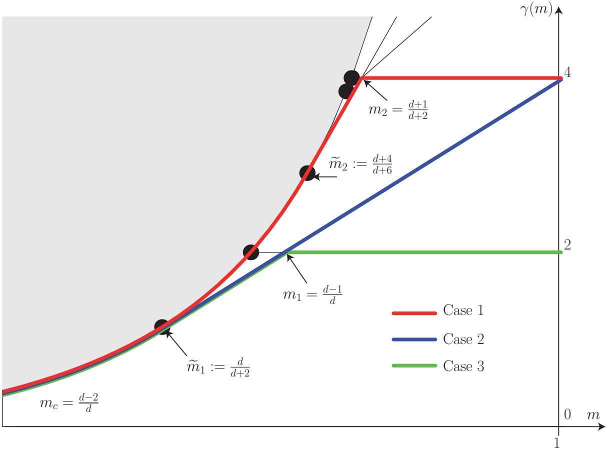
with $\alpha := \frac{1}{m-1} < 0$, $\mathrm{d}\mu_\alpha := h_\alpha\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

**Lemma (already a stability result)**

*Under assumption* (H), $\mathscr{I}[v] \ge (4+\eta)\,\mathscr{F}[v]$ *for some* $\eta \in (0, 2(\gamma(m)-2))$

Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

# Spectral gap and the asymptotic time layer




$$\mathscr{F}[v(t,\cdot)] \leq C\, e^{-2\gamma(m)\,t} \quad \forall\, t \geq 0$$

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

# The asymptotic time layer improvement

*Linearized free energy* and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,\mathrm{d}x \quad\text{and}\quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,\mathrm{d}x$$

*Hardy-Poincaré inequality*. Let $d\ge1$, $m\in(m_1,1)$ and $g\in\mathrm{L}^2(\mathbb{R}^d,\mathscr{B}^{2-m}\,dx)$ such that $\nabla g\in\mathrm{L}^2(\mathbb{R}^d,\mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,\mathrm{d}x=0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,\mathrm{d}x=0$

$$\mathsf{I}[g]\ge 4\,\alpha\,\mathsf{F}[g] \quad\text{where}\quad \alpha=2-d\,(1-m)$$

## Proposition

*Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/3,1)$ if $d=1$, $\eta=2\,d\,(m-m_1)$ and $\chi=m/(266+56\,m)$. If $\int_{\mathbb{R}^d} v\,\mathrm{d}x=\mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,\mathrm{d}x=0$ and*

$$(1-\varepsilon)\,\mathscr{B}\le v\le(1+\varepsilon)\,\mathscr{B}$$

*for some $\varepsilon\in(0,\chi\,\eta)$, then*

$$\mathscr{Q}[v] := \frac{\mathscr{I}[v]}{\mathscr{F}[v]}\ge 4+\eta$$

# The initial time layer improvement: backward estimate

Rephrasing the *carré du champ* method, $\mathscr{Q}[v] := \frac{\mathscr{I}[v]}{\mathscr{F}[v]}$ is such that

$$\frac{d\mathscr{Q}}{dt} \le \mathscr{Q}\,(\mathscr{Q} - 4)$$

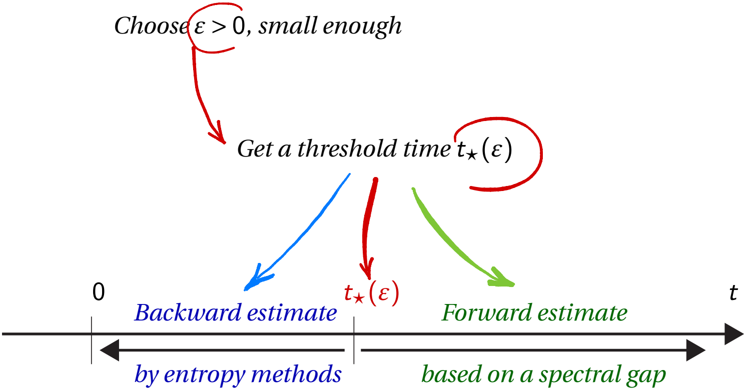
## Lemma

*Assume that* $m > m_1$ *and* $v$ *is a solution to* (2) *with nonnegative initial datum* $v_0$*. If for some* $\eta > 0$ *and* $T > 0$*, we have* $\mathscr{Q}[v(T,\cdot)] \ge 4 + \eta$*, then*

$$\mathscr{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,T}}{4 + \eta - \eta\,e^{-4\,T}} \quad \forall\, t \in [0, T]$$

# Regularity and stability

*Our strategy*




Initial time layer

Asymptotic time layer

# Uniform convergence in relative error

## Theorem

*Assume that* $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$ *and let* $\varepsilon \in (0, 1/2)$, *small enough,* $A > 0$, *and* $G > 0$ *be given. There exists an explicit time* $t_\star \geq 0$ *such that, if* $u$ *is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

*with nonnegative initial datum* $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *satisfying*

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u_0 \, dx \leq A < \infty \tag{$\mathrm{H}_A$}$$

$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathcal{B} \, \mathrm{d}x = \mathcal{M}$ *and* $\mathcal{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq t_\star$$

# The threshold time

## Proposition

*Let* $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/3,1)$ *if* $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ *and* $G>0$

$$t_\star=\mathsf{c}_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

*where* $\mathsf{a}=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$ *and* $\vartheta=\nu/(d+\nu)$

$$\mathsf{c}_\star=\mathsf{c}_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

# Improved entropy-entropy production inequality

**Theorem**

*Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. Then there is a positive number $\zeta$ such that*

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

*for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,\mathrm{d}x=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,\mathrm{d}x=0$ and $v$ satisfies $(H_A)$*

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\varepsilon_\star)\,,\quad \varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad T=\frac12\log R(t_\star)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge T$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,T]\,,\quad\text{where}\quad\zeta=\frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}}$$

# Two consequences

$$\zeta = \mathsf{Z}(A, \mathcal{F}[u_0])\,, \quad \mathsf{Z}(A, G) := \frac{\zeta_\star}{1 + A^{(1-m)\,\frac{2}{\alpha}} + G}\,, \quad \zeta_\star := \frac{4\,\eta}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}} c_\alpha$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

## Corollary

*Let* $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/2, 1)$ *if* $d = 1$, $A > 0$ *and* $G > 0$. *If* $v$ *is a solution of* (2) *with nonnegative initial datum* $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *such that* $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,\mathrm{d}x = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,\mathrm{d}x = 0$ *and* $v_0$ *satisfies* $(H_A)$, *then*

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \geq 0$$

▷ The stability in the entropy - entropy production estimate $\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

# A general stability result

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}\,,\quad \kappa[f] := \frac{\mathscr{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathrm{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2p}\,dx$$

$$\mathrm{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\,\mathrm{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,f^{2p} - \mathrm{g}^{2p}\right)\right)\mathrm{d}x$$

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathrm{Z}(\mathrm{A}[f],\mathrm{E}[f])$$

**Theorem**

*Let $d \ge 1$ and $p \in (1,p^*)$. For any $f \in \mathscr{W}$, we have*

$$\left(\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\right)^{2p\gamma} \ge \mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathrm{E}[f]$$

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