L^2 hypocoercivity, inequalities and applications

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Classical and Quantum Mechanical Models of Many-Particle Systems

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Outline

- **Diffusion rates and functional inequalities**
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  - ▷ Nash inequality
- L^2 **Hypocoercivity**
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  - ▷ Refined decay rates in the whole space
- **Decay and convergence rates for kinetic equations**
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  - ▷ Without confinement: Nash inequality
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Diffusion, rates and functional inequalities

Fokker-Planck equations and Poincaré inequalities

If $u \geq 0$ is a solution of the *Fokker-Planck equation*

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \, \nabla V) \quad \text{in} \quad \mathbb{R}^d$$

with initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ (of mass 1), if $\mu = e^{-V}$ is the density of a probability measure such that the *Poincaré inequality*

$$\boxed{\int_{\mathbb{R}^d} |u - \bar{u}|^2 \, d\mu \leq \frac{1}{\lambda} \int_{\mathbb{R}^d} |\nabla u|^2 \, d\mu \quad \forall \, u \in \mathcal{H}^1(\mathbb{R}^d, d\mu)}$$

then $u = u/\mu$ solves the Ornstein-Uhlenbeck equation

$$\frac{\partial u}{\partial t} = \Delta u - \nabla u \cdot \nabla V$$

and $\|u(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d, d\mu)} = \|u(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d, d\mu)} = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d, d\mu)} = \bar{u}$,

$$\frac{d}{dt} \|u(t,\cdot) - \bar{u}\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\mu)} = -\, 2 \, \|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\mu)} \leq -2 \, \lambda \, \|u(t,\cdot) - \bar{u}\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\mu)}$$

and $$\int_{\mathbb{R}^d} |u(t,\cdot) - \bar{u}|^2 \, d\mu \leq \int_{\mathbb{R}^d} |u_0 - \bar{u}|^2 \, d\mu \, e^{-2 \, \lambda \, t} \quad \forall \, t \geq 0$$

Proofs of Poincaré inequalities

▷ Compactness methods
▷ Direct computation of the spectral gap (*e.g.* when V is radial)
▷ Bakry-Emery or *carré du champ* method: prove

$$\frac{d}{dt}\|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\mu)} \le -2\,\lambda\,\|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\mu)}$$

▷ Equivalence between Fokker-Planck and Schrödinger spectral estimates: with $v = e^{V/2}\,u$, the Poincaré inequality is equivalent to

$$\int_{\mathbb{R}^d}\left(|\nabla v|^2 + W\,|v|^2\right)dx \ge \lambda\int_{\mathbb{R}^d}|v|^2\,dx$$

then use Persson's lemma

$$0 < \inf\sigma_{\mathrm{ess}}(-\Delta + W) = \lim_{R\to+\infty}\inf_{\mathrm{supp}(v)\subset B_R^c}\frac{\int_{\mathbb{R}^d}\left(|\nabla v|^2 + W\,|v|^2\right)dx}{\int_{\mathbb{R}^d}|v|^2\,dx}$$

▷ Constructive method: the IMS truncation method
▷ Lyapunov criterion

The decay rate of the heat equation

If u is a solution of the *heat equation*

$$\frac{\partial u}{\partial t} = \Delta u \quad \text{in} \quad \mathbb{R}^d$$

with initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$, then

$$\|u(t,\cdot)\|_{\mathrm{L}^1(\mathbb{R}^d,dx)} = \|u_0\|_{\mathrm{L}^1(\mathbb{R}^d,dx)}$$

$$\frac{d}{dt}\|u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)} = -\,2\,\|\nabla u(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d,dx)} \le -\,\mathcal{C}\,\|u(t,\cdot)\|^{2+\frac{4}{d}}_{\mathrm{L}^2(\mathbb{R}^d,dx)}$$

by Nash's inequality

$$\boxed{\|u\|_2^{2+\frac{4}{d}} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|_1^{\frac{4}{d}}\,\|\nabla u\|_2^2}$$

and so

$$\|u(t,\cdot)\|_{\mathrm{L}^2(\mathbb{R}^d,dx)} \le \mathcal{C}\,\|u_0\|_{\mathrm{L}^2(\mathbb{R}^d,dx)}\,(1+t)^{-d/2}$$

Proofs of Nash's inequality

▷ Nash's proof (Stein): use Fourier variables, optimize on $R > 0$

$$\|u\|_2^2 = \int_{\mathbb{R}^d} |\hat{u}(\xi)|^2 \, d\xi \le \underbrace{\int_{|\xi|<R} \|\hat{u}\|_\infty^2 \, d\xi}_{= \omega_d \, R^d \, \|u\|_1^2} + \underbrace{\frac{1}{R^2} \int_{\mathbb{R}^d} |\xi|^2 \, |\hat{u}(\xi)|^2 \, d\xi}_{= R^{-2} \|\nabla u\|_2^2}$$

▷ The optimal constant is given by the spectral gap of the Laplace operator on a ball with Neumann boundary conditions (Carlen, Loss 91)

▷ (Bouin, JD, Schmeiser) As a limit case of the Gagliardo-Nirenberg inequalities

$$\|u\|_2 \le \mathcal{C}_{\rm GN} \, \|\nabla u\|_2^\theta \, \|u\|_p^{1-\theta}$$

as $p \to 1_+$. Up to normalizations, optimal functions solve $-\Delta u = u - u^{p-1}$, are radial and have compact support if $p < 2$ (Pucci, Serrin, Zou 99) so that $v = u - 1$ solves $-\Delta v = v$ with Neumann boundary conditions on its support (a ball)

L^2 Hypocoercivity

▷ Abstract statement, diffusion limit

▷ Mode-by-mode analysis in Fourier variables

▷ Refined decay rates in the whole space

Collaboration with C. Mouhot and C. Schmeiser
+ E. Bouin, S. Mischler

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$
* denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$

Π is the orthogonal projection onto the null space of L

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle\mathsf{L}F, F\rangle \le -\,\lambda_m\,\|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially
$\Leftarrow$ *microscopic coercivity*

Formal macroscopic / diffusion limit

$F = F(t,x,v)$, $\mathsf{T} = v\cdot\nabla_x$, L good collision operator. Scaled evolution equation

$$\varepsilon\,\frac{dF}{dt} + \mathsf{T}F = \frac{1}{\varepsilon}\,\mathsf{L}F$$

on the Hilbert space $\mathcal{H}$. $F_\varepsilon = F_0 + \varepsilon\,F_1 + \varepsilon^2\,F_2 + \mathcal{O}(\varepsilon^3)$ as $\varepsilon \to 0_+$

$$\begin{aligned}
\varepsilon^{-1}: &\qquad \mathsf{L}F_0 = 0\,,\\
\varepsilon^{0}: &\qquad \mathsf{T}F_0 = \mathsf{L}F_1\,,\\
\varepsilon^{1}: &\qquad \tfrac{dF_0}{dt} + \mathsf{T}F_1 = \mathsf{L}F_2
\end{aligned}$$

The first equation reads as $u = F_0 = \Pi F_0$
The second equation is simply solved by $F_1 = -\,(\mathsf{T}\Pi)\,F_0$
After projection, the third equation is

$$\tfrac{d}{dt}\,(\Pi F_0) -\, \Pi\mathsf{T}\,(\mathsf{T}\Pi)\,F_0 = \Pi\mathsf{L}F_2 = 0$$

$$\partial_t u + (\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)\,u = 0$$

is such that $\frac{d}{dt}\|u\|^2 = -\,2\,\|(\mathsf{T}\Pi)\,u\|^2 \le -\,2\,\lambda_M\,\|u\|^2$
$\Leftarrow$ *Macroscopic coercivity*

The macro part and the Poincaré inequality

▷ Free transport operator: $\mathsf{T}F = v \cdot \nabla_x F$
If $F_0(x,v) = u(x)\,\mathcal{M}(v)$ with $\mathcal{M}(v) = (2\pi)^{-d/2}\,e^{-|v|^2/2}$ then

$$(\mathsf{T}\Pi)^*\,(\mathsf{T}\Pi)F_0 = (-\Delta_x u)\,\mathcal{M}$$

and we obtain the heat equation (*e.g.* on $\mathbb{T}^d$)

$$\partial_t u = \Delta\, u$$

▷ With an external potential V so that $\mathsf{T}F = v \cdot \nabla_x F - \nabla_x V \cdot \nabla_v F$
we obtain the Fokker-Planck equation

$$\partial_t u = \Delta\, u + \nabla \cdot (u\,\nabla V)$$

The operator $\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$ is such that

$$\langle \mathsf{AT}\Pi F, F\rangle \ge \frac{\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

if the Poincaré inequality $\int_{\mathbb{R}^d} |\nabla u|^2\,d\mu \ge \lambda_M \int_{\mathbb{R}^d} |u - \bar{u}|^2\,d\mu$ holds

The assumptions in the compact case

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi \mathsf{T} \Pi \, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{A}\mathsf{T}(1-\Pi)F\| + \|\mathsf{A}\mathsf{L}F\| \leq C_M \, \|(1-\Pi)F\| \tag{H4}$$

Equivalence and entropy decay

For some $\delta > 0$ to be chosen, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\tfrac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \tfrac{2+\delta}{4}\,\|F\|^2$$

Entropy decay: $\frac{d}{dt}\mathsf{H}[F] = -\,\mathsf{D}[F]$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\mathsf{D}[F] \ge \lambda\,\mathsf{H}[F]$$

Theorem

Under (H1)–(H4), *for any* $t \ge 0$,

$$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda\,t}$$

$$\|F(t,\cdot)\|^2 \le \mathcal{C}\,\|F_0\|^2\,e^{-\lambda\,t} \quad \textit{with} \quad \mathcal{C} = \tfrac{2+\delta}{2-\delta}$$

Basic examples

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad f(0,x,v) = f_0(x,v)$$

L is the *Fokker-Planck operator* L_1 or the *linear BGK operator* L_2

$$\mathsf{L}_1 f := \Delta_v f + \nabla_v \cdot (v\,f) \quad \text{and} \quad \mathsf{L}_2 f := \rho_f\,\mathcal{M} - f$$

$\mathcal{M}(v) = \frac{e^{-\frac{1}{2}\,|v|^2}}{(2\,\pi)^{d/2}}$ is the normalized Gaussian function

$\rho_f := \int_{\mathbb{R}^d} f\,dv$ is the spatial density

$$d\gamma := \gamma(v)\,dv \quad \text{where} \quad \gamma := \frac{1}{\mathcal{M}}$$

$$\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} := \iint_{\mathcal{X}\times\mathbb{R}^d} |f(x,v)|^2\,dx\,d\gamma$$

where either $\mathcal{X} = \mathbb{R}^d$ or $\mathcal{X} = \mathbb{T}^d$

Fourier variables: mode-by-mode hypocoercivity

Let us consider the Fourier transform in x, denote by $\xi \in \mathbb{R}^d$ the Fourier variable, so that $F = \hat{f}$ solves

$$\partial_t F + \mathsf{T} F = \mathsf{L} F\,, \quad F(0,\xi,v) = \hat{f}_0(\xi,v)\,, \quad \mathsf{T} F = i\,(v\cdot\xi) F$$

Goal: apply the abstract method with ξ considered as a parameter

$$\mathcal{H} = \mathrm{L}^2\,(d\gamma)\;, \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2\,d\gamma\,, \quad \Pi F = \mathcal{M}\int_{\mathbb{R}^d} F\,dv = \mathcal{M}\,\rho_F$$

The operator A is now defined as

$$(\mathsf{A}F)(v) = \frac{-\,i\,\xi}{1+|\xi|^2}\cdot\int_{\mathbb{R}^d} w\,F(w)\,dw\,\mathcal{M}(v)$$

and, with $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$, we have that

$$|\mathrm{Re}\langle \mathsf{A}F,F\rangle| \le \frac{|\xi|}{1+|\xi|^2}\,X\,Y\,, \quad \|F\|^2 = X^2+Y^2$$

$$\frac12\left(1-\frac{\delta\,|\xi|}{1+|\xi|^2}\right)(X^2+Y^2) \le \mathsf{H}[F] \le \frac12\left(1+\frac{\delta\,|\xi|}{1+|\xi|^2}\right)(X^2+Y^2)$$

Entropy production

$$-\langle \mathsf{L}F,F\rangle+\delta\,\langle \mathsf{AT\Pi}F,F\rangle\ge X^2+\frac{\delta\,|\xi|^2}{1+|\xi|^2}\,Y^2$$

$$\begin{aligned}\mathsf{D}[F]&=-\langle \mathsf{L}F,F\rangle+\delta\,\langle \mathsf{AT\Pi}F,F\rangle+\delta\,(\ldots)\\&\ge(\lambda_m-\,\delta)\,X^2+\frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2-\,\delta\,C_M\,X\,Y\end{aligned}$$

with $\lambda_m=1\,,\quad \Lambda_M=|\xi|^2=:s^2\,,\quad C_M=\dfrac{s\,(1+\sqrt{3}\,s)}{1+s^2}$

$\mathsf{D}[F]-\lambda\,\mathsf{H}[F]$

$$\ge\left(1-\tfrac{\delta\,s^2}{1+s^2}-\tfrac{\lambda}{2}\right)X^2-\tfrac{\delta\,s}{1+s^2}\left(1+\sqrt{3}\,s+\lambda\right)X\,Y+\left(\tfrac{\delta\,s^2}{1+s^2}-\tfrac{\lambda}{2}\right)Y^2$$

is (for any $s=|\xi|>0$) a nonnegative quadratic form of X and Y iff...

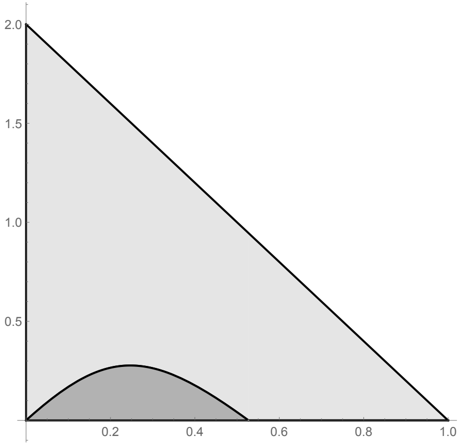


Figure: Horizontal axis: δ, vertical axis: λ. Admissible region: grey triangle. Negative discriminant: dark grey area, shown here for $s=5$

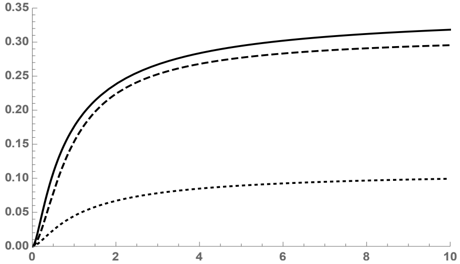
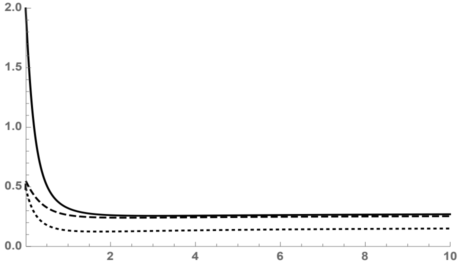


Figure: $s \mapsto \lambda(s)$



Comments

▷ Not much originality so far, *cf.* (JD, Mouhot, Schmeiser) and (Bouin, JD, Mischler, Mouhot, Schmeiser)
▷ Last curves are part of a joint work (Arnold, JD, Schmeiser, Wöhrer) in progress intended to compare L^2 hypocoercivity methods with a ***twist*** induced by the analysis of the diffusion limit, *i.e.*, given by $\delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$, and results based on *Lyapunov matrix inequalities:* *cf.* Anton's lecture of yesterday
▷ The methods are very close with the Lyapunov matrix inequality based on the deformation matrix P, the *twisted Euclidean norm* $|F\|_P^2 := \langle F, P\,F\rangle$ and the computation

$$\frac{d}{dt}\|F\|_P^2 = -\,\langle F, (C^*P + PC)F\rangle \le -\,2\,\mu\,\|F\|_P^2$$

Estimate even coincide in some cases (Goldstein-Taylor model)
▷ Orders of magnitude are ok (estimate of the rate λ)
▷ Estimates are compatible with diffusion limits and optimal in some asymptotic regimes

Decay in the whole space

If $s \mapsto \lambda(s)$ is a positive non-decreasing bounded function on $\mathbb{R}^+$, let

$$h_\lambda(M,R,s) := \lambda(R)\left(\omega_d\,R^d\,M^2 - s\right)\,,\quad \lambda^*(M,s) := -\min_{R>0} h_\lambda(M,R,s)$$

$$\psi_{\lambda,M}(s) := -\int_1^s \frac{dz}{\lambda^*(M,z)}$$

Lemma

If $\lim_{s\to0_+}\psi_{\lambda,\mu}(s) = +\infty$ *and if* $u \in C(\mathbb{R}^+, \mathrm{L}^1\cap\mathrm{L}^2(dx))$ *is such that*

$$\|u(t,\cdot)\|_{\mathrm{L}^1(dx)} \le M\,,\quad |\hat u(t,\xi)|^2 \le |\hat u(0,\xi)|^2\,e^{-2\,\lambda(|\xi|)\,t}\quad \forall\,(t,\xi)$$

then

$$\|u(t,\cdot)\|^2_{\mathrm{L}^2(dx)} \le \psi^{-1}_{\lambda,M}\left(2\,t + \psi_{\lambda,M}\left(\|u(0,\cdot)\|^2_{\mathrm{L}^2(dx)}\right)\right)\quad \forall\,t\in\mathbb{R}^+$$

Lemma

Under the previous assumptions, if for some bounded continuous function C with $C(s) \ge 1$ for any $s > 0$,

$$|\hat{u}(t,\xi)|^2 \le C(|\xi|)\,|\hat{u}(0,\xi)|^2\,e^{-\,2\,\lambda(|\xi|)\,t} \quad \forall\,(t,\xi)$$

then

$$\|u(t,\cdot)\|^2_{\mathrm{L}^2(dx)} \le \Psi_{M,Q}(t)$$

where $M := \|u_0\|_{\mathrm{L}^1(dx)}$, $Q := \|u(0,\cdot)\|_{\mathrm{L}^2(dx)}$ *and* $\Psi_{M,Q}(t)$*is defined as*

$$\inf_{R>0}\left(\int_0^R C(s)\,e^{-\,2\,\lambda(s)\,t}\,s^{d-1}\,ds\,d\,\omega_d\,M^2+\sup_{s\ge R} C(s)\,e^{-\,2\,\lambda(R)\,t}\,Q^2\right)$$

Theorem

Assume that $f_0 \in \mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d, dx\, d\gamma) \cap \mathrm{L}^2\big(\mathbb{R}^d, d\gamma; \mathrm{L}^1(\mathbb{R}^d, dx)\big)$ *and* $\mathsf{L} = \mathsf{L}_1$ *or* $\mathsf{L} = \mathsf{L}_2$*, then we have the estimate*

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d, dx\, d\gamma)} \le (2\,\pi)^{-d}\, \Psi_{M,Q}(t)$$

using $C(s) = \frac{2+\delta(s)}{2-\delta(s)}$
$M = \|f_0\|_{\mathrm{L}^2\left(\mathbb{R}^d, d\gamma; \mathrm{L}^1(\mathbb{R}^d, dx)\right)}$ *and* $Q = \|f_0\|_{\mathrm{L}^2(\mathbb{R}^d \times \mathbb{R}^d, dx\, d\gamma)}$

Here λ and δ are estimates arising from the hypocoercivity method and they must satisfy some conditions (which are fulfilled in the two examples)

Decay and convergence rates for kinetic equations

What can we do when at least one of the coercivity conditions is missing ? microscopic coercivity (H1) *or macroscopic coercivity* (H2)

In collaboration with Emeric Bouin, Stéphane Mischler, Clément Mouhot, Christian Schmeiser + Laurent Lafleche

The global picture: from diffusive to kinetic

- Depending on the local equilibria and on the external potential (H1) and (H2) (which are Poincaré type inequalities) can be replaced by other functional inequalities:

▷ *microscopic coercivity* (H1)

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \, \|(1-\Pi)F\|^2$$

$\Longrightarrow$ ***weak Poincaré inequalities*** or ***Hardy-Poincaré inequalities***

▷ *macroscopic coercivity* (H2)

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \, \|\Pi F\|^2$$

$\Longrightarrow$ ***Nash inequality***, ***weighted Nash*** or ***Caffarelli-Kohn-Nirenberg inequalities***

- This can be done at the level of the *diffusion equation* (homogeneous case) or at the level of the *kinetic equation* (non-homogeneous case)

Diffusion (Fokker-Planck) equations

Potential	$V = 0$	$V(x) = \gamma \log \lvert x \rvert$ $\gamma < d$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 1: $\partial_t u = \Delta u + \nabla \cdot (u \nabla V)$

Kinetic Fokker-Planck equations

B = Bouin, L = Lafleche, M = Mouhot, MM = Mischler, Mouhot
S = Schmeiser

Potential	$V = 0$	$V(x) = \gamma \log \lvert x\rvert$ $\gamma < d$	$V(x) = \lvert x\rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x\rvert^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v\rangle^\beta}$, $\beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v\rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta =$ $\min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v\rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 1: $\partial_t f + v\cdot\nabla_x f = F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v\rangle = \sqrt{1+|v|^2}$

A result based on Nash's inequality

$$\boxed{\partial_t f + v \cdot \nabla_x f = \mathsf{L} f\,, \quad (t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d}$$

$$\mathsf{D}[f] = -\frac{d}{dt}\mathsf{H}[f] \ge \mathsf{a}\left(\|(1-\Pi)f\|^2 + 2\,\langle \mathsf{AT\Pi}f, f\rangle\right)$$

We observe that

$$\mathsf{A}^* f = \mathsf{T\Pi}\left(1 + (\mathsf{T\Pi})^*\mathsf{T\Pi}\right)^{-1} f$$
$$= \mathsf{T}\left(1 + (\mathsf{T\Pi})^*\mathsf{T\Pi}\right)^{-1} \Pi f = M\,\mathsf{T}\,u_f = v\,M \cdot \nabla_x u_f$$

if u_f is the solution in $\mathrm{H}^1(\mathbb{R}^d)$ of $u_f - \Theta\,\Delta u_f = \rho_f$, and

$$\|u_f(t,\cdot)\|_{\mathrm{L}^1(dx)} = \|\rho_f(t,\cdot)\|_{\mathrm{L}^1(dx)} = \|f_0\|_{\mathrm{L}^1(dx\,dv)}$$

$$\|u_f\|^2_{\mathrm{L}^2(dx)} \le \|\rho_f\|^2_{\mathrm{L}^2(dx)}\,, \quad \|\nabla_x u_f\|^2_{\mathrm{L}^2(dx)} \le \frac{1}{\Theta}\,\langle \mathsf{AT\Pi}f, f\rangle$$

$$\|\rho_f\|^2_{\mathrm{L}^2(dx)} = \|\Pi f\|^2 \le \|u_f\|^2_{\mathrm{L}^2(dx)} + 2\,\langle \mathsf{AT\Pi}f, f\rangle$$

Nash's inequality

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}} \, \|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)} \, \|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)} \quad \forall\, u \in \mathrm{L}^1 \cap \mathrm{H}^1(\mathbb{R}^d)$$

Use $\|\Pi f\|^2 \le \Phi^{-1}\big(2\,\langle \mathsf{AT\Pi} f, f\rangle\big)$ with $\Phi^{-1}(y) := y + \big(\frac{y}{\mathsf{c}}\big)^{\frac{d}{d+2}}$ to get

$$\|(1-\Pi) f\|^2 + 2\,\langle \mathsf{AT\Pi} f, f\rangle \ge \Phi(\|f\|^2) \ge \Phi\big(\tfrac{2}{1+\delta}\,\mathsf{H}[f]\big)$$

$$\mathsf{D}[f(t,\cdot)] = -\frac{d}{dt}\mathsf{H}[f(t,\cdot)] \ge \mathsf{a}\,\Phi\big(\tfrac{2}{1+\delta}\,\mathsf{H}[f(t,\cdot)]\big)$$

As $s \to 0_+$, $\Phi(s) \sim s^{1+\frac{d}{2}}$ + Grönwall: $\mathsf{H}[f(t,\cdot)] \sim t^{-d/2}$ as $t \to +\infty$

$$\mathsf{H}[f] := \tfrac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \delta\,\langle \mathsf{A} f, f\rangle_{dx\,d\gamma}$$

Theorem

There exists a constant $C > 0$ such that, for any $t \ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

Very weak confinement: Caffarelli-Kohn-Nirenberg

$$\frac{\partial u}{\partial t} = \Delta_x u + \nabla_x \cdot (\nabla_x V\, u) = \nabla_x \left(e^{-V}\, \nabla_x \left(e^V\, u\right)\right)$$

Here $x \in \mathbb{R}^d$, $d \geq 3$, and V is a potential such that $e^{-V} \notin \mathrm{L}^1(\mathbb{R}^d)$ corresponding to a *very weak confinement*

Two examples

$$V_1(x) = \gamma\, \log|x| \quad \text{and} \quad V_2(x) = \gamma\, \log\langle x\rangle$$

with $\gamma < d$ and $\langle x\rangle := \sqrt{1+|x|^2}$ for any $x \in \mathbb{R}^d$

In collaboration with Emeric Bouin and Christian Schmeiser

Potential	$V=0$	$V(x)=\gamma\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\geq 1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 2: $\partial_t u = \Delta u + \nabla\cdot(u\,\nabla V)$

Actually, this is more complicated, because the rate depends on the functional space (and of the range of the parameters)...

Using moments

Theorem

Let $d \geq 1$, $0 < \gamma < d$, $V = V_1$ *or* $V = V_2$, *and* $u_0 \in \mathrm{L}^1_+ \cap \mathrm{L}^2(e^V)$ *with* $\||x|^k u_0\|_1 < \infty$ *for some* $k \geq \max\{2, \gamma/2\}$

$$\forall\, t \geq 0\,, \quad \|u(t,\cdot)\|^2_{\mathrm{L}^2(e^V\,dx)} \leq \|u_0\|^2_{\mathrm{L}^2(e^V\,dx)}\,(1 + c\,t)^{-\frac{d-\gamma}{2}}$$

for some c *depending on* d, γ, k, $\|u_0\|_{\mathrm{L}^2(e^V\,dx)}$, $\|u_0\|_1$, *and* $\||x|^k u_0\|_1$

Proof

Growth of the moment

$$M_k(t) := \int_{\mathbb{R}^d} |x|^k u \, dx$$

From the equation

$$M_k' = k\,(d+k-2-\gamma)\int_{\mathbb{R}^d} u\,|x|^{k-2}\,dx \le k\,(d+k-2-\gamma)\,M_0^{\frac{2}{k}}\;M_k^{1-\frac{2}{k}}$$

then use the *Caffarelli-Kohn-Nirenberg inequality*

$$\boxed{\int_{\mathbb{R}^d} |x|^\gamma\,u^2\,dx \le \mathcal{C}\left(\int_{\mathbb{R}^d} |x|^{-\gamma}\;|\nabla\left(|x|^\gamma u\right)|^2\,dx\right)^a \left(\int_{\mathbb{R}^d} |x|^k\,|u|\,dx\right)^{2(1-a)}}$$

Kinetic Fokker-Planck equation, very weak confinement

Let us consider the kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

where $\mathsf{L} f$ is one of the two following collision operators

(a) a Fokker-Planck operator

$$\mathsf{L} f = \nabla_v \cdot \Big(F \, \nabla_v \left(F^{-1} \, f \right) \Big)$$

(b) a scattering collision operator

$$\mathsf{L} f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \left(f(v') \, F(\cdot) - f(\cdot) \, F(v') \right) dv'$$

$$\boxed{V(x) \sim \gamma \log |x| \,, \quad \gamma \in (0, d)}$$

Potential	$V = 0$	$V(x) = \gamma \log \lvert x \rvert$ $\gamma < d$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \in (0,1)$	$V(x) = \lvert x \rvert^\alpha$ $\alpha \geq 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v) = e^{-\langle v \rangle^\beta}$, $\beta \geq 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b < 1$, $\beta = 2$ convergence	DMS, Mischler-Mouhot $e^{-\lambda t}$ convergence
$F(v) = e^{-\langle v \rangle^\beta}$, $\beta \in (0,1)$	BDLS: $t^{-\zeta}$, $\zeta = \min\left\{\frac{d}{2}, \frac{k}{\beta}\right\}$ decay			
$F(v) = \langle v \rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 2: $\partial_t f + v \cdot \nabla_x f = F\, \nabla_v \big(F^{-1}\, \nabla_v f\big)$. Notation: $\langle v \rangle = \sqrt{1 + |v|^2}$

Decay rates

$$\forall\,(x,v)\in\mathbb{R}^d\times\mathbb{R}^d\,,\quad \mathcal{F}(x,v)=M(v)\,e^{-V(x)}\,,\quad M(v)=(2\pi)^{-\frac{d}{2}}\,e^{-\frac{1}{2}\,|v|^2}$$

$$\textbf{(H1)}\quad 1\le\sigma(v,v')\le\overline{\sigma}\,,\quad \forall\,v\,,\,v'\in\mathbb{R}^d\,,\quad \text{for some}\quad \overline{\sigma}\ge 1$$

$$\textbf{(H2)}\quad \int_{\mathbb{R}^d}\big(\sigma(v,v')-\sigma(v',v)\big)\,M(v')\,dv'=0\quad \forall\,v\in\mathbb{R}^d$$

+ *Caffarelli-Kohn-Nirenberg inequalities*

Theorem

Let $d\ge 1$, $V=V_2$ *with* $\gamma\in[0,d)$, $k>\max\{2,\gamma/2\}$ *and* $f_0\in\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)$ *such that*

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}\langle x\rangle^k\,f_0\,dx\,dv+\iint_{\mathbb{R}^d\times\mathbb{R}^d}|v|^k\,f_0\,dx\,dv<+\infty$$

If (H1)–(H2) *hold, then there exists* $C>0$ *such that*

$$\forall\,t\ge 0\,,\quad \|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(\mathcal{M}^{-1}dx\,dv)}\le C\,(1+t)^{-\frac{d-\gamma}{2}}$$

Sub-exponential equilibria

▷ We consider the ***homogeneous Fokker-Planck equation***

$$\partial_t g = \nabla_v \cdot \Big(\mathcal{M}\,\nabla_v\big(\mathcal{M}^{-1}\,g\big)\Big)$$

associated with ***sub-exponential equilibria***

$$\mathcal{M}(v) = C_\alpha\,e^{-\langle v\rangle^\alpha}\,,\quad \alpha \in (0,1)$$

or the corresponding Ornstein-Uhlenbeck equation for $h = g/\mathcal{M}$
– decay rates based on the weak Poincaré inequality (Kavian, Mischler)
– decay rates based on a weighted Poincaré / Hardy-Poincaré inequality

In collaboration with Emeric Bouin, Laurent Lafleche and Christian Schmeiser

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge1$
Inequality	Nash	Caffarelli-Kohn -Nirenberg	Weak Poincaré or Weighted Poincaré	Poincaré
Asymptotic behavior	$t^{-d/2}$ decay	$t^{-(d-\gamma)/2}$ decay	$t^{-\mu}$ or $t^{-\frac{k}{2\,(1-\alpha)}}$ convergence	$e^{-\lambda t}$ convergence

Table 3: $\partial_t u=\Delta u+\nabla\cdot(u\,\nabla V)$

Weak Poincaré inequality

$$\int_{\mathbb{R}^d} \left|h-\widetilde{h}\right|^2 d\xi \le \mathcal{C}_{\alpha,\tau} \left(\int_{\mathbb{R}^d} |\nabla h|^2\, d\xi\right)^{\frac{\tau}{1+\tau}} \left\|h-\widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^{\frac{2}{1+\tau}}$$

for some explicit positive constant $\mathcal{C}_{\alpha,\tau}$, $\widetilde{h} := \int_{\mathbb{R}^d} h\, d\xi$. Using

$$\frac{d}{dt}\int_{\mathbb{R}^d} \left|h(t,\cdot)-\widetilde{h}\right|^2 d\xi = -\,2\int_{\mathbb{R}^d} |\nabla_v h|^2\, d\xi$$

where $h = g/\mathcal{M}$ and $d\xi = \mathcal{M}\, dv$ + Hölder's inequality

$$\int_{\mathbb{R}^d} \left|h-\widetilde{h}\right|^2 d\xi \le \left(\int_{\mathbb{R}^d} \left|h-\widetilde{h}\right|^2 \langle v\rangle^{-\beta}\, d\xi\right)^{\frac{\tau}{\tau+1}} \left(\int_{\mathbb{R}^d} \left\|h-\widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^2 \langle v\rangle^{\beta\,\tau}\, d\xi\right)^{\frac{1}{1+\tau}}$$

with $(\tau+1)/\tau = \beta/\eta$, then for with $\mathcal{M} = \sup_{s\in(0,t)} \left\|h(s,\cdot)-\widetilde{h}\right\|_{L^\infty(\mathbb{R}^d)}^{2/\tau}$

$$\int_{\mathbb{R}^d} \left|h(t,\cdot)-\widetilde{h}\right|^2 d\xi \le \left(\left(\int_{\mathbb{R}^d} \left|h(0,\cdot)-\widetilde{h}\right|^2 d\xi\right)^{-\frac{1}{\tau}} + \frac{2\,\tau^{-1}}{\mathcal{C}_{\alpha,\tau}^{1+1/\tau}\,\mathcal{M}}\, t\right)^{-\tau}$$

Weighted Poincaré inequality

There exists a constant $\mathcal{C} > 0$ such that

$$\boxed{\int_{\mathbb{R}^d} |\nabla h|^2 \, F \, dv \geq \mathcal{C} \int_{\mathbb{R}^d} \left| h - \widetilde{h} \right|^2 \langle v \rangle^{-\beta} \, F \, dv}$$

with $\beta = 2\,(1-\alpha)$, $\widetilde{h} := \int_{\mathbb{R}^d} h \, F \, dv$ and $F(v) = C_\alpha \, e^{-\langle v \rangle^\alpha}$ and $\alpha \in (0,1)$

Written in terms of $g = h \, \mathcal{M}$, the inequality becomes

$$\boxed{\int_{\mathbb{R}^d} \left| \nabla_v \big(\mathcal{M}^{-1} \, g \big) \right|^2 \mathcal{M}^2 \, d\mu \geq \mathcal{C} \int_{\mathbb{R}^d} |g - \overline{g}|^2 \, \langle v \rangle^{-2\,(1-\alpha)} \, d\mu}$$

where $d\mu = \mathcal{M} \, dv$ and $\overline{g} := \left(\int_{\mathbb{R}^d} g \, dv \right) \mathcal{M}$

$$\frac{d}{dt}\int_{\mathbb{R}^d}|h(t,v)|^2\,\langle v\rangle^k\,\mathcal{M}\,\mathrm{d}v+\,2\int_{\mathbb{R}^d}|\nabla_v h|^2\,\langle v\rangle^k\,\mathcal{M}\,\mathrm{d}v$$

$$=-\int_{\mathbb{R}^d}\nabla_v(h^2)\cdot\left(\nabla_v\langle v\rangle^k\right)\mathcal{M}\,\mathrm{d}v$$

With $\ell=2-\alpha$, $a\in\mathbb{R}$, $b\in(0,+\infty)$

$$\nabla_v\cdot\left(\mathcal{M}\,\nabla_v\langle v\rangle^k\right)=\frac{k}{\langle v\rangle^4}\left(d+(k+d-2)\,|v|^2-\alpha\,\langle v\rangle^\alpha\,|v|^2\right)\le a-b\,\langle v\rangle^{-\ell}$$

Proposition (Weighted L^2 norm)

There exists a constant $\mathcal{K}_k>0$ such that, if h solves the Ornstein-Uhlenbeck equation, then

$$\forall\,t\ge0\qquad\|h(t,\cdot)\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}\xi)}\le\mathcal{K}_k\,\left\|h^{\mathrm{in}}\right\|_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}\xi)}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\left|h(t,\cdot)-\widetilde{h}\right|^2 \mathrm{d}\xi = -\,2\int_{\mathbb{R}^d}|\nabla_v h|^2\,\mathrm{d}\xi \le -\,2\,\mathcal{C}\int_{\mathbb{R}^d}\left|h-\widetilde{h}\right|^2\,\langle v\rangle^{-\beta}\,\mathrm{d}\xi$$

$+$ Hölder

Theorem

Assume that $\alpha\in(0,1)$. *Let* $g^{\rm in}\in\mathrm{L}^1_+(\mathrm{d}\mu)\cap\mathrm{L}^2(\langle v\rangle^k\mathrm{d}\mu)$ *for some* $k>0$ *and consider the solution* g *to the homogeneous Fokker-Planck equation with initial datum* $g^{\rm in}$. *If* $\overline{g}=\left(\int_{\mathbb{R}^d} g\,\mathrm{d}v\right)\mathcal{M}$, *then*

$$\int_{\mathbb{R}^d}|g(t,\cdot)-\overline{g}|^2\,\mathrm{d}\mu \le \left(\left(\int_{\mathbb{R}^d}\left|g^{\rm in}-\overline{g}\right|^2\mathrm{d}\mu\right)^{-\beta/k}+\frac{2\,\beta\,\mathcal{C}}{k\,\mathcal{K}^{\beta/k}}\,t\right)^{-k/\beta}$$

with $\beta=2\,(1-\alpha)$ *and* $\mathcal{K}:=\mathcal{K}_k^2\,\left\|g^{\rm in}\right\|^2_{\mathrm{L}^2(\langle v\rangle^k\,\mathrm{d}\mu)}+\Theta_k\left(\int_{\mathbb{R}^d}g^{\rm in}\,\mathrm{d}v\right)^2$

Kinetic Fokker-Planck, no confinement and sub-exponential local equilibria

- the *Fokker-Planck* operator

$$\mathsf{L}_1 f = \nabla_v \cdot \Big(\mathcal{M} \, \nabla_v \big(\mathcal{M}^{-1} \, f \big) \Big)$$

- the *scattering* collision operator

$$\mathsf{L}_2 f = \int_{\mathbb{R}^d} \sigma(\cdot, v') \Big(f(v') \, \mathcal{M}(\cdot) - f(\cdot) \, \mathcal{M}(v') \Big) \, \mathrm{d}v'$$

under assumptions (H1)–(H2)

$$V = 0 \qquad \boxed{F(v) = e^{-\langle v \rangle^\beta} \quad \beta \in (0,1)}$$

Potential	$V=0$	$V(x)=\gamma\,\log\lvert x\rvert$ $\gamma<d$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\in(0,1)$	$V(x)=\lvert x\rvert^\alpha$ $\alpha\ge 1$, or $\mathbb{T}^d$ Macro Poincaré
Micro Poincaré $F(v)=e^{-\langle v\rangle^\beta}$, $\beta\ge 1$	BDMMS: $t^{-d/2}$ decay	BDS: $t^{-(d-\gamma)/2}$ decay	Cao: e^{-t^b}, $b<1$, $\beta=2$ convergence	DMS, Mischler- Mouhot $e^{-\lambda t}$ convergence
$F(v)=e^{-\langle v\rangle^\beta}$, $\beta\in(0,1)$	BDLS: $t^{-\zeta}$, $\zeta=$ $\min\left\{\frac{d}{2},\frac{k}{\beta}\right\}$ decay			
$F(v)=\langle v\rangle^{-d-\beta}$	BDLS, fractional in progress			

Table 3: $\partial_t f+v\cdot\nabla_x f=F\,\nabla_v\big(F^{-1}\,\nabla_v f\big)$. Notation: $\langle v\rangle=\sqrt{1+\lvert v\rvert^2}$

The decay rate with sub-exponential local equilibria

Theorem

Let $\alpha\in(0,1)$, $\beta>0$, $k>0$ and let $\mathcal{M}(v)=C_\alpha\,e^{-\langle v\rangle^\alpha}$. Assume that either $\mathsf{L}=\mathsf{L}_1$ and $\beta=2\,(1-\alpha)$, or $\mathsf{L}=\mathsf{L}_2$ + Assumptions. There exists a numerical constant $\mathcal{C}>0$ such that any solution f of

$$\partial_t f+v\cdot\nabla_x f=\mathsf{L} f\,,\quad f(0,\cdot,\cdot)=f^{\rm in}\in\mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu)\cap\mathrm{L}^1_+(\mathrm{d}x\,\mathrm{d}v)$$

satisfies

$$\forall\,t\ge0\,,\quad\|f(t,\cdot,\cdot)\|^2=\iint_{\mathbb{R}^d\times\mathbb{R}^d}\big|f(t,x,v)\big|^2\,\mathrm{d}x\,\mathrm{d}\mu\le\mathcal{C}\,\frac{\big\|f^{\rm in}\big\|^2}{(1+\kappa\,t)^\zeta}$$

with rate $\zeta=\min\{d/2,k/\beta\}$, for some positive κ which is an explicit function of the two quotients, $\big\|f^{\rm in}\big\|\,/\,\big\|f^{\rm in}\big\|_k$ and $\big\|f^{\rm in}\big\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}\,/\,\big\|f^{\rm in}\big\|$

Proof (1/2)

$$\begin{aligned}\mathsf{D}[f] := & -\langle \mathsf{L} f, f\rangle + \delta\,\langle \mathsf{AT\Pi} f, \mathsf{\Pi} f\rangle \\ & + \delta\,\langle \mathsf{AT}(\mathrm{Id}-\mathsf{\Pi}) f, \mathsf{\Pi} f\rangle - \delta\,\langle \mathsf{TA}(\mathrm{Id}-\mathsf{\Pi}) f, (\mathrm{Id}-\mathsf{\Pi}) f\rangle \\ & - \delta\,\langle \mathsf{AL}(\mathrm{Id}-\mathsf{\Pi}) f, \mathsf{\Pi} f\rangle\end{aligned}$$

- ***microscopic coercivity***. If $\mathsf{L}=\mathsf{L}_1$, we rely on the ***weighted Poincaré inequality***

$$\boxed{\langle \mathsf{L} f, f\rangle \le -\,\mathcal{C}\,\|(\mathrm{Id}-\Pi) f\|_{-\beta}^2}$$

If $\mathsf{L}=\mathsf{L}_2$, we assume that there exists a constant $\mathcal{C}>0$ such that

$$\int_{\mathbb{R}^d}\left|h-\widetilde{h}\right|^2\langle v\rangle^{-\beta}\,\mathcal{M}\,dv \le \mathcal{C}\iint_{\mathbb{R}^d\times\mathbb{R}^d}\sigma(v,v')\left|h'-h\right|^2\mathcal{M}\,\mathcal{M}'\,dv\,dv'$$

- ***Weighted*** L^2 ***norms***. Let $k>0$, $f^{\rm in}\in\mathrm{L}^2(\langle v\rangle^k\,dx\,d\mu)$ a solution $\exists\,\mathcal{K}_k>1$ such that

$$\forall\,t\ge 0\,,\quad \|f(t,\cdot,\cdot)\|_{\mathrm{L}^2(\langle v\rangle^k\,dx\,d\mu)}\le\mathcal{K}_k\,\left\|f^{\rm in}\right\|_{\mathrm{L}^2(\langle v\rangle^k\,dx\,d\mu)}$$

Proof (2/2)

$$\mathsf{H}_\delta[f] := \frac{1}{2}\,\|f\|^2 + \delta\,\langle \mathsf{A}f, f\rangle\,, \quad \frac{d}{dt}\mathsf{H}_\delta[f] = -\,\mathsf{D}[f]$$

- There exists $\kappa > 0$ such that $\forall\, f \in \mathrm{L}^2\left(\langle v\rangle^{-\beta}\,\mathrm{d}x\,\mathrm{d}\mu\right) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\mathsf{D}[f] \geq \kappa\left(\|(\mathrm{Id}-\Pi)f\|_{-\beta}^2 + \langle \mathsf{AT\Pi}f, \Pi f\rangle\right)$$

- For any $f \in \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^2(\mathrm{d}x\,\mathrm{d}v)$,

$$\langle \mathsf{AT\Pi}f, \Pi f\rangle \geq \Phi\left(\|\Pi f\|^2\right)$$

$$\Phi^{-1}(y) := 2\,y + \left(\frac{y}{\mathsf{c}}\right)^{\frac{d}{d+2}}\,, \quad \mathsf{c} = \Theta\,\mathcal{C}_{\mathrm{Nash}}^{-\frac{d+2}{d}}\,\|f\|_{\mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)}^{-\frac{4}{d}}$$

- For any $f \in \mathrm{L}^2(\langle v\rangle^k\mathrm{d}x\,\mathrm{d}\mu) \cap \mathrm{L}^1(\mathrm{d}x\,\mathrm{d}v)$,

$$\|(\mathrm{Id}-\Pi)f\|_{-\beta}^2 \geq \Psi\left(\|(\mathrm{Id}-\Pi)f\|^2\right)$$

$$\Psi(y) := C_0\,y^{1+\beta/k}\,, \quad C_0 := \left(\mathcal{K}_k\,\left(1+\Theta_k\right)\|f^{\mathrm{in}}\|_k\right)^{-\frac{2\,\beta}{k}}$$

More references

- E. Bouin, J. Dolbeault, S. Mischler, C. Mouhot, and C. Schmeiser. Hypocoercivity without confinement. Pure and Applied Analysis, 2 (2): 203-232, May 2020. Nash
- E. Bouin, J. Dolbeault, and C. Schmeiser. Diffusion and kinetic transport with very weak confinement. Kinetic & Related Models, 13 (2): 345-371, 2020. Weighted Nash / Caffarelli-Kohn-Nirenberg inequalities
- E. Bouin, J. Dolbeault, and C. Schmeiser. A variational proof of Nash's inequality. Atti della Accademia Nazionale dei Lincei. Rendiconti Lincei. Matematica e Applicazioni, 31: 211-223, April 2020. Nash
- E. Bouin, J. Dolbeault, L. Lafleche, and C. Schmeiser. Hypocoercivity and sub-exponential local equilibria. Monatshefte für Mathematik, November 2020. Weighted Poincaré inequalities
- E. Bouin, J. Dolbeault, L. Lafleche, and C. Schmeiser. Fractional hypocoercivity. arXiv: 1911.11020 Weighted fractional Poincaré inequalities

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