Hypocoercivity without confinement: mode-by-mode analysis and decay rates in the Euclidean space

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▷ In collaboration with E. Bouin, S. Mischler, C. Mouhot, C. Schmeiser

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An abstract hypocoercivity result

▷ Abstract statement

▷ A toy model

An abstract evolution equation

Let us consider the equation

$$\frac{dF}{dt} + \mathsf{T}F = \mathsf{L}F \tag{1}$$

In the framework of kinetic equations, T and L are respectively the transport and the collision operators

We assume that T and L are respectively anti-Hermitian and Hermitian operators defined on the complex Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$

$$\mathsf{A} := \big(1 + (\mathsf{T}\Pi)^*\mathsf{T}\Pi\big)^{-1}(\mathsf{T}\Pi)^*$$

* denotes the adjoint with respect to $\langle\cdot,\cdot\rangle$

Π is the orthogonal projection onto the null space of L

The assumptions

λ_m, λ_M, and C_M are positive constants such that, for any $F \in \mathcal{H}$

▷ *microscopic coercivity:*

$$-\langle \mathsf{L}F, F\rangle \geq \lambda_m \,\|(1-\Pi)F\|^2 \tag{H1}$$

▷ *macroscopic coercivity:*

$$\|\mathsf{T}\Pi F\|^2 \geq \lambda_M \,\|\Pi F\|^2 \tag{H2}$$

▷ *parabolic macroscopic dynamics:*

$$\Pi\mathsf{T}\Pi\, F = 0 \tag{H3}$$

▷ *bounded auxiliary operators:*

$$\|\mathsf{AT}(1-\Pi)F\| + \|\mathsf{AL}F\| \leq C_M \,\|(1-\Pi)F\| \tag{H4}$$

The estimate

$$\frac{1}{2}\,\frac{d}{dt}\|F\|^2 = \langle \mathsf{L}F, F\rangle \leq -\,\lambda_m \,\|(1-\Pi)F\|^2$$

is not enough to conclude that $\|F(t,\cdot)\|^2$ decays exponentially

Equivalence and entropy decay

For some $\delta > 0$ to be determined later, the L^2 entropy / Lyapunov functional is defined by

$$\mathsf{H}[F] := \tfrac{1}{2}\,\|F\|^2 + \delta\,\mathrm{Re}\langle \mathsf{A}F, F\rangle$$

as in (J.D.-Mouhot-Schmeiser) so that $\langle \mathsf{AT}\Pi F, F\rangle \sim \|\Pi F\|^2$ and

$$-\frac{d}{dt}\mathsf{H}[F] =: \mathsf{D}[F]$$
$$= -\,\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT}\Pi F, F\rangle$$
$$-\,\delta\,\mathrm{Re}\langle \mathsf{TA}F, F\rangle + \delta\,\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle - \delta\,\mathrm{Re}\langle \mathsf{AL}F, F\rangle$$

▷ *entropy decay rate:* for any $\delta > 0$ small enough and $\lambda = \lambda(\delta)$

$$\lambda\,\mathsf{H}[F] \le \mathsf{D}[F]$$

▷ *norm equivalence* of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

Exponential decay of the entropy

$$\lambda = \frac{\lambda_M}{3\,(1+\lambda_M)}\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\},\ \delta=\frac{1}{2}\,\min\left\{1,\lambda_m,\frac{\lambda_m\,\lambda_M}{(1+\lambda_M)\,C_M^2}\right\}$$

$$h_1(\delta,\lambda) := (\delta\,C_M)^2 - 4\left(\lambda_m-\delta-\frac{2+\delta}{4}\,\lambda\right)\left(\frac{\delta\,\lambda_M}{1+\lambda_M}-\frac{2+\delta}{4}\,\lambda\right)$$

Theorem

Let L *and* T *be closed linear operators (respectively Hermitian and anti-Hermitian) on* $\mathcal{H}$*. Under* (H1)–(H4)*, for any* $t\ge 0$

$$\mathsf{H}[F(t,\cdot)] \le \mathsf{H}[F_0]\,e^{-\lambda_\star\,t}$$

where $\lambda_\star$ *is characterized by*

$$\lambda_\star := \sup\left\{\lambda>0\,:\,\exists\,\delta>0\ s.t.\ h_1(\delta,\lambda)=0\,,\ \lambda_m-\delta-\tfrac{1}{4}\,(2+\delta)\,\lambda>0\right\}$$

Sketch of the proof

- Since $\mathsf{AT\Pi} = \big(1 + (\mathsf{T\Pi})^*\mathsf{T\Pi}\big)^{-1} (\mathsf{T\Pi})^*\mathsf{T\Pi}$, from (H1) and (H2)

$$-\langle \mathsf{L}F, F\rangle + \delta\,\langle \mathsf{AT\Pi}F, F\rangle \geq \lambda_m\,\|(1-\Pi)F\|^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,\|\Pi F\|^2$$

- By (H4), we know that

$$|\mathrm{Re}\langle \mathsf{AT}(1-\Pi)F, F\rangle + \mathrm{Re}\langle \mathsf{AL}F, F\rangle| \leq C_M\,\|\Pi F\|\,\|(1-\Pi)F\|$$

- The equation $G = \mathsf{A}F$ is equivalent to $(\mathsf{T\Pi})^* F = G + (\mathsf{T\Pi})^*\,\mathsf{T\Pi}\,G$

$$\langle \mathsf{TA}F, F\rangle = \langle G, (\mathsf{T\Pi})^*\,F\rangle = \|G\|^2 + \|\mathsf{T\Pi}G\|^2 = \|\mathsf{A}F\|^2 + \|\mathsf{TA}F\|^2$$

By the Cauchy-Schwarz inequality, for any $\mu > 0$

$$\langle G, (\mathsf{T\Pi})^*\,F\rangle \leq \|\mathsf{TA}F\|\,\|(1-\Pi)F\| \leq \frac{1}{2\,\mu}\,\|\mathsf{TA}F\|^2 + \frac{\mu}{2}\,\|(1-\Pi)F\|^2$$

$$\|\mathsf{A}F\| \leq \frac{1}{2}\,\|(1-\Pi)F\|\,,\ \|\mathsf{TA}F\| \leq \|(1-\Pi)F\|\,,\ |\langle \mathsf{TA}F, F\rangle| \leq \|(1-\Pi)F\|^2$$

- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$

$$\mathsf{D}[F] - \lambda\,\mathsf{H}[F] \geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{2+\delta}{4}\,\lambda\,(X^2+Y^2)$$

Hypocoercivity

Corollary

For any $\delta \in (0,2)$*, if* $\lambda(\delta)$ *is the largest positive root of* $h_1(\delta,\lambda)=0$ *for which* $\lambda_m - \delta - \frac{1}{4}(2+\delta)\,\lambda > 0$*, then for any solution* F *of* (1)

$$\|F(t)\|^2 \le \frac{2+\delta}{2-\delta}\, e^{-\lambda(\delta)\,t}\, \|F(0)\|^2 \quad \forall\, t \ge 0$$

From the norm equivalence of $\mathsf{H}[F]$ and $\|F\|^2$

$$\frac{2-\delta}{4}\,\|F\|^2 \le \mathsf{H}[F] \le \frac{2+\delta}{4}\,\|F\|^2$$

We use $\frac{2-\delta}{4}\,\|F_0\|^2 \le \mathsf{H}[F_0]$ so that $\lambda_\star \ge \sup_{\delta\in(0,2)} \lambda(\delta)$

A toy problem

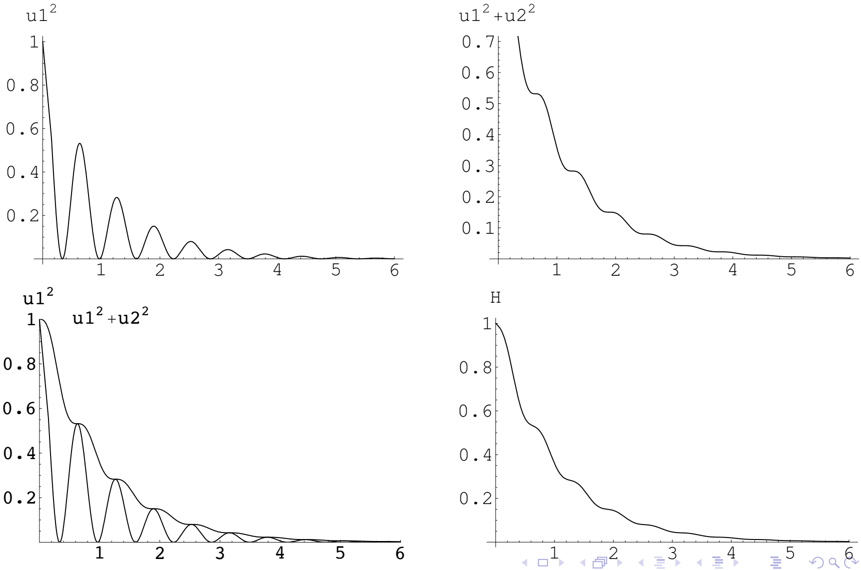
$$\frac{du}{dt} = (\mathsf{L}-\mathsf{T})\,u\,,\quad \mathsf{L} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}\,,\quad \mathsf{T} = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}\,,\quad k^2 \ge \Lambda > 0$$

Non-monotone decay, a well known picture:
see for instance (Filbet, Mouhot, Pareschi, 2006)

- H-theorem: $\frac{d}{dt}|u|^2 = -\,2\,u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -\,k^2\,u_1$
- generalized entropy: $\mathsf{H}(u) = |u|^2 - \frac{\delta\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned} \frac{d\mathsf{H}}{dt} &= -\left(2 - \frac{\delta\,k^2}{1+k^2}\right)u_2^2 - \frac{\delta\,k^2}{1+k^2}\,u_1^2 + \frac{\delta\,k}{1+k^2}\,u_1\,u_2 \\ &\le -(2-\delta)\,u_2^2 - \frac{\delta\Lambda}{1+\Lambda}\,u_1^2 + \frac{\delta}{2}\,u_1 u_2 \end{aligned}$$

Plots for the toy problem



Mode-by-mode hypocoercivity

- ▷ Fokker-Planck equation and scattering collision operators
- ▷ A mode-by-mode hypocoercivity result
- ▷ Enlargement of the space by factorization
- ▷ Application to the torus and some numerical results

(Bouin, J.D., Mischler, Mouhot, Schmeiser)

Fokker-Planck equation with general equilibria

We consider the Cauchy problem

$$\partial_t f + v \cdot \nabla_x f = \mathsf{L} f \,, \quad f(0,x,v) = f_0(x,v) \tag{2}$$

for a distribution function $f(t,x,v)$, with *position* variable $x \in \mathbb{R}^d$ or $x \in \mathbb{T}^d$ the flat d-dimensional torus

Fokker-Planck collision operator with a general equilibrium M

$$\mathsf{L} f = \nabla_v \cdot \Big[M \, \nabla_v \left(M^{-1} \, f \right) \Big]$$

Notation and assumptions: an *admissible local equilibrium* M is positive, radially symmetric and

$$\int_{\mathbb{R}^d} M(v) \, dv = 1 \,, \quad d\gamma = \gamma(v) \, dv := \frac{dv}{M(v)}$$

γ is an *exponential weight* if

$$\lim_{|v| \to \infty} \frac{|v|^k}{\gamma(v)} = \lim_{|v| \to \infty} M(v) \, |v|^k = 0 \quad \forall \, k \in (d, \infty)$$

Definitions

$$\Theta = \frac{1}{d} \int_{\mathbb{R}^d} |v|^2 \, M(v) \, dv = \int_{\mathbb{R}^d} (v \cdot \mathsf{e})^2 \, M(v) \, dv$$

for an arbitrary $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\int_{\mathbb{R}^d} v \otimes v \, M(v) \, dv = \Theta \, \mathrm{Id}$$

Then

$$\theta = \frac{1}{d} \, \|\nabla_v M\|^2_{\mathrm{L}^2(d\gamma)} = \frac{4}{d} \int_{\mathbb{R}^d} \left|\nabla_v \sqrt{M}\right|^2 dv < \infty$$

If $M(v) = \frac{e^{-\frac{1}{2}|v|^2}}{(2\pi)^{d/2}}$, then $\Theta = 1$ and $\theta = 1$

$$\overline{\sigma} := \frac{1}{2} \sqrt{\theta / \Theta}$$

Microscopic coercivity property (Poincaré inequality): for all $u = M^{-1} F \in \mathrm{H}^1(M \, dv)$

$$\int_{\mathbb{R}^d} |\nabla u|^2 \, M \, dv \geq \lambda_m \int_{\mathbb{R}^d} \left(u - \int_{\mathbb{R}^d} u \, M \, dv \right)^2 M \, dv$$

Scattering collision operators

Scattering collision operator

$$\mathsf{L}f=\int_{\mathbb{R}^d}\sigma(\cdot,v')\,\big(f(v')\,M(\cdot)-f(\cdot)\,M(v')\big)\,dv'$$

Main assumption on the *scattering rate* σ: for some positive, finite $\overline{\sigma}$

$$1\le\sigma(v,v')\le\overline{\sigma}\quad\forall\,v,\,v'\in\mathbb{R}^d$$

Example: linear BGK operator

$$\mathsf{L}f=M\rho_f-f\,,\quad\rho_f(t,x)=\int_{\mathbb{R}^d}f(t,x,v)\,dv$$

Local mass conservation

$$\int_{\mathbb{R}^d}\mathsf{L}f\,dv=0$$

and we have

$$\int_{\mathbb{R}^d}|\mathsf{L}f|^2\,d\gamma\le 4\,\overline{\sigma}^2\int_{\mathbb{R}^d}|M\rho_f-f|^2\,d\gamma$$

The symmetry condition

$$\int_{\mathbb{R}^d} \big(\sigma(v,v') - \sigma(v',v)\big)\, M(v')\, dv' = 0 \quad \forall\, v \in \mathbb{R}^d$$

implies the *local mass conservation* $\int_{\mathbb{R}^d} \mathsf{L} f\, dv = 0$

Micro-reversibility, *i.e.*, the symmetry of σ, is not required

The null space of L is spanned by the local equilibrium M
L only acts on the velocity variable

Microscopic coercivity property: for some $\lambda_m > 0$

$$\frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \sigma(v,v')\, M(v)\, M(v')\, \big(u(v) - u(v')\big)^2\, dv'\, dv \\ \geq \lambda_m \int_{\mathbb{R}^d} \big(u - \rho_{u\,M}\big)^2\, M\, dv$$

holds according to Proposition 2.2 of (Degond, Goudon, Poupaud, 2000) for all $u = M^{-1} F \in \mathrm{L}^2(M\, dv)$. If $\sigma \equiv 1$, then $\lambda_m = 1$

Fourier modes

In order to perform a *mode-by-mode hypocoercivity* analysis, we introduce the Fourier representation with respect to x,

$$f(t,x,v)=\int_{\mathbb{R}^d}\hat{f}(t,\xi,v)\,e^{-i\,x\cdot\xi}\,d\mu(\xi)$$

$d\mu(\xi)=(2\pi)^{-d}\,d\xi$ and $d\xi$ is the Lesbesgue measure if $x\in\mathbb{R}^d$
$d\mu(\xi)=(2\pi)^{-d}\sum_{z\in\mathbb{Z}^d}\delta(\xi-z)$ is discrete for $x\in\mathbb{T}^d$

Parseval's identity if $\xi\in\mathbb{Z}^d$ and Plancherel's formula if $x\in\mathbb{R}^d$ read

$$\|f(t,\cdot,v)\|_{\mathrm{L}^2(dx)}=\left\|\hat{f}(t,\cdot,v)\right\|_{\mathrm{L}^2(d\mu(\xi))}$$

The Cauchy problem is now decoupled in the ξ-direction

$$\partial_t\hat{f}+\mathsf{T}\hat{f}=\mathsf{L}\hat{f}\,,\quad \hat{f}(0,\xi,v)=\hat{f}_0(\xi,v)$$

$$\mathsf{T}\hat{f}=i\,(v\cdot\xi)\,\hat{f}$$

For any fixed $\xi \in \mathbb{R}^d$, let us apply the abstract result with

$$\mathcal{H} = \mathrm{L}^2\,(d\gamma)\;, \quad \|F\|^2 = \int_{\mathbb{R}^d} |F|^2\,d\gamma\,, \quad \Pi F = M \int_{\mathbb{R}^d} F\,dv = M\,\rho_F$$

and $\mathsf{T}\hat{f} = i\,(v\cdot\xi)\,\hat{f}$, $\mathsf{T}\Pi F = i\,(v\cdot\xi)\,\rho_F\,M$,

$$\|\mathsf{T}\Pi F\|^2 = |\rho_F|^2 \int_{\mathbb{R}^d} |v\cdot\xi|^2\,M(v)\,dv \;= \Theta\,|\xi|^2\,|\rho_F|^2 = \Theta\,|\xi|^2\,\|\Pi F\|^2$$

(H2) *Macroscopic coercivity* $\|\mathsf{T}\Pi F\|^2 \ge \lambda_M\,\|\Pi F\|^2 \,:\, \lambda_M = \Theta\,|\xi|^2$

(H3) $\int_{\mathbb{R}^d} v\,M(v)\,dv = 0$

The operator A is given by

$$\mathsf{A}F = \frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d} v'\,F(v')\,dv'}{1+\Theta\,|\xi|^2}\,M$$

A mode-by-mode hypocoercivity result

$$\|\mathsf{A}F\| = \|\mathsf{A}(1-\Pi)F\| \le \frac{1}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}\frac{|(1-\Pi)F|}{\sqrt{M}}\,|v\cdot\xi|\,\sqrt{M}\,dv$$
$$\le \frac{1}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|\left(\int_{\mathbb{R}^d}(v\cdot\xi)^2\,M\,dv\right)^{1/2}$$
$$= \frac{\sqrt{\Theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

- Scattering operator $\|\mathsf{L}F\|^2 \le 4\,\overline{\sigma}^2\,\|(1-\Pi)F\|^2$
- Fokker-Planck (FP) operator

$$\|\mathsf{AL}F\| \le \frac{2}{1+\Theta\,|\xi|^2}\int_{\mathbb{R}^d}\frac{|(1-\Pi)F|}{\sqrt{M}}\,|\xi\cdot\nabla_v\sqrt{M}|\,dv \le \frac{\sqrt{\theta}\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

In both cases with $\kappa = \sqrt{\theta}$ (FP) or $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ we obtain

$$\|\mathsf{AL}F\| \le \frac{\kappa\,|\xi|}{1+\Theta\,|\xi|^2}\,\|(1-\Pi)F\|$$

$$\mathsf{TA}F(v) = -\frac{(v \cdot \xi)\, M}{1 + \Theta\, |\xi|^2} \int_{\mathbb{R}^d} (v' \cdot \xi)\, (1 - \Pi) F(v')\, dv'$$

is estimated by

$$\|\mathsf{TA}F\| \le \frac{\Theta\, |\xi|^2}{1 + \Theta\, |\xi|^2}\, \|(1 - \Pi) F\|$$

(H4) holds with $C_M = \frac{\kappa\, |\xi| + \Theta\, |\xi|^2}{1 + \Theta\, |\xi|^2}$

Two elementary estimates

$$\frac{\Theta\, |\xi|^2}{1 + \Theta\, |\xi|^2} \ge \frac{\Theta}{\max\{1, \Theta\}}\, \frac{|\xi|^2}{1 + |\xi|^2}$$

$$\frac{\lambda_M}{(1 + \lambda_M)\, C_M^2} = \frac{\Theta\, \left(1 + \Theta\, |\xi|^2\right)}{\left(\kappa + \Theta\, |\xi|\right)^2} \ge \frac{\Theta}{\kappa^2 + \Theta}$$

Mode-by-mode hypocoercivity with exponential weights

Theorem

Let us consider an admissible M and a collision operator L satisfying Assumption (H)*, and take $\xi \in \mathbb{R}^d$. If $\hat f$ is a solution such that $\hat f_0(\xi,\cdot) \in \mathrm{L}^2(d\gamma)$, then for any $t \geq 0$, we have*

$$\left\|\hat f(t,\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma)} \leq 3\, e^{-\,\mu_\xi\, t} \left\|\hat f_0(\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma)}$$

where

$$\mu_\xi := \frac{\Lambda\,|\xi|^2}{1+|\xi|^2} \quad \textit{and} \quad \Lambda = \frac{\Theta}{3\,\max\{1,\Theta\}}\,\min\left\{1, \frac{\lambda_m\,\Theta}{\kappa^2+\Theta}\right\}$$

with $\kappa = 2\,\overline{\sigma}\,\sqrt{\Theta}$ for scattering operators
and $\kappa = \sqrt{\theta}$ for (FP) operators

Enlargement of the space by factorization

A simple case (factorization of order 1) of the *factorization method* of (Gualdani, Mischler, Mouhot)

Theorem

Let $\mathcal{B}_1$, $\mathcal{B}_2$ be Banach spaces and let $\mathcal{B}_2$ be continuously imbedded in $\mathcal{B}_1$, i.e., $\|\cdot\|_1 \le c_1 \|\cdot\|_2$. Let $\mathfrak{B}$ and $\mathfrak{A}+\mathfrak{B}$ be the generators of the strongly continuous semigroups $e^{\mathfrak{B}\,t}$ and $e^{(\mathfrak{A}+\mathfrak{B})\,t}$ on $\mathcal{B}_1$. If for all $t \ge 0$,

$$\left\|e^{(\mathfrak{A}+\mathfrak{B})\,t}\right\|_{2\to 2} \le c_2\, e^{-\lambda_2\,t}\,, \quad \left\|e^{\mathfrak{B}t}\right\|_{1\to 1} \le c_3\, e^{-\lambda_1\,t}\,, \quad \|\mathfrak{A}\|_{1\to 2} \le c_4$$

where $\|\cdot\|_{i\to j}$ denotes the operator norm for linear mappings from $\mathcal{B}_i$ to $\mathcal{B}_j$. Then there exists a positive constant $C = C(c_1,c_2,c_3,c_4)$ such that, for all $t \ge 0$,

$$\left\|e^{(\mathfrak{A}+\mathfrak{B})\,t}\right\|_{1\to 1} \le \begin{cases} C\left(1+|\lambda_1-\lambda_2|^{-1}\right)\, e^{-\min\{\lambda_1,\lambda_2\}\,t} & \text{for } \lambda_1 \ne \lambda_2 \\ C\,(1+t)\, e^{-\lambda_1\,t} & \text{for } \lambda_1 = \lambda_2 \end{cases}$$

Integrating the identity $\frac{d}{ds}\left(e^{(\mathfrak{A}+\mathfrak{B})\,s}\,e^{\mathfrak{B}\,(t-s)}\right)=e^{(\mathfrak{A}+\mathfrak{B})\,s}\,\mathfrak{A}\,e^{\mathfrak{B}\,(t-s)}$ with respect to $s\in[0,t]$ gives

$$e^{(\mathfrak{A}+\mathfrak{B})\,t}=e^{\mathfrak{B}\,t}+\int_0^t e^{(\mathfrak{A}+\mathfrak{B})\,s}\,\mathfrak{A}\,e^{\mathfrak{B}\,(t-s)}\,ds$$

The proof is completed by the straightforward computation

$$\begin{aligned}\left\|e^{(\mathfrak{A}+\mathfrak{B})\,t}\right\|_{1\to1}&\le c_3\,e^{-\lambda_1\,t}+c_1\int_0^t\left\|e^{(\mathfrak{A}+\mathfrak{B})\,s}\,\mathfrak{A}\,e^{\mathfrak{B}\,(t-s)}\right\|_{1\to2}ds\\&\le c_3\,e^{-\lambda_1\,t}+c_1\,c_2\,c_3\,c_4\,e^{-\lambda_1\,t}\int_0^t e^{(\lambda_1-\lambda_2)\,s}\,ds\end{aligned}$$

Weights with polynomial growth

Let us consider the measure

$$d\gamma_k := \gamma_k(v)\,dv \quad \text{where} \quad \gamma_k(v) = \pi^{d/2}\,\frac{\Gamma((k-d)/2)}{\Gamma(k/2)}\,\left(1+|v|^2\right)^{k/2}$$

for an arbitrary $k \in (d, +\infty)$

We choose $\mathcal{B}_1 = \mathrm{L}^2(d\gamma_k)$ and $\mathcal{B}_2 = \mathrm{L}^2(d\gamma)$

Theorem

Let $\Lambda = \frac{\Theta}{3\,\max\{1,\Theta\}}\,\min\left\{1, \frac{\lambda_m\,\Theta}{\kappa^2+\Theta}\right\}$ *and* $k \in (d, \infty]$. *For any* $\xi \in \mathbb{R}^d$ *if* $\hat{f}$ *is a solution with initial datum* $\hat{f}_0(\xi,\cdot) \in \mathrm{L}^2(d\gamma_k)$, *then there exists a constant* $C = C(k, d, \overline{\sigma})$ *such that*

$$\left\|\hat{f}(t,\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma_k)} \le C\,e^{-\mu_\xi\,t}\,\left\|\hat{f}_0(\xi,\cdot)\right\|^2_{\mathrm{L}^2(d\gamma_k)} \quad \forall\, t \ge 0$$

- Fokker-Planck: $\mathfrak{A}F = N\,\chi_R F$ and $\mathfrak{B}F = -\,i\,(v\cdot\xi)\,F + \mathsf{L}F - \mathfrak{A}F$

N and R are two positive constants, χ is a smooth cut-off function and $\chi_R := \chi(\cdot/R)$

For any R and N large enough, according to Lemma 3.8 of (Mischler, Mouhot, 2016)

$$\int_{\mathbb{R}^d}(\mathsf{L}-\mathfrak{A})(F)F\,d\gamma_k \le -\,\lambda_1\int_{\mathbb{R}^d}F^2\,d\gamma_k$$

for some $\lambda_1>0$ if $k>d$, and $\lambda_2=\mu_\xi/2\le 1/4$

- Scattering operator:

$$\begin{aligned}\mathfrak{A}F(v) &= M(v)\int_{\mathbb{R}^d}\sigma(v,v')\,F(v')\,dv'\\ \mathfrak{B}F(v) &= -\left[i\,(v\cdot\xi)+\int_{\mathbb{R}^d}\sigma(v,v')\,M(v')\,dv'\right]F(v)\end{aligned}$$

Boundedness: $\|\mathfrak{A}F\|_{\mathrm{L}^2(d\gamma)}\le\overline{\sigma}\left(\int_{\mathbb{R}^d}\gamma_k^{-1}\,dv\right)^{1/2}\|F\|_{\mathrm{L}^2(d\gamma_k)}$

$\lambda_1=1$ and $\lambda_2=\mu_\xi/2\le 1/4$

Exponential convergence to equilibrium in $\mathbb{T}^d$

The unique global equilibrium in the case $x \in \mathbb{T}^d$ is given by

$$f_\infty(x,v) = \rho_\infty\, M(v) \quad \text{with} \quad \rho_\infty = \frac{1}{|\mathbb{T}^d|} \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0\, dx\, dv$$

Theorem

Assume that $k \in (d,\infty]$ and γ has an exponential growth if $k = \infty$. We consider an admissible M, a collision operator L satisfying Assumption (H)*, and Λ given by* (3)
There exists a positive constant C_k such that the solution f of (2) *on $\mathbb{T}^d \times \mathbb{R}^d$ with initial datum $f_0 \in \mathrm{L}^2(dx\, d\gamma_k)$ satisfies*

$$\|f(t,\cdot,\cdot) - f_\infty\|_{\mathrm{L}^2(dx\, d\gamma_k)} \le C_k\, \|f_0 - f_\infty\|_{\mathrm{L}^2(dx\, d\gamma_k)}\, e^{-\frac{1}{4}\Lambda\, t} \quad \forall\, t \ge 0$$

If we represent the flat torus $\mathbb{T}^d$ by the box $[0,2\pi)^d$ with periodic boundary conditions, the Fourier variable satisfies $\xi \in \mathbb{Z}^d$. For $\xi = 0$, the microscopic coercivity implies

$$\left\| \hat{f}(t,0,\cdot) - \hat{f}_\infty(0,\cdot) \right\|_{\mathrm{L}^2(d\gamma)} \le \left\| \hat{f}_0(0,\cdot) - \hat{f}_\infty(0,\cdot) \right\|_{\mathrm{L}^2(d\gamma)} e^{-t}$$

Otherwise $\mu_\xi \ge \Lambda/2$ for any $\xi \neq 0$

Parseval's identity applies, with measure $d\gamma(v)$ and $C_\infty = \sqrt{3}$
The result with weight γ_k follows from the factorization result for some $C_k > 0$

Computation of the constants

▷ A more numerical point of view

Two simple examples: L denotes either the *Fokker-Planck operator*

$$\mathsf{L}_1 f := \Delta_v f + \nabla_v \cdot (v\, f)$$

or the *linear BGK operator*

$$\mathsf{L}_2 f := \Pi f - f$$

$\Pi f = \rho_f\, M$ is the projection operator on the normalized Gaussian function

$$M(v) = \frac{e^{-\frac{1}{2}\,|v|^2}}{(2\,\pi)^{d/2}}$$

and $\rho_f := \int_{\mathbb{R}^d} f\, dv$ is the spatial density

Where do we have space for improvements ?

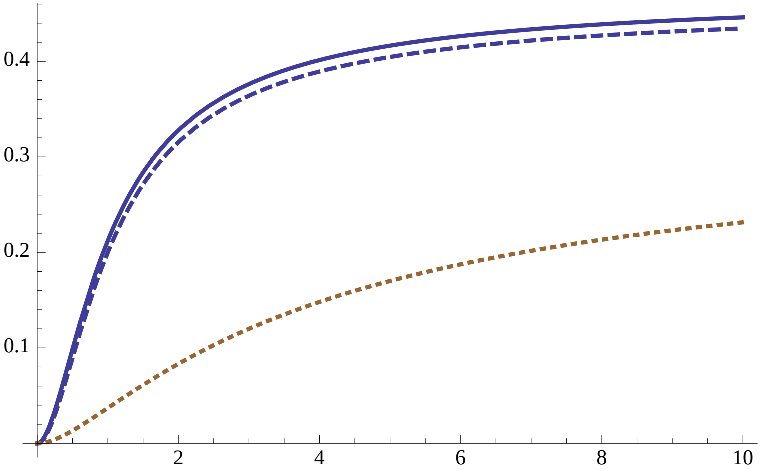
- With $X := \|(1-\Pi)F\|$ and $Y := \|\Pi F\|$, we wrote

$$\mathsf{D}[F] - \lambda\,\mathsf{H}[F]$$
$$\geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right)$$
$$\geq (\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{2+\delta}{4}\,\lambda\,(X^2+Y^2)$$

- We can directly study the positivity condition for the quadratic form

$$(\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right)$$

$\lambda_m = 1$, $\lambda_M = |\xi|^2$ and $C_M = |\xi|\,(1+|\xi|)/(1+|\xi|^2)$


0.4
0.3
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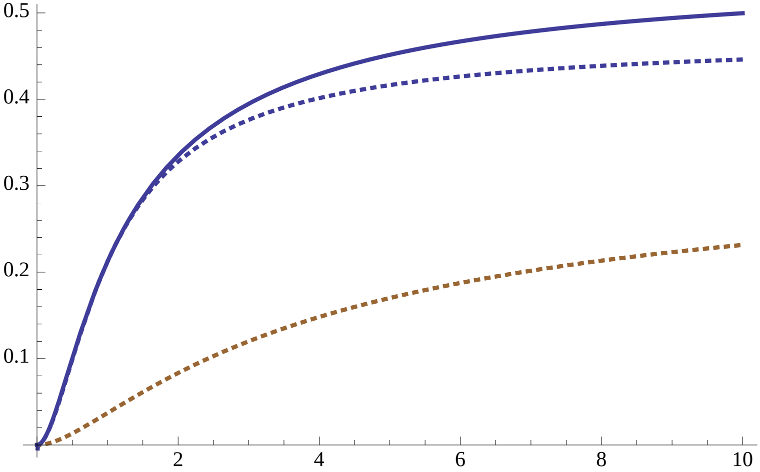
With $\lambda_m = 1$, $\lambda_M = |\xi|^2$ and $C_M = |\xi|\,(1+|\xi|)/(1+|\xi|^2)$, we optimize λ under the condition that the quadratic form

$$(\lambda_m - \delta)\,X^2 + \frac{\delta\,\lambda_M}{1+\lambda_M}\,Y^2 - \delta\,C_M\,X\,Y - \frac{\lambda}{2}\left(X^2 + Y^2 + \delta\,X\,Y\right)$$

is positive, thus getting a $\lambda(\xi)$

- By taking also $\delta = \delta(\xi)$ where ξ is seen as a parameter, we get a better estimate of $\lambda(\xi)$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By taking $\delta=\delta(\xi)$, for each value of ξ we build a different Lyapunov function, namely

$$\mathsf{H}_\xi[F]:=\tfrac12\,\|F\|^2+\delta(\xi)\,\mathrm{Re}\langle\mathsf{A}F,F\rangle$$

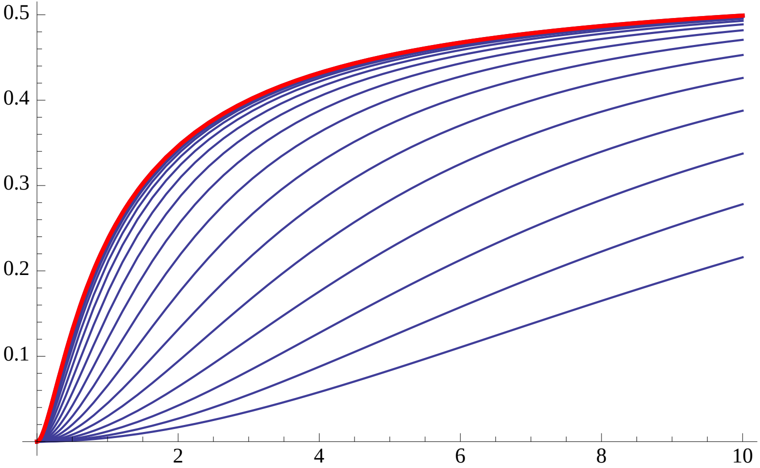
where the operator A is given by

$$\mathsf{A}F=\frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d}v'\,F(v')\,dv'}{1+|\xi|^2}\,M$$

- We can consider

$$\mathsf{A}_\varepsilon F=\frac{-\,i\,\xi\cdot\int_{\mathbb{R}^d}v'\,F(v')\,dv'}{\varepsilon+|\xi|^2}\,M$$

and look for the optimal value of ε...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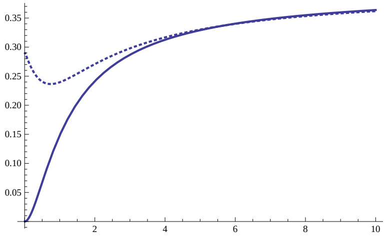
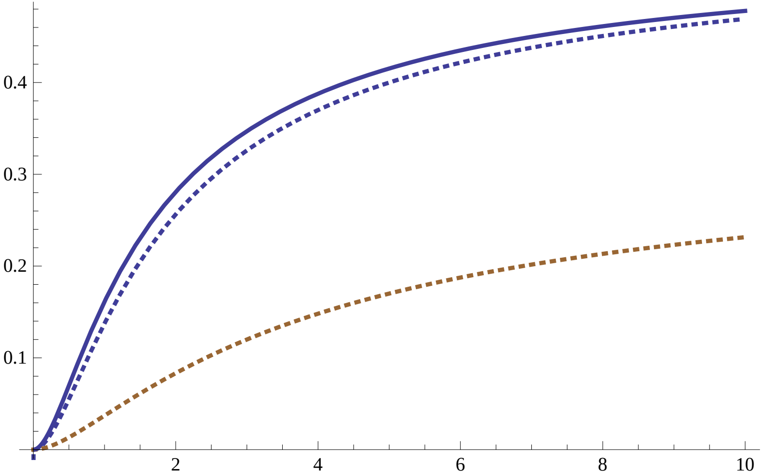
The dependence of λ in ε is monotone, and the limit as $\varepsilon \to 0_+$ gives the optimal estimate of λ. The operator

$$\mathsf{A}_0 F = \frac{-\,i\,\xi \cdot \int_{\mathbb{R}^d} v'\,F(v')\,dv'}{|\xi|^2}\,M$$

is not bounded anymore, but estimates still make sense and $\lim_{\xi\to0}\delta(\xi) = 0$ (see below)

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Theorem (Hypocoercivity on $\mathbb{T}^d$ with exponential weight)

Assume that $\mathsf{L} = \mathsf{L}_1$ *or* $\mathsf{L} = \mathsf{L}_2$. *If* f *is a solution, then*

$$\|f(t,\cdot,\cdot) - f_\infty\|^2_{\mathrm{L}^2(dx\,d\gamma)} \le \mathcal{C}_\star \, \|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma)} \, e^{-\lambda_\star t} \quad \forall\, t \ge 0$$

with $f_\infty(x,v) = M(v) \iint_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x,v)\,dx\,dv$

$\mathcal{C}_\star \approx 1.75863$ *and* $\lambda_\star = \frac{2}{13}(5 - 2\sqrt{3}) \approx 0.236292.$

Warning: work in progress

Some comments on recent works

A more algebraic approach based on the spectral analysis of symmetric and non-symmetric operators

- On BGK models
(Achleitner, Arnold, Carlen)

- On Fokker-Planck models
(Arnold, Stürzer)
(Arnold, Erb)
(Arnold, Einav, Wöhrer)

Decay rates in the whole space

Algebraic decay rates in $\mathbb{R}^d$

On the whole Euclidean space, we can define the entropy

$$\mathsf{H}[f] := \frac{1}{2}\,\|f\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} + \delta\,\langle \mathsf{A} f, f\rangle_{dx\,d\gamma_k}$$

Replacing the *macroscopic coercivity* condition by *Nash's inequality*

$$\|u\|^2_{\mathrm{L}^2(dx)} \le \mathcal{C}_{\mathrm{Nash}}\,\|u\|^{\frac{4}{d+2}}_{\mathrm{L}^1(dx)}\,\|\nabla u\|^{\frac{2\,d}{d+2}}_{\mathrm{L}^2(dx)}$$

proves that

$$\mathsf{H}[f] \le C\left(\mathsf{H}[f_0] + \|f_0\|^2_{\mathrm{L}^1(dx\,dv)}\right)(1+t)^{-\frac{d}{2}}$$

Theorem

Assume that γ_k has an exponential growth ($k=\infty$) or a polynomial growth of order $k>d$

There exists a constant $C>0$ such that, for any $t\ge 0$

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le C\left(\|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(dx))}\right)(1+t)^{-\frac{d}{2}}$$

A direct proof... Recall that $\mu_\xi = \frac{\Lambda\,|\xi|^2}{1+|\xi|^2}$

By the Plancherel formula

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le C \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} e^{-\,\mu_\xi\,t}\,|\hat{f}_0|^2\,d\xi \right) d\gamma_k$$

- if $|\xi|<1$, then $\mu_\xi \ge \frac{\Lambda}{2}\,|\xi|^2$

$$\int_{|\xi|\le 1} e^{-\,\mu_\xi\,t}\,|\hat{f}_0|^2\,d\xi \le C\,\|f_0(\cdot,v)\|^2_{\mathrm{L}^1(dx)} \int_{\mathbb{R}^d} e^{-\,\frac{\Lambda}{2}\,|\xi|^2\,t}\,d\xi \le C\,\|f_0(\cdot,v)\|^2_{\mathrm{L}^1(dx)}\;t^{-\frac{d}{2}}$$

- if $|\xi|\ge 1$, then $\mu_\xi \ge \Lambda/2$ when $|\xi|\ge 1$

$$\int_{|\xi|>1} e^{-\,\mu_\xi\,t}\,|\hat{f}_0|^2\,d\xi \le C\,e^{-\frac{\Lambda}{2}\,t}\,\|f_0(\cdot,v)\|^2_{\mathrm{L}^2(dx)}$$

Improved decay rate for zero average solutions

Theorem

Assume that $f_0 \in \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d \times \mathbb{R}^d)$ *with* $\iint_{\mathbb{R}^d \times \mathbb{R}^d} f_0(x,v)\,dx\,dv = 0$ *and*
$\mathcal{C}_0 := \|f_0\|^2_{\mathrm{L}^2(d\gamma_{k+2};\,\mathrm{L}^1(dx))} + \|f_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(|x|\,dx))} + \|f_0\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} < \infty$

Then there exists a constant $c_k > 0$ *such that*

$$\|f(t,\cdot,\cdot)\|^2_{\mathrm{L}^2(dx\,d\gamma_k)} \le c_k\,\mathcal{C}_0\,(1+t)^{-\left(1+\frac{d}{2}\right)}$$

Step 1: Decay of the average in space, factorization

- *x-average in space*

$$f_\bullet(t,v) := \int_{\mathbb{R}^d} f(t,x,v)\,dx$$

with $\int_{\mathbb{R}^d} f_\bullet(t,v)\,dv = 0$ and observe that $f_\bullet$ solves a *Fokker-Planck* equation

$$\partial_t f_\bullet = \mathsf{L} f_\bullet$$

From the *microscopic coercivity property*, we deduce that

$$\|f_\bullet(t,\cdot)\|^2_{\mathrm{L}^2(d\gamma)} \le \|f_\bullet(0,\cdot)\|^2_{\mathrm{L}^2(d\gamma)}\;e^{-\lambda_m\,t}$$

- *Factorisation*

$$\|f_\bullet(t,\cdot)\|^2_{\mathrm{L}^2(|v|^2\,d\gamma_k)} \le C\;\|f_0\|^2_{\mathrm{L}^2(|v|^2\,d\gamma_k\,;\mathrm{L}^1(dx))}\;e^{-\lambda\,t}$$

Step 2: Improved decay of f

Let us define $g := f - f_\bullet\,\varphi$, with $\varphi(x) := (2\pi)^{-d/2}\,e^{-|x|^2/2}$
The Fourier transform $\hat g$ solves

$$\partial_t \hat g + \mathsf{T}\hat g = \mathsf{L}\hat g - f_\bullet\,\mathsf{T}\hat\varphi \text{ with } \mathsf{T}\hat\varphi = i\,(v\cdot\xi)\,\hat\varphi$$

Duhamel's formula

$$\hat g = \underbrace{e^{i(\mathsf{L}-\mathsf{T})\,t}\,\hat g_0}_{C\,e^{-\frac12\,\mu_\xi\,t}\,\|\hat g_0(\xi,\cdot)\|_{\mathrm{L}^2(d\gamma_k)}} + \int_0^t \underbrace{e^{i(\mathsf{L}-\mathsf{T})\,(t-s)}\left(-f_\bullet(s,v)\,\mathsf{T}\hat\varphi(\xi)\right)}_{C\,e^{-\frac{\mu_\xi}{2}\,(t-s)}\|f_\bullet(s,\cdot)\|_{\mathrm{L}^2(|v|^2\,d\gamma_k)}|\xi|\,|\hat\varphi(\xi)|}\,ds$$

- $\hat g_0(\xi,v) = \underbrace{\hat g_0(0,v)}_{=0} + \int_0^{|\xi|}\frac{\xi}{|\xi|}\cdot\nabla_\xi \hat g_0\left(\eta\,\frac{\xi}{|\xi|},v\right)d\eta$ yields

$$|\hat g_0(\xi,v)| \le |\xi|\;\|\nabla_\xi \hat g_0(\cdot,v)\|_{\mathrm{L}^\infty(dv)} \le |\xi|\;\|g_0(\cdot,v)\|_{\mathrm{L}^1(|x|\,dx)}$$

- $\mu_\xi = \Lambda\,|\xi|^2/(1+|\xi|^2) \ge \Lambda/2$ if $|\xi|>1$ (contribution $O(e^{-\frac{\Lambda}{2}t})$) and

$$\int_{|\xi|\le 1}\int_{\mathbb{R}^d}\left|e^{i(\mathsf{L}-\mathsf{T})\,t}\hat g_0\right|^2 d\gamma_k\,d\xi \le \underbrace{\int_{\mathbb{R}^d}|\xi|^2\,e^{-\frac{\Lambda}{2}\,|\xi|^2\,t}\,d\xi}_{=O(t^{-(d+2)})}\;\|g_0\|^2_{\mathrm{L}^2(d\gamma_k;\,\mathrm{L}^1(|x|\,dx))}$$

These slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Lectures/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/
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For final versions, use `Dolbeault` as login and `Jean` as password

Thank you for your attention !