Asymptotic behaviour for small mass in the two-dimensional parabolic-elliptic Keller-Segel model

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Introduction

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2\,,\ t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\ t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

- $\int_{\mathbb{R}^2} n_0\, dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2\, n_0\, dx < \infty$: blow-up in finite time
- $\int_{\mathbb{R}^2} n_0\, dx \leq 8\pi$: global existence [Jäger-Luckhaus], [J.D., Perthame], [Blanchet, JD, Perthame], [Blanchet, Carrillo, Masmoudi] based on the free energy

$$F[u(\cdot,t)] + \int_0^t \int_{\mathbb{R}^2} u \left| \nabla (\log u) - \nabla v \right|^2 dx\, ds \leq F[u_0]$$

$F[u] := \int_{\mathbb{R}^2} u \log u\; dx - \frac{1}{2} \int_{\mathbb{R}^2} u\, v\; dx$

The parabolic-elliptic Keller and Segel system

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Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx)\,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx)\,, \quad M := \int_{\mathbb{R}^2} n_0(x)\,dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\,dx$ for any $t \geq 0$, see [Jäger-Luckhaus], [Blanchet, JD, Perthame]

In dimension $d = 2$: $v = -\frac{1}{2\pi}\,\log|\cdot| * u$

...global invariance under scalings

Rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot \left(n\left(\nabla c - x\right)\right) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi} \log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[Blanchet, JD, Perthame]

$$\lim_{t\to\infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

The stationary solution in self-similar variables

$$n_\infty = M\, \frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\, dx} = -\Delta c_\infty\ , \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- Radial symmetry [Naito]
- Uniqueness [Biler,Karch, Laurençot, Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [Blanchet, JD, Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M, it follows that

$$\lim_{M\to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[Joseph, Lundgreen] [JD, Stańczy]

Main result

Theorem 1. *There exists a positive constant M^* such that, for any initial data $n_0 \in L^2(n_\infty^{-1}\,dx)$ of mass $M < M^*$ satisfying the above assumptions, there is a unique solution $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ for any $\tau > 0$*

Moreover, there are two positive constants, C and δ, such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \,\frac{dx}{n_\infty} \le C\, e^{-\delta\, t} \quad \forall\, t > 0$$

As a function of M, δ is such that $\lim_{M\to 0_+} \delta(M) = 1$

The condition $M \le 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- *Spectral gap* of a linearised operator $\mathcal{L}$

A more general functional framework

Same results in the more general framework

$$n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2))$$
$$n \, \log n \,, \; n \, |x|^2 \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2))$$
$$2\nabla\sqrt{n} + x\,\sqrt{n} - \sqrt{n}\,\nabla c \in L^1(\mathbb{R}^+, L^2(\mathbb{R}^2))$$

an approach based on standard but tedious truncation methods, see [Blanchet, JD, Perthame]

The parabolic-elliptic Keller and Segel system

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Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx) \,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \,, \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\,\pi$$

First step: the trap

Decay Estimates of $u(t)$ in $L^\infty(\mathbb{R}^2)$

Lemma 2. *For any* $M < M_1$*, there exists* $C = C(M)$ *such that, for any solution* $u \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}^+_{\text{loc}} \times \mathbb{R}^2)$

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C\, t^{-1} \quad \forall\, t > 0$$

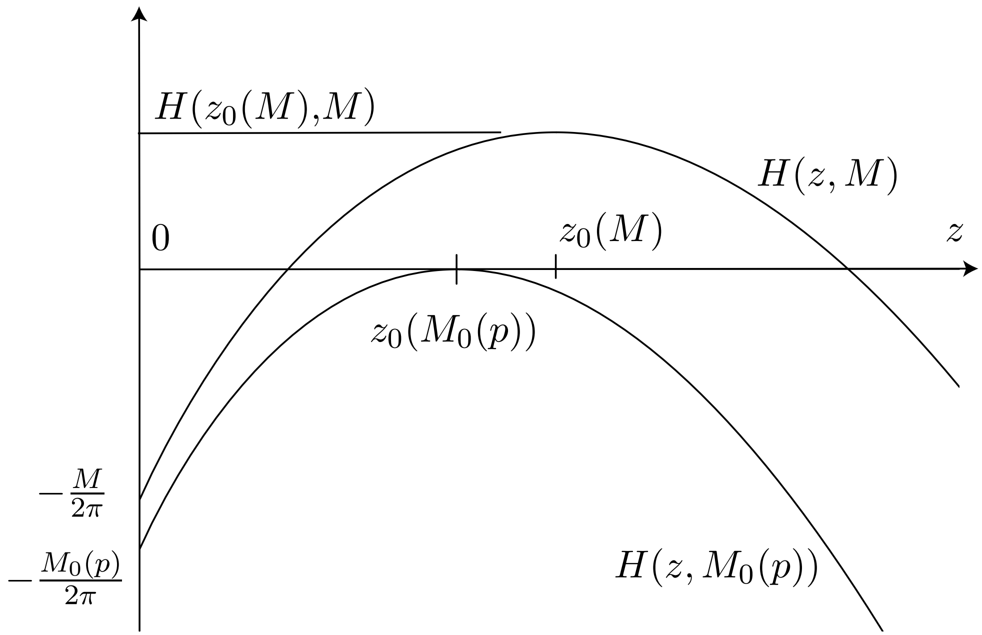
The *method of the trap...* prove that

$$H(\psi(t), M) \leq 0 \quad \text{where} \quad \psi(t) := t\, \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)}$$

where $z \mapsto H(z, M)$ is a continuous function which is
- negative on $[0, z_1)$
- positive on (z_1, z_2) for some z_1, z_2 such that $0 < z_1 < z_2 < \infty$

ψ is continuous and $\psi(0) = 0 \Longrightarrow \psi(t) \leq z_1 \leq z_0(M)$ for any $t \geq 0$ if

$H(z_0(M), M) = \sup_{z \in [z_1, z_2]} H(z, M) \geq 0$



The *method of the trap* amounts to prove that $H(z,M) \leq 0$ implies that $z = \psi(t)$ is bounded by $z_0(M)$ as long as $H(z_0(M),M) > 0$

Duhamel's formula:

$$u(x,t_0+t) - \int_{\mathbb{R}^2} N(x-y,t)\,u(y,t_0)\;dy$$
$$= \int_0^t \int_{\mathbb{R}^2} N(x-y,t-s)\;\nabla\cdot\left[u(y,t_0+s)\,\nabla v(y,t_0+s)\right]\;dy\;ds$$

where $N(x,t) = \frac{1}{4\pi t}\,e^{-|x|^2/(4t)}$. Let $\kappa_\sigma = \left\|\partial N/\partial x_i\,(\cdot,1)\right\|_{L^\sigma(\mathbb{R}^2)}$

$$\|u(\cdot,t_0+t)\|_{L^\infty(\mathbb{R}^2)} - \frac{1}{4\pi t}\,\|u(\cdot,t_0)\|_{L^1(\mathbb{R}^2)}$$
$$\leq \sum_{i=1,2} \int_0^t \left\| \frac{\partial N}{\partial x_i}(\cdot,t-s) * \left[\left(u\,\frac{\partial v}{\partial x_i}\right)(\cdot,t_0+s)\right]\right\|_{L^\infty(\mathbb{R}^2)}\;ds$$
$$\leq \sum_{i=1,2} \kappa_\sigma \int_0^t (t-s)^{-(1-\frac{1}{\sigma})-\frac{1}{2}} \left\| \left(u\,\frac{\partial v}{\partial x_i}\right)(\cdot,t_0+s)\right\|_{L^\rho(\mathbb{R}^2)}\;ds$$

HLS inequality + Hölder and take $t_0 = t$

$$2t\ \|u(\cdot,2t)\|_{L^\infty(\mathbb{R}^2)} - \frac{M}{2\pi}$$
$$\leq \frac{2\,\kappa_\sigma\,C_{\rm HLS}}{\pi}\,M^{\frac{1}{p}+\frac{1}{r}}\,t\int_0^t (t-s)^{\frac{1}{\sigma}-\frac{3}{2}}\,(t+s)^{\frac{1}{p}+\frac{1}{r}-2}\left[\psi(t)\right]^{2-\frac{1}{p}-\frac{1}{r}}ds$$

with $\psi(t) := \sup_{0\leq s\leq t} 2s\ \|u(\cdot,2s)\|_{L^\infty(\mathbb{R}^2)}$ and

$$t\int_0^t (t-s)^{\frac{1}{\sigma}-\frac{3}{2}}\,(t+s)^{\frac{1}{p}+\frac{1}{r}-2}ds \ = \ \frac{\sigma}{2-\sigma}$$

$\psi(t) \leq \dfrac{M}{2\pi} + C_0\left(\psi(t)\right)^\theta$ with $C_0 = \dfrac{2\,\kappa_\sigma\,C_{\rm HLS}}{\pi}\,M^{\frac{1}{p}+\frac{1}{r}}\,\dfrac{\sigma}{2-\sigma}\,,\quad \theta = 2 - \dfrac{1}{p} - \dfrac{1}{r}$

Choice: $H(z,M) = z - C_0\,z^\theta - M/(2\pi)$

How small is the mass ?

The exponents σ, ρ, p, q and r are related by

$$\begin{cases} \frac{1}{\sigma} + \frac{1}{\rho} = 1\,, & 1 < \sigma < 2 \\ \frac{1}{p} + \frac{1}{q} = \frac{1}{\rho}\,, & p\,,\; q > 2 \\ \frac{1}{r} - \frac{1}{q} = \frac{1}{2}\,, & r > 1 \end{cases}$$

For the choice $r = 4/3$, $q = 4$, $C_{\mathrm{HLS}} = 2\sqrt{\pi}$

$$C_0 = \tfrac{4\kappa_\sigma}{\sqrt{\pi}}\, M^{\frac{1}{p}+\frac{1}{4}}\, \frac{\sigma}{2-\sigma} \text{ with } \sigma = \tfrac{4p}{3p-4}$$

... there exists $M_0(p)$ such that $H(z_0(M), M) > 0$ if and only if $M < M_0(p)$

and $\sup_{p\in(4,+\infty)} M_0(p) = \lim_{p\to+\infty} M_0(p) \approx 0.822663 < 8\pi \approx 25.1327$

Other norms: interpolation

Corollary 3. *For any mass* $M < M_1$ *and all* $p \in [1, \infty]$*, there exists a positive constant* $C = C(p, M)$ *with* $\lim_{M \to 0_+} C(p, M) = 0$*, such that*

$$\|u(t)\|_{L^p(\mathbb{R}^2)} \leq C\, t^{-(1-\frac{1}{p})} \quad \forall\, t > 0$$

Remark 1*.* The above rates are optimal as can easily be checked using the self-similar solutions $(n_\infty,\, c_\infty)$

Second step: weighted H^1 estimates

L^p and H^1 estimates in the self-similar variables

Consider the solution of

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

For any $p \in (1, \infty]$

$$\|n(t)\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \forall\, t > 0$$

for some positive constant C_1, and for $p > 2$

$$2\pi\ \|\nabla c(t)\|_{L^\infty} \leq \underbrace{\sup_{x\in\mathbb{R}^2} \int_{|x-y|\geq 1} \frac{n(t,y)}{|x-y|}\,dy}_{\leq M} + \underbrace{\sup_{x\in\mathbb{R}^2} \int_{|x-y|\leq 1} \frac{n(t,y)}{|x-y|}\,dy}_{\leq \left(2\pi\,\frac{p-1}{p-2}\right)^{\frac{p}{p-1}}\ \|n\|_{L^p(\mathbb{R}^2)}}$$

$$\|n(t)\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \text{and} \quad \|\nabla c(t)\|_{L^\infty(\mathbb{R}^2)} \leq C_2 \quad \forall\, t > 0$$

Lemma 4. *The constants C_1 and C_2 depend on M and are such that*

$$\lim_{M \to 0_+} C_i(M) = 0 \quad i = 1\,,\ 2$$

Exponential weights

With $K = K(x) = e^{|x|^2/2}$, let us rewrite the equation for n as

$$\frac{\partial n}{\partial t} - \frac{1}{K} \nabla \cdot (K \, \nabla n) = -\nabla c \cdot \nabla n + 2n + n^2$$

Proposition 5. *For any mass $M \in (0, M_1)$, there is a positive constant C such that*

$$\|n(t)\|_{H^1(K)} \leq C \quad \forall \, t > 0$$

First ingredient [M. Escobedo and O. Kavian]: for any $q > 2$ and $\varepsilon > 0$, there exists a positive constant $C(\varepsilon, q)$ such that

$$\int_{\mathbb{R}^2} n^2 \, K \, dx \leq \varepsilon \int_{\mathbb{R}^2} |\nabla n|^2 \, K \, dx + C(\varepsilon, q) \, \|n\|^2_{L^q(\mathbb{R}^2)}$$

$L^2(K)$ estimate

Multiply the equation by $n\,K$ and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|n|^2\,K\,dx+\int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx$$
$$=-\int_{\mathbb{R}^2}n\nabla c\cdot\nabla n\,K\,dx+2\int_{\mathbb{R}^2}n^2\,K\,dx+\int_{\mathbb{R}^2}n^3\,K\,dx$$
$$\leq\varepsilon\int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx+C$$

and so

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2}|n|^2\,K\,dx+(1-\varepsilon)\underbrace{\int_{\mathbb{R}^2}|\nabla n|^2\,K\,dx}_{\geq\int_{\mathbb{R}^2}|n|^2\,K\,dx}\leq C$$

(expand the square in $\int_{\mathbb{R}^2}|\nabla(n\,K)|^2\,K^{-1}\,dx\geq 0$)

$H^1(K)$ estimate (1/2)

Let $S(t)$ be the semi-group generated by $-K^{-1}\nabla\cdot(K\ \nabla\cdot)$ on $L^2(K)$

$$n(t,x) = S(t)\,n_0(x) - \int_0^t S(t-s)\,(\nabla c\cdot\nabla n)(s)\,ds + \int_0^t S(t-s)\,(2n+n^2)(s)\,ds$$

$$\|n(t)\|_{H^1(K)} - \|S(t)\,n_0\|_{H^1(K)}$$
$$\leq \int_0^t \|S(t-s)\,(\nabla c\cdot\nabla n)(s)\|_{H^1(K)}\,ds + \int_0^t \|S(t-s)\,(2n+n^2)(s)\|_{H^1(K)}\,ds$$

Second ingredient: $\|S(t)\,h\|_{H^1(K)} \leq \kappa\,(1+t^{-1/2})\,\|h\|_{L^2(K)}$

$$\frac{1}{\kappa}\left(\|n(t)\|_{H^1(K)} - \|S(t)\,n_0\|_{H^1(K)}\right)$$
$$\leq C_2\int_0^t \left(1+\tfrac{1}{\sqrt{t-s}}\right)\|\nabla n(s)\|_{L^2(K)}\,ds + (2+C_1)\int_0^t \left(1+\tfrac{1}{\sqrt{t-s}}\right)\|n(s)\|_{L^2(K}$$

$H^1(K)$ estimate (2/2)

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)} \leq \left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1 + C_3\int_0^t \left(1+\tfrac{1}{\sqrt{t-s}}\right)\|n(s+\tau)\|_{H^1(K)}\,ds$$

Let $H(T) = \sup_{t\in(0,T)} \int_0^t \left(1+\frac{1}{\sqrt{t-s}}\right)\|n(s+\tau)\|_{H^1(K)}\,ds$ and choose $T>0$ such that $\frac{1}{2\kappa} = C_3\int_0^T\left(1+\frac{1}{\sqrt{T-s}}\right)ds = C_3\left(T+2\sqrt{T}\right)$

$$\frac{1}{\kappa}\,H(T) \leq \left(\pi+4\sqrt{T}+T\right)\,C_1 + \frac{1}{2\kappa}\,H(T) \quad\Longrightarrow\quad H(T)\leq 2\left(\pi+4\sqrt{T}+T\right)\kappa\, C$$

For any $t\in(0,T)$

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)} \leq \left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1 + C_3\,H(T) \leq \left(1+\tfrac{1}{\sqrt{t}}\right)\,C_1 + 2\left(\pi+4\sqrt{T}+T\right)$$

Conclusion: bound in $H^1(K)$

The estimate

$$\frac{1}{\kappa}\,\|n(t+\tau)\|_{H^1(K)} \leq \left(1+\frac{1}{\sqrt{t}}\right)\,C_1+C_3\,H(T) \leq \left(1+\frac{1}{\sqrt{t}}\right)\,C_1+2\left(\pi+4\sqrt{T}+T\right)$$

for any $t \in (0,T)$ gives a bound on $\|n(T+\tau)\|_{H^1(K)}$ for any $\tau > 0$

Lemma 6.

$$\|n(t)\|_{H^1(K)} \leq C\,\max\left\{1,\frac{\sqrt{T}}{\sqrt{t}}\right\} \quad \forall\, t>0$$

Actually $n(t)$ can be bounded also in $H^1(n_\infty^{-1})$ but further estimates are needed...

Third step: linearization and spectral gap

A linearized operator

Introduce f and g defined by

$$n(x,t) = n_\infty(x)(1+f(x,t)) \qquad \text{and} \qquad c(x,t) = c_\infty(x)(1+g(x,t))$$

(f,g) is solution of the non-linear problem

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mathcal{L}(t,x,f,g) = -\dfrac{1}{n_\infty} \nabla \cdot [f\, n_\infty \nabla (g\, c_\infty)] & x \in \mathbb{R}^2\,,\ t>0 \\ -\Delta(c_\infty\, g) = f\, n_\infty & x \in \mathbb{R}^2\,,\ t>0 \end{cases}$$

where $\mathcal{L}$ is the linear operator given by

$$\mathcal{L}(t,x,f,g) = \frac{1}{n_\infty} \nabla \cdot [n_\infty \nabla (f - g\, c_\infty)]$$

The conservation of mass is replaced here by $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$

A spectral gap estimate

Proposition 7. *For any* $M \in (0, M_2)$, *for any* $f \in H^1(n_\infty\, dx)$ *such that*

$$\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0 \quad \Longrightarrow \quad \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx \geq \Lambda(M) \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

for some $\Lambda(M) > 0$ *and* $\lim_{M \to 0_+} \Lambda(M) = 1$

Let $h = \sqrt{n_\infty}\, f = \sqrt{\lambda}\, e^{-|x|^2/4 + c_\infty/2}\, f$

$$\lambda\, |\nabla f|^2\, n_\infty = |\nabla h|^2 + \frac{|x|^2}{4}\, h^2 + \frac{1}{4}\, |\nabla c_\infty|^2\, h^2 + h\, \nabla h \cdot (x - \nabla c_\infty) - \frac{1}{2}\, x \cdot \nabla c_\infty\, h^2$$

(integrations by parts)

$$\int_{\mathbb{R}^2} h\, \nabla h \cdot x\, dx = -\int_{\mathbb{R}^2} h^2\, dx$$

$$\int_{\mathbb{R}^2} h\, \nabla h \cdot \nabla c_\infty\, dx = \tfrac{1}{2} \int_{\mathbb{R}^2} h^2\, (-\Delta c_\infty)\, dx \leq \tfrac{1}{2}\, \|n_\infty\|_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^2} h^2\, dx$$

$$\tfrac{1}{2} \int_{\mathbb{R}^2} x \cdot \nabla c_\infty\, h^2\, dx \leq \tfrac{\sigma^2 - 1}{\sigma^2} \int_{\mathbb{R}^2} \tfrac{|x|^2}{4}\, h^2\, dx + \tfrac{1}{4}\, \tfrac{\sigma^2}{\sigma^2 - 1} \int_{\mathbb{R}^2} |\nabla c_\infty|^2\, h^2\, dx$$

$H^1(n_\infty^{-1})$ estimate

Assume that $n_0/n_\infty \in L^2(n_\infty)$
There exists a constant $C > 0$ such that

$$|x| > 1 \quad \Longrightarrow \quad \Big| c_\infty + M/(2\pi) \log |x| \Big| \le C$$

$n_\infty \, K = e^{c_\infty}$ behaves like $O(|x|^{-M/(2\pi)})$ as $|x| \to \infty$

$$\frac{\partial n}{\partial t} - n_\infty \, \nabla \cdot \left(\frac{1}{n_\infty} \, \nabla n \right) = (\nabla c_\infty - \nabla c) \cdot \nabla n + 2n + n^2$$

Corollary 8. *If $M < M_2$, then any solution n is bounded in*

$$L^\infty(\mathbb{R}^+, L^2(n_\infty^{-1}\, dx)) \cap L^\infty((\tau, \infty), H^1(n_\infty^{-1}\, dx))$$

for any $\tau > 0$

Fourth step: collecting the estimates

Proof of the exponential rate of convergence

$$\begin{cases} \dfrac{\partial f}{\partial t} - \mathcal{L}(t,x,f,g) = -\dfrac{1}{n_\infty} \nabla \cdot [f\, n_\infty \nabla\, (g\, c_\infty)] & x \in \mathbb{R}^2\,,\ t > 0 \\ -\Delta (c_\infty\, g) = f\, n_\infty & x \in \mathbb{R}^2\,,\ t > 0 \end{cases}$$

Multiply by $f\, n_\infty$ and integrate by parts

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx + \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx$$
$$= \underbrace{\int_{\mathbb{R}^2} \nabla f \cdot \nabla\, (g\, c_\infty)\; n_\infty\; dx}_{=\mathrm{I}} + \underbrace{\int_{\mathbb{R}^2} \nabla f \cdot \nabla (g\, c_\infty)\; f\, n_\infty\; dx}_{=\mathrm{II}}$$

Cauchy-Schwarz' inequality

$$\mathrm{I} = \int_{\mathbb{R}^2} \nabla f \cdot \nabla\, (g\, c_\infty)\; n_\infty\; dx \le \|\nabla f\|_{L^2(n_\infty\, dx)}\; \|\nabla (g\, c_\infty)\|_{L^2(n_\infty\, dx)}$$

first term

Hölder's inequality (with $q > 2$)

$$\|\nabla(g\,c_\infty)\|_{L^2(n_\infty\,dx)} \leq M^{1/2-1/q}\,\|n_\infty\|^{1/q}_{L^\infty(\mathbb{R}^2)}\,\|\nabla(g\,c_\infty)\|_{L^q(\mathbb{R}^2)}$$

HLS inequality (with $1/p = 1/2 + 1/q$)

$$\|\nabla(g\,c_\infty)\|_{L^q(\mathbb{R}^2)} \leq \frac{1}{2\pi}\left(\int_{\mathbb{R}^2}\left|(f\,n_\infty) * \tfrac{1}{|\cdot|}\right|^q dx\right)^{\frac{1}{q}} \leq \frac{C_{\mathrm{HLS}}}{2\pi}\,\|f\,n_\infty\|_{L^p(\mathbb{R}^2)}$$

Hölder's inequality: $\|f\,n_\infty\|_{L^p(\mathbb{R}^2)} \leq \|f\|_{L^2(n_\infty\,dx)}\;\|n_\infty\|^{1/2}_{L^{q/2}(\mathbb{R}^2)}$

$$\mathrm{I} = \int_{\mathbb{R}^2}\nabla f\cdot\nabla(g\,c_\infty)\,f\,n_\infty\,dx \leq C_*(M)\,\|f\|_{L^2(n_\infty\,dx)}\,\|\nabla f\|_{L^2(n_\infty\,dx)}$$

$C_*(M) := C_{\mathrm{HLS}}\,(2\pi)^{-1}\,M^{1/2-1/q}\;\|n_\infty\|^{1/2}_{L^{q/2}(\mathbb{R}^2)}\;\|n_\infty\|^{1/q}_{L^\infty(\mathbb{R}^2)} \to 0$ as $M \to 0$

second term and conclusion

Use $g\,c_\infty = c - c_\infty$ and the Cauchy-Schwarz inequality

$$\int_{\mathbb{R}^2} \nabla f \cdot \nabla (g\,c_\infty)\, f\, n_\infty\, dx \le \underbrace{\|\nabla c - \nabla c_\infty\|_{L^\infty(\mathbb{R}^2)}}_{\le 2\, C_2(M) \searrow 0} \|f\|_{L^2(n_\infty\, dx)} \|\nabla f\|_{L^2(n_\infty\, dx)}$$

Spectral gap estimate

$$\underbrace{\sqrt{\Lambda(M)}}_{\to 1} \|f\|_{L^2(n_\infty\, dx)} \le \|\nabla f\|_{L^2(n_\infty\, dx)}$$

With $\gamma(M) := \frac{C_*(M) + 2\, C_2(M)}{\sqrt{\Lambda(M)}} \searrow 0$,

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx \le -\,[1 - \gamma(M)] \int_{\mathbb{R}^2} |\nabla f|^2\, n_\infty\, dx$$

Uniqueness

If n_1 and n_2 are two solutions in $C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ for any $\tau > 0$, with $f = (n_2 - n_1)/n_\infty$ we also get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx \leq -\left[1-\gamma(M)\right]\Lambda(M) \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

As a consequence, if the initial condition is the same, then $n_1 = n_2$