Nonlinear diffusion equations as diffusion limits of kinetic equations
An application in astrophysics

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Motivations (1)

(1) From global Gibbs states (energy profiles) to local Gibbs states ... when relaxation phenomena occur on a short time scale (collisions). A very standard assumption in semiconductor theory (Fermi-Dirac distributions) or in stellar dynamics (polytropic distribution functions)

(2) Non monotone energy profiles means nonlinear instabilities. Monotonically decreasing energy profile provides a convex Lyapunov functional (total energy + convex nonlinear entropy)... characterization of the global Gibbs state as its unique minimizer

(3) Goal : derive the nonlinear diffusion limit consistently with the Gibbs state : consider a 'projection' onto the local Gibbs state with the same spatial density (local Lagrange multiplier = chemical potential). The collision kernel is a relaxation-time kernel

Motivations (2)

- collisions : short time scale
- Gibbs states are usually better known than collision kernels
- Gibbs states $\Longleftrightarrow$ generalized entropies
- nonlinear diffusion equations are difficult to justify directly
- global Gibbs states have the same macroscopic density at the kinetic / diffusion levels
- they have the 'same' Lyapunov functionals

[Ben Abdallah, J.D.], [Chavanis, Laurençot, Lemou], [Degond, Ringhofer]

Models

Power law Gibbs states are well known

- in astrophysics : [Binney-Tremaine,Guo-Rein,Chavanis et al.]
- in two dimensional turbulence models

$$\gamma(E) = (E_2 - E)_+^k$$

... compactly supported solutions

Fermi-Dirac statistics :

- semiconductors models : [Goudon-Poupaud]

$$\gamma(E) = \frac{1}{\alpha + e^E}$$

First, we consider only the three-dimensional case

Equations

$$\epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f - \epsilon \nabla_x V(x) \cdot \nabla_v f = Q[f] \qquad (1)$$

$$Q[f] := G_f - f$$

$$G_f := \gamma\Big(\frac{1}{2}|v|^2 + V(x) - \mu_{\rho_f}(x,t)\Big)$$

Local Fermi level : μ_{ρ_f} is implicitly determined by the condition

$$\int_{\mathbb{R}^3} G_f \, dv = \rho_f := \int_{\mathbb{R}^3} f \, dv$$

The collision operator can be rewritten as

$$\mu_{\rho_f}(x,t) = V(x) + \bar{\mu}(\rho_f(x,t)) \quad \text{and} \quad \int_{\mathbb{R}^3} \gamma\Big(\frac{1}{2}|v|^2 - \bar{\mu}(\rho_f)\Big)\, dv = \rho_f$$

$$Q[f] = G_f - f\,, \quad G_f = \gamma\Big(\frac{1}{2}|v|^2 - \bar{\mu}(\rho_f)\Big)$$

Assumptions on the energy profile

The energy profile $\gamma : (E_1, E_2) \to \mathbb{R}_+$ is a nonincreasing, nonnegative $\mathcal{C}^1$ function, $-\infty \le E_1 < E_2 \le \infty$, $\lim_{E \to E_2} \gamma(E) = 0$. If $E_2 < \infty$, we extend γ to $[E_2, \infty)$ by 0 and assume that there are constants $k > 0$ and $C > 0$ such that

$$\gamma(E) \le C(E_2 - E)^k \quad \text{on} \quad (\hat{E}, E_2)$$

If $E_2 = \infty$ we require

$$\gamma(E) = O(E^{-5/2}) \quad \text{as} \quad E \to \infty$$

to ensure the existence of second velocity moments

+ technical assumptions on γ close to E_2

Formal asymptotics as $\epsilon \to 0$ (1)

$$f = \sum_{i=0}^{\infty} f^i \epsilon^i \quad \rho^i := \int_{\mathbb{R}^3} f^i \, dv \quad \rho = \sum_{i=1}^{\infty} \rho^i \epsilon^i$$

Let $G^i := \mathrm{sign}(\rho^i)\, \gamma(|v|^2/2 - \bar{\mu}(|\rho^i|))$, $G_f \approx \sum_{i=1}^{\infty} G^i \epsilon^i$, $\mu^0 := \mu_{f^0}$

$$\epsilon^0 : \qquad G^0(x,v,t) = G_{f^0} = \gamma(|v|^2/2 + V(x) - \mu^0(x,t)) = f^0$$

$$\epsilon^1 : \qquad v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = G^1 - f^1$$

$$\epsilon^2 : \qquad \partial_t f^0 + v \cdot \nabla_x f^1 - \nabla_x V \cdot \nabla_v f^1 = G^2 - f^2$$

$$\partial_t \int_{\mathbb{R}^3} f^0 \, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f^1 dv = O(\epsilon)$$

$$f^1 = v \cdot \nabla_x \mu^0 \, \gamma' \left(\frac{1}{2} v^2 + V(x) - \mu^0(x,t) \right) + G^1$$

Formal asymptotics as $\epsilon \to 0$ (2)

$$G_{f^0} = f^0 \,, \qquad \mu^0 = \mu_{f^0} = V + \bar\mu(\rho^0)$$

$$f^1 = v \cdot \nabla_x \mu^0 \, \gamma'\Big(\frac{1}{2}v^2 + V(x) - \mu^0(x,t)\Big) + G^1$$

$$\int_{\mathbb{R}^3} v f^1 \, dv = -\rho^0 \nabla_x \mu^0$$

$$\partial_t \int_{\mathbb{R}^3} f^0 \, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f^1 dv = O(\epsilon)$$

Collecting these estimates, we get, for $\rho^0(x,t) = \int_{\mathbb{R}^3} f^0(x,v,t)\, dv$,

$$\partial_t \rho^0 = \nabla \cdot (\rho^0 \nabla \mu^0)$$

Use : $\mu^0 = \bar\mu(\rho^0) + V$ to recover the expected drift-diffusion equation :

$$\partial_t \rho^0 = \nabla \cdot \Big[\rho^0 (\nabla \bar\mu(\rho^0) + \nabla V(x))\Big] \qquad (2)$$

An explicit example : porous media

$$\epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f - \epsilon \nabla_x V(x) \cdot \nabla_v f = G_f - f$$

$$G_f := \gamma\left(\frac{1}{2}|v|^2 + V(x) - \mu_{\rho_f}(x,t)\right) = \gamma\left(\frac{1}{2}|v|^2 - \bar{\mu}_{\rho_f}(x,t)\right)$$

$$\gamma(E) = (-E)_+^k$$

$$\bar{\mu}(\rho) = Const \cdot \rho^{\frac{1}{k+3/2}} \quad \text{where} \quad \alpha(k) = 4\pi\sqrt{2}\int_0^1 \sqrt{u}(1-u)^k \, du$$

$$\partial_t \rho^0 = \nabla \cdot \left[\rho^0 (\nabla \bar{\mu}(\rho^0) + \nabla V(x))\right]$$

$\nu(\rho) := \int_0^\rho s\, \bar{\mu}'(s)\, ds = \Theta \rho^m$ with $m = \frac{2k+5}{2k+3}$

$$\partial_t \rho = \nabla \cdot \left(\Theta \nabla \left(\rho^m\right) + \rho \nabla V\right)$$

Example 1. Power law case

$\gamma(E) := DE^{-k}$, $D > 0$, $k > 5/2$ (existence of second velocity moments)

$$\bar{\mu}(\rho) = -\left(\frac{\rho}{D\beta(k)}\right)^{\frac{1}{\frac{3}{2}-k}}, \quad \text{where} \quad \beta(k) := 4\pi\sqrt{2}\int_0^\infty \frac{\sqrt{s}}{(s+1)^k}\,ds$$

Fast diffusion equations :

$$\partial_t \rho = \nabla\cdot\left(\Theta\nabla(\rho^{\frac{k-5/2}{k-3/2}}) + \rho\nabla V\right), \quad \text{where} \quad \Theta := \frac{1}{k-\frac{5}{2}}\left(\frac{1}{D\beta(k)}\right)^{\frac{1}{3/2-k}}$$

Outside of a finite ball, the potential has to grow faster than a power

$$V(x) \geq C|x|^q, \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > \frac{3}{k-5/2}$$

Exemple 2. Maxwell distribution

$\gamma(E) = \exp(-E)$

$$\bar{\mu}(\rho) = \log \rho - \frac{3}{2} \log(2\pi)$$

Linear drift-diffusion equation : $\nu(\rho) = \rho$

$$\partial_t \rho = \nabla \cdot \big(\nabla \rho + \rho \nabla V \big)$$

”the linear case“

Growth assumption on the potential

$$V(x) \geq q \log(|x|), \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > 3$$

Example 3. Power law with positive exponent

Let γ be a cut-off power with positive exponent :

$$\gamma(E) = (E_2 - E)_+^k := \begin{cases} D(E_2 - E)^k & \text{for} \quad E < E_2, \quad D > 0, \quad k > 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\mu}(\rho) = \left(\frac{\rho}{D\alpha(k)}\right)^{\frac{1}{k+\frac{3}{2}}} - E_2 \,, \quad \text{where} \quad \alpha(k) = 4\pi\sqrt{2}\int_0^1 \sqrt{u}(1-u)^k \, du$$

Porous medium equations : $\nu(\rho) = \Theta \rho^{\frac{2k+5}{2k+3}}$

$$\partial_t \rho = \nabla \cdot \left(\Theta \nabla \left(\rho^{\frac{k+5/2}{k+3/2}}\right) + \rho \nabla V\right) \,, \quad \text{where} \quad \Theta := \frac{1}{k+\frac{5}{2}}\left(\frac{1}{D\alpha(k)}\right)^{\frac{1}{k+3/2}}$$

Growth condition on the potential : if μ^* is the upper bound for the Fermi energy

$$(E_2 + \mu^* - V(x))_+ = O\left(\frac{1}{|x|^q}\right) \quad \text{a.e. as} \quad |x| \to \infty, \quad q > \frac{3}{k + \frac{3}{2}}$$

This is an assumption which is compatible with the behavior of a self-consistent gravitational field

Example 4. Fermi-Dirac distribution

$\gamma(E) = \frac{1}{\exp(E)+\alpha}$

$$(\bar{\mu}^{-1})(\theta) = \frac{4\pi\sqrt{2}}{\alpha}\int_0^\infty \frac{\sqrt{p}\,dp}{\exp(p-\theta-\log\alpha)+1} = -\frac{(2\pi)^{\frac{3}{2}}}{\alpha}\,\mathrm{Li}_{3/2}\big(-\alpha\exp(\theta)\big)$$

$$\bar{\mu}(\rho) = \log\left(-\frac{1}{\alpha}(\mathrm{Li}_{3/2}^{-1})\big(-\frac{\alpha\rho}{(2\pi)^{3/2}}\big)\right), \quad \mathrm{Li}_n(z) := \sum_{k=1}^\infty \frac{z^k}{k^n}$$

Macroscopic equation : $\partial_t \rho = \nabla\cdot\Big(\big(D(\rho)\nabla\rho + \rho\nabla V\big)\Big)$

$$D(\rho) = \nu'(\rho) = \rho\,\bar{\mu}'(\rho) = \frac{-\alpha}{(2\pi)^{3/2}}\,\frac{\rho}{\mathrm{Li}_{1/2}\big((\mathrm{Li}_{3/2}^{-1})(\frac{-\alpha\rho}{(2\pi)^{3/2}})\big)}$$

Moreover the expansion of $D(\rho)$ at $\rho = 0$ gives

$$D(\rho) = 1 + \frac{\sqrt{2}}{4}\,\frac{\alpha\rho}{(2\pi)^{3/2}} + \left(\frac{3}{8} - \frac{2\sqrt{3}}{9}\right)\frac{\alpha^2\rho^2}{(2\pi)^3} + O(\rho^3)$$

Example 5. Bose-Einstein distribution

$\gamma(E) = \frac{1}{\exp(E)-\alpha}$

$$(\bar{\mu}^{-1})(\theta) = \frac{4\pi\sqrt{2}}{\alpha}\int_0^\infty \frac{\sqrt{p}\,dp}{\exp(p-\theta-\log\alpha)-1} = \frac{(2\pi)^{\frac{3}{2}}}{\alpha}\,\mathrm{Li}_{3/2}\big(\alpha\exp(\theta)\big)$$

$$\bar{\mu}(\rho) = \log\left(\frac{1}{\alpha}\,(\mathrm{Li}_{3/2}^{-1})\,\big(\frac{\alpha\rho}{(2\pi)^{3/2}}\big)\right)$$

Macroscopic equation : $\partial_t \rho = \nabla\cdot\Big(\big(D(\rho)\nabla\rho + \rho\nabla V\big)\Big)$

$$D(\rho) = \nu'(\rho) = \rho\bar{\mu}'(\rho) = \frac{\alpha}{(2\pi)^{3/2}}\,\frac{\rho}{\mathrm{Li}_{1/2}\big((\mathrm{Li}_{3/2}^{-1})(\frac{\alpha\rho}{(2\pi)^{3/2}})\big)}$$

Observe that $\lim_{\rho\to\bar{\rho}}\nu'(\rho) = 0$ ($\bar{\rho}$ is the maximal density) and
$\lim_{\rho\to 0}\nu'(\rho) = 1$

Expression of $\bar\mu$

$$Q[f] = G_f - f\,, \quad G_f = \gamma\Big(\frac{1}{2}|v|^2 - \bar\mu(\rho_f)\Big)$$

$\bar\mu^{-1} : (-E_2, -E_1) \to (0, \infty)$ is such that

$$(\bar\mu^{-1})(\theta) = 4\pi\sqrt{2}\int_0^\infty \gamma(p-\theta)\sqrt{p}\;dp$$

We extend $\bar\mu^{-1}$ by the value 0 on $(-\infty, -E_2)$. Differentiation with respect to θ leads to the Abelian equation

$$\frac{(\bar\mu^{-1})'(\theta)}{2\pi\sqrt{2}} = \int_{-\infty}^{\theta} \frac{\gamma(-q)}{\sqrt{\theta - q}}\;dq$$

and gives an explicit expression of γ in terms of $\bar\mu^{-1}$

$$\gamma(E) = \frac{1}{\sqrt{2}\;2\pi^2}\,\frac{d^2}{dE^2}\int_{-\infty}^{-E} \frac{(\bar\mu^{-1})(\theta)}{\sqrt{-E-\theta}}\;d\theta$$

References

Formal expansions (generalized Smoluchowski equation)

[Ben Abdallah, J.D.]
[Chavanis-Laurençot, Lemou]
[Chavanis et al.]

Rigorous justification with nonlinear diffusion limits

[J.D., P. Markowich, D. Ölz and C. Schmeiser]

Assumptions on the initial data

We assume that there is a constant *Fermi level* μ^* such that

$$0 \leq f_I(x,v) \leq f^*(x,v) := \gamma\Big(\frac{1}{2}|v|^2 + V(x) - \mu^*\Big) \quad \forall (x,v) \in \mathbb{R}^6$$

Maximal macroscopic density :

$$\bar{\rho} := \lim_{\theta \to -E_1^+} \int_{\mathbb{R}^3} \gamma\Big(\frac{1}{2}|v|^2 - \theta\Big)\, dv$$

If $\bar{\rho} < \infty$ we assume

$$\bar{\mu}^{-1}(\mu^*) \leq \bar{\rho}$$

Compatibility assumption on the potential

$\nabla_x V \in W^{1,\infty}\mathbb{R}^3)$ and V is bounded from below

$$\inf_{x\in\mathbb{R}^3} V(x) = V_{\text{min}} = 0$$

Confinement condition : with $f^*(x,v) := \gamma\Big(\frac{1}{2}|v|^2 + V(x) - \mu^*\Big)$

$$f^* \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \quad \text{and} \quad \iint_{\mathbb{R}^6} \Big(\frac{1}{2}|v|^2 + V(x)\Big)\, f^*(x,v)\, dv\, dx < \infty$$

Observe that this implies $\|f^*\|_{L^1} \geq \|f_I\|_{L^1} = M$

A *compatibility assumption :* Given a Gibbs state (a function γ), impose some minimal growth conditions on V.

Existence and uniqueness

Proposition 1. For any $p \in (1, \infty)$, Eq. (**??**) has a unique weak solution in

$$\mathcal{V} := \{ f \in \mathcal{C}(0, \infty; (L^1 \cap L^p)(\mathbb{R}^6)) : 0 \leq f \leq f^*,\ \forall t > 0 \text{ a.e.} \}$$

Proof : Cf. [Poupaud-Schmeiser, 1991] : define the map $f \mapsto \Gamma[f] = g$

$$\epsilon^2 \partial_t g + \epsilon\, v \cdot \nabla_x g - \epsilon\, \nabla_x V \cdot \nabla_v g = G_f - g$$

$$g(t = 0) = f_I$$

Γ maps $\mathcal{V}$ into itself and is a contraction for sufficiently small time intervals.

Free energy

$$\mathcal{F}[f] := \iint_{\mathbb{R}^6} \left[\left(\frac{1}{2}|v|^2 + V \right) f + \beta_\gamma[f] \right] dv\, dx$$

$$\beta_\gamma[f] := \int_0^f -\gamma^{-1}(s)\, ds$$

$-\gamma^{-1}$ is monotonically increasing $\Longrightarrow$ β_γ is a convex function

Microscopic energy associated to a distribution function f :

$$E_f(x,v,t) := \frac{1}{2}|v|^2 + V(x) - \mu_{\rho_f}(x,t) = \frac{1}{2}|v|^2 - \bar{\mu}\big(\rho_f(x,t)\big) = -\gamma^{-1}[G_f]$$

$$\epsilon^2 \frac{d}{dt} \mathcal{F}\big(f(.,.,t)\big) = \iint_{\mathbb{R}^6} (G_f - f)\big(\gamma^{-1}[G_f] - \gamma^{-1}[f]\big)\, dv\, dx := -D[f] \leq 0$$

Free energy and local Gibbs states (1)

$$\mathcal{F}_{\text{loc}}[f](x,t) := \int_{\mathbb{R}^3} \left[\left(\frac{1}{2}|v|^2 + V(x) - \mu_f(x,t) \right) f(x,v,t) + \beta_\gamma\big(f(x,v,t)\big) \right] dv$$

is convex. Minimum if and only if $f = G_f$

$$0 = \frac{1}{2}|v|^2 + V(x) - \mu_f(x,t) + \beta'_\gamma[f] = \frac{1}{2}|v|^2 + V(x) - \mu_f(x,t) - \gamma^{-1}[f]$$

Summarizing :

$$\mathcal{F}_{\text{loc}}[f](x,.,t) = \int_{\mathbb{R}^3} \left[\beta_\gamma(f) - \beta'_\gamma(G_f)(f - G_f) \right] dv + R(x,t)$$

$$\mathcal{F}[f](t) = \int_{\mathbb{R}^3} \mathcal{F}_{\text{loc}}[f](x,t)\, dx$$

Free energy of a local Gibbs states (2)

$$\mathcal{F}_{\rm loc}[f](x,t) \geq \mathcal{F}_{\rm loc}[G_f](x,t) \quad \text{and} \quad \mathcal{F} \geq \mathcal{F}[G_f]$$

If g is a local Gibbs state : $g(t,x,v) = \gamma\Big(\frac{1}{2}|v|^2 - \mu(x,t)\Big)$, then

$$\mathcal{F}[g] = \iint_{\mathbb{R}^6} \gamma\Big(\frac{1}{2}|v|^2 + V - \mu\Big)\Big(\mu - \frac{|v|^2}{3}\Big)\,dv\,dx = \int_{\mathbb{R}^3} \big(\mu\rho_g - \nu(\rho_g)\big)\,dx$$

with $\quad \rho_g = \bar\mu^{-1}(\mu - V) \quad$ and $\quad \nu(\rho_g) := \frac{1}{3}\int_{\mathbb{R}^3} |v|^2\gamma\Big(\frac{1}{2}|v|^2 - \bar\mu(\rho_g)\Big)\,dv$

$$\mathcal{F}[g] = \int_{\mathbb{R}^3} \big(\bar\mu(\rho_g)\rho_g - \nu(\rho_g) + V\rho_g\big)\,dx$$

Notice that if $\partial_t\rho = \nabla\cdot\big[\nabla\nu(\rho) + \rho\nabla V\big]$, then (Tsallis entropy)

$$\frac{d}{dt}\int_{\mathbb{R}^3} \big(\bar\mu(\rho)\rho - \nu(\rho) + V\rho\big)\,dx = -\int_{\mathbb{R}^3} \rho\,\big|\nabla\bar\mu(\rho) + \nabla V\big|^2\,dx$$

Consequences

If β_γ is convex, then $\mathcal{F}$ is bounded if V satisfies the compatibility assumption with γ

$$\epsilon^2 \frac{d}{dt}\mathcal{F}\big(f(.,.,t)\big) = \iint_{\mathbb{R}^6} (G_f - f)(\gamma^{-1}[G_f] - \gamma^{-1}[f]\big) \, dv \, dx := -D[f] \leq 0$$

$$\epsilon^2 \Big[\mathcal{F}\big(f(,.,.,t)\big) - \mathcal{F}\big(f_I\big)\Big] = -\int_0^T D[f](t) \, dt$$

To prove the convergence of f^ϵ to $G_{f^\epsilon} = \gamma(E_{f^\epsilon})$ consider the partition of the support of f^* according to

$$\Omega_+ := \left\{(x,v,t) \in \text{supp}\, f^* \subset \mathbb{R}^6 \times (0,T) : E_f = \frac{1}{2}|v|^2 - \bar{\mu}(\rho_f(x,t)) < E_2\right\}$$

$$\Omega_0 := \left\{(x,v,t) \in \text{supp}\, f^* \subset \mathbb{R}^6 \times (0,T) : E_f = \frac{1}{2}|v|^2 - \bar{\mu}(\rho_f(x,t)) \geq E_2\right\}$$

Main result

Theorem 2. *For any $\varepsilon > 0$, the equation has a unique weak solution $f^\varepsilon \in C(0,\infty; L^1 \cap L^p(\mathbb{R}^6))$ for all $p < \infty$. As $\varepsilon \to 0$, f^ε weakly converges to a local Gibbs state f^0 given by*

$$f^0(x,v,t) = \gamma\left(\frac{1}{2}|v|^2 + V(x) - \bar{\mu}(\rho(x,t))\right) \quad \forall\, (x,v,t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$$

where ρ is a solution of the nonlinear diffusion equation

$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho \nabla_x V(x))$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v)\,dv$

$$\nu(\rho) = \int_0^\rho s\,\bar{\mu}'(s)\,ds$$

Moreover $\int_{\mathbb{R}^3} f^\varepsilon\,dv$ strongly converges to ρ in L^p_{loc} as $\varepsilon \to 0$

Proof (1)

$$\Omega_+^{x,t} := \{v \in \mathbb{R}^3 : (x,v,t) \in \Omega_+\} \quad \text{and} \quad \Omega_0^{x,t} := \{v \in \mathbb{R}^3 : (x,v,t) \in \Omega_0\}$$

Notice that $\Omega_+^{x,t} = \mathbb{R}^3$ if $E_2 = \infty$

Lemma 1. *For any nonnegative function $f \le f^*$ there exists a constant, which does not depend on x and t, such that*

$$\int_{\Omega_+^{x,t}} v_i^{2m} \frac{G_f - f}{\gamma^{-1}[f] - E_f}\, dv \le \mathcal{M}\ ,$$

for any $m = 1, 2$, $i = 1, 2, 3$.

Proof (2)

Scaled perturbations of the first and second moments

$$j^{\epsilon} := \int_{\mathbb{R}^3} v \, \frac{f^{\epsilon} - G_{f^{\epsilon}}}{\epsilon} \, dv \quad \text{and} \quad \kappa^{\epsilon} := \int_{\mathbb{R}^3} v \otimes v \, \frac{f^{\epsilon} - G_{f^{\epsilon}}}{\epsilon} \, dv$$

Lemma 2. *For any bounded, open set $U \subset \mathbb{R}^3 \times [0, T)$, there are two constants $\mathcal{M}_U^1$ and $\mathcal{M}_U^2$, which do not depend on ϵ, such that*

$$\|j^{\epsilon}\|_{L^2_{x,t}(U)} \leq \mathcal{M}_U^1 \quad \textit{and} \quad \|\kappa^{\epsilon}\|_{L^2_{x,t}(U)} \leq \mathcal{M}_U^2 \quad \textit{as} \quad \epsilon \to 0 \, .$$

If $g(x,v,t) := \gamma(|v|^2/2 - \bar{\mu}(\rho(x,t)))$, then

$$\int_{\mathbb{R}^3} v \otimes v \, g \, dv = \nu(\rho) \, \mathrm{Id}^{3\times 3} \quad \text{where} \quad \nu(\rho) := \int_0^{\rho} \sigma \bar{\mu}'(\sigma) \, d\sigma$$

Proof (3)

$$\int_U \left(\int_{\Omega_+^{x,t}} |v|^m \frac{|f^\varepsilon - G_{f^\varepsilon}|}{\varepsilon} \, dv \right)^2 dx \, dt$$

$$\leq \int_U \left(\int_{\Omega_+^{x,t}} |v|^{2m} \frac{G_{f^\varepsilon} - f^\varepsilon}{\gamma^{-1}(f^\varepsilon) - E_{f^\varepsilon}} \, dv \right)$$

$$\cdot \left(\frac{1}{\varepsilon^2} \int_{\Omega_+^{x,t}} (G_{f^\varepsilon} - f^\varepsilon)(\gamma^{-1}(f^\varepsilon) - E_{f^\varepsilon}) \, dv \right) dx \, dt$$

$$\leq C_1$$

Proof (4)

Proposition 3. $\rho^\epsilon \to \rho^0$ in L^p_{loc} strongly for all $p \in (1,\infty)$.

Div-Curl Lemma as in [Goudon-Poupaud, 2001]. Integrate (**??**) with respect to dv and $v\,dv$

$$\begin{cases} \partial_t \rho^\epsilon + \nabla_x \cdot j^\epsilon = 0 \\ \epsilon^2 \partial_t j^\epsilon + \nabla_x \cdot \displaystyle\int_{\mathbb{R}^3} v \otimes v f^\epsilon \, dv = -j^\epsilon - \rho^\epsilon \nabla_x V \end{cases}$$

Split the second moments of f

$$\int_{\mathbb{R}^3} v \otimes v f^\epsilon \, dv = \int_{\mathbb{R}^3} v \otimes v G^{f^\epsilon} \, dv + \int_{\mathbb{R}^3} v \otimes v (f^\epsilon - G^{f^\epsilon}) \, dv = \nu(\rho^\epsilon) I^{3\times 3} + \epsilon \kappa^\epsilon$$

$$\begin{cases} \partial_t \rho^\epsilon + \nabla_x \cdot j^\epsilon = 0 \\ \nabla_x \nu(\rho^\epsilon) = -j^\epsilon - \rho^\epsilon \nabla_x V - \epsilon \nabla_x \cdot \kappa^\epsilon - \epsilon^2 \partial_t j^\epsilon \end{cases}$$

Proof (5)

Apply the Div-Curl Lemma to

$$U^\epsilon := (\rho^\epsilon, j^\epsilon), \quad V^\epsilon := (\nu(\rho^\epsilon), 0, 0, 0)$$

With $(\mathrm{curl}\, w)_{ij} = w^i_{x_j} - w^j_{x_i}$ and

$$\begin{cases} \mathrm{div}_{t,x} U^\epsilon = 0 \;, \\ (\mathrm{curl}_{t,x} V^\epsilon)_{1,2\ldots 4} = -j^\epsilon - \rho^\epsilon \nabla_x V - \epsilon \nabla_x \cdot \kappa^\epsilon - \epsilon^2 \partial_t j^\epsilon \end{cases}$$

we obtain the convergence of $U^{\epsilon_i} \cdot V^{\epsilon_i} = \rho^{\epsilon_i} \nu(\rho^{\epsilon_i})$

As in [Marcati-Milani, 1990], we deduce using Young measures that the convergence of ρ^{ϵ_i} is strong. The strict convexity assumption is replaced by the strict monotonicity of the function ν in $\rho\,\nu(\rho)$.

Application to a flat rotating system of gravitating particles

Preliminaries : a kinetic description

Consider the gravitational Vlasov-Poisson-Boltzmann system

$$\partial_t F + v \cdot \nabla_x F - \nabla_x \psi \cdot \nabla_v F = \mathcal{Q}_\omega(F)$$

where the potential ψ is given as a solution of the Poisson equation

$$\Delta \psi = \int_{\mathbb{R}^2 \times \mathbb{R}} F \, dv \, dw$$

the distribution function is concentrated on $\{((x,z),(v,w)) \in (\mathbb{R}^2 \times \mathbb{R}) \times (\mathbb{R}^2 \times \mathbb{R}) \; : \; z = 0 \,, \; w = 0\}$ and $\mathcal{Q}_\omega(F)$ is a collision kernel which depends on the angular velocity ω, to be specified later

$$\psi(t,x) = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} F(t,x,v) \, dv$$

Rotation at constant angular speed

Reduced problem in $\mathbb{R}^2$

$$(x,v) \mapsto (x\,e^{i\omega t}, (v+i\omega x)\,e^{i\omega t}) =: \mathcal{R}_{\omega,t}(x,v)$$

$$F(t,x,v) =: f(t, x\,e^{i\omega t}, (v+i\omega x)\,e^{i\omega t})) = f \circ \mathcal{R}_{\omega,t}(x,v)\ .$$

The equation satisfied by f can be written as

$$\partial_t f + v\cdot\nabla_x f + \omega^2\,x\cdot\nabla_v f + 2\,\mathrm{Re}\left(i\,\omega\,v\,\overline{\nabla_v f}\right) - \nabla_x\phi\cdot\nabla_v f = Q(f)$$

where the collision kernel Q is defined by $Q(f) := \mathcal{Q}_\omega(F)\circ\mathcal{R}_{\omega,t}^{-1}$ and the potential ϕ is given by

$$\phi(t,x) = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f(t,x,v)\,dv$$

in the rotating reference frame....

Written in cartesian coordinates, the equation satisfied by f is

$$\partial_t f + v \cdot \nabla_x f + \omega^2 \, x \cdot \nabla_v f + 2\,\omega\, v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = Q(f)$$

$$\phi = -\frac{1}{4\pi |x|} * \int_{\mathbb{R}^2} f \, dv$$

where

$a \wedge b := a^\perp \cdot b = (-a_2, a_1) \cdot (b_1, b_2) = a_1 b_2 - a_2 b_1 = \mathrm{Re}(i(a_1 + i\, a_2)(b_1 - i\, b_2))$

Local Gibbs state and collision kernel :

$$G_f(t,x,v) = \gamma\left(\frac{1}{2}\,|v|^2 + \phi(t,x) - \frac{1}{2}\,\omega^2\,|x|^2 + \mu_f(t,x)\right)$$

$$\int_{\mathbb{R}^2} G_f(t,x,v)\,dv = \int_{\mathbb{R}^2} f(t,x,v)\,dv\,, \qquad Q(f) = G_f - f$$

Polytropes

$$\partial_t f + v \cdot \nabla_x f + \omega^2\, x \cdot \nabla_v f + 2\,\omega\, v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = G_f - f$$

$$\phi = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f\; dv$$

For simplicity : case of the *polytropic gases,* or *polytropes :*

$$\gamma(s) := \left(\frac{-s}{k+1}\right)_+^k \quad \text{and} \quad \bar{\mu}(\rho) = -(k+1)\left(\frac{\rho}{2\pi}\right)^{\frac{1}{k+1}}$$

$$G(s) := \int_{\mathbb{R}^2} \gamma\Big(\frac{1}{2}\,|v|^2 - s\Big)\, ds = 2\pi\left(\frac{-s}{k+1}\right)_+^{k+1}$$

A priori estimates

Mass :

$$M = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f \, dx \, dv > 0$$

Free energy functional : with $\beta(s) = \int_s^0 \gamma^{-1}(\sigma) \, d\sigma$

$$\mathcal{F}[f] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left[f \left(\frac{1}{2} |v|^2 - \frac{1}{2} \omega^2 |x|^2 + \frac{1}{2} \phi \right) + \beta(f) \right] dx \, dv$$

is such that

$$\frac{d}{dt} \mathcal{F}[f(t, \cdot, \cdot)] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \Big(G_f - f \Big) \Big(\gamma^{-1}(G_f) - \gamma^{-1}(f) \Big) \, dx \, dv$$

Critical points

Local Lagrange multiplier

$$\mu_f(t,x) = \frac{1}{2}\,\phi - \frac{1}{2}\,\omega^2\,|x|^2 - \bar{\mu}(\rho)$$

"Global" Gibbs state (on a ball)

$$f^\infty(x,v) := \gamma\left(\frac{1}{2}\,|v|^2 + \phi^\infty(x) - \frac{1}{2}\,\omega^2\,|x|^2 - C\right)$$

with $\phi^\infty(x) := -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f^\infty(x,v)\,dv$

f^∞ is a critical point of $\mathcal{F}$ under the constraint

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} f^\infty(x,v)\,dx\,dv = M$$

Stationary solutions

$$\beta(f) = \frac{f^q}{q-1} \quad \text{with} \quad k = \frac{1}{q-1} \quad \Longleftrightarrow \quad q = 1 + \frac{1}{k}$$

$$\phi = -\frac{1}{4\pi|x|} * \rho \quad \text{with} \quad \rho = G\left(\phi - \frac{1}{2}\,\omega^2\,|x|^2 - C\right)$$

C is determined by the condition : $\int_{\mathbb{R}^2} \rho\, dx = M$

$$G(s) = 2\pi\left(-\frac{q-1}{q}\,s\right)^{\frac{q}{q-1}} \quad \Longleftrightarrow \quad -\frac{q}{q-1}\,\rho^{q-1} + \phi_{\rm eff} - C = 0$$

on the support of ρ, where the effective potential is

$$\phi_{\rm eff}(x) := -\frac{1}{4\pi|x|} * \rho - \frac{1}{2}\,\omega^2\,|x|^2\,.$$

Reduced variational problem and diffusion limit

Free energy of a local Gibbs state

$$\mathcal{F}[G_\rho] =: \mathcal{G}[\rho] \quad \text{with} \quad G_\rho(x,v) := \gamma\left(\frac{1}{2}\,|v|^2 + \bar{\mu}(\rho)\right)$$

Reduced variational problem takes the form

$$\mathcal{G}[\rho] = \int_{\mathbb{R}^2} \left[h(\rho) + \left(\phi(x) - \frac{1}{2}\,\omega^2\,|x|^2\right)\rho\right] dx$$

$$\begin{aligned} h(\rho) \quad &:= \quad \int_{\mathbb{R}^2} \left[(\beta\circ\gamma)\left(\frac{1}{2}\,|v|^2 + \bar{\mu}(\rho)\right) + \frac{1}{2}\,|v|^2\,\gamma\left(\frac{1}{2}\,|v|^2 + \bar{\mu}(\rho)\right)\right] dv \\ &= 2\pi \int_0^\infty \left[(\beta\circ\gamma)(s+\bar{\mu}(\rho)) + s\,\gamma(s+\bar{\mu}(\rho))\right] ds \end{aligned}$$

Polytropes : $h(\rho) = \frac{\kappa}{m-1}\,\rho^m$ with $m = 2 - \frac{1}{q}$

Results

$\omega = 0$: [Rein] Under the mass constraint, both functionals $\mathcal{F}$ and $\mathcal{G}$ have a radial minimizer
[Schaeffer] : the radial minimizer is unique
[J.D., Ben Abdallah,...], [J.D., J. Fernández] : dynamical stability holds for both models

$\omega \neq 0$: [J.D., J. Fernández] (work in progress)

Theorem 1. *For any $M > 0$, there exists an angular velocity $\tilde{\omega}(M)$ such that for any $\omega \in (0, \tilde{\omega}(M))$, there is a stationary solution, which is a minimizer of the localized energy. This solution is never radially symmetric*

Schwarz foliated symmetry
For any $M > 0$, there exists an angular velocity $\hat{\omega}(M)$ such that for any $\omega \in (0, \hat{\omega}(M))$, there is a radial stationary solution

Open questions

Dynamical stability for $\omega \neq 0$?

Systematic construction of stationary solutions with higher Morse indices ?

Dynamical stability of these solutions (at the diffusion level and at the kinetic level) ?