Nonlinear diffusion equations as diffusion limits of kinetic equations
An application in astrophysics

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Motivations (1)

(1) From global Gibbs states (energy profiles) to local Gibbs states ... when relaxation phenomena occur on a short time scale (collisions). A very standard assumption in semiconductor theory (Fermi-Dirac distributions) or in stellar dynamics (polytropic distribution functions).

(2) Non monotone energy profiles means nonlinear instabilities. Monotonically decreasing energy profile provides a convex Lyapunov functional (total energy + convex nonlinear entropy) ... characterization of the global Gibbs state as its unique minimizer.

(3) Goal : derive the nonlinear diffusion limit consistently with the Gibbs state: consider a ‘projection’ onto the local Gibbs state with the same spatial density (local Lagrange multiplier = chemical potential). The collision kernel is a relaxation-time kernel.
Motivations (2)

- collisions: short time scale
- Gibbs states are usually better known than collision kernels
- Gibbs states $\iff$ generalized entropies
- nonlinear diffusion equations are difficult to justify directly
- global Gibbs states have the same macroscopic density at the kinetic/diffusion levels
- they have the ‘same’ Lyapunov functionals

[Ben Abdallah, J.D.], [Chavanis, Laurençot, Lemou], [Degond, Ringhofer]
Models

Power law Gibbs states are well known
- in astrophysics : [Binney-Tremaine,Guo-Rein,Chavanis et al.]
- in two dimensional turbulence models

\[ \gamma(E) = (E_2 - E)^k \]

... compactly supported solutions

Fermi-Dirac statistics :
- semiconductors models : [Goudon-Poupaud]

\[ \gamma(E) = \frac{1}{\alpha + e^E} \]

First, we consider only the three-dimensional case
\[ \epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f - \epsilon \nabla_x V(x) \cdot \nabla_v f = Q[f] \]  

\[ Q[f] := G_f - f \]

\[ G_f := \gamma \left( \frac{1}{2} |v|^2 + V(x) - \mu_{\rho_f}(x, t) \right) \]

Local Fermi level: \( \mu_{\rho_f} \) is implicitly determined by the condition

\[ \int_{\mathbb{R}^3} G_f \, dv = \rho_f := \int_{\mathbb{R}^3} f \, dv \]

The collision operator can be rewritten as

\[ \mu_{\rho_f}(x, t) = V(x) + \bar{\mu}(\rho_f(x, t)) \quad \text{and} \quad \int_{\mathbb{R}^3} \gamma \left( \frac{1}{2} |v|^2 - \bar{\mu}(\rho_f) \right) \, dv = \rho_f \]

\[ Q[f] = G_f - f, \quad G_f = \gamma \left( \frac{1}{2} |v|^2 - \bar{\mu}(\rho_f) \right) \]
Assumptions on the energy profile

The energy profile \( \gamma : (E_1, E_2) \to \mathbb{R}_+ \) is a nonincreasing, nonnegative \( C^1 \) function, \(-\infty \leq E_1 < E_2 \leq \infty \), \( \lim_{E \to E_2} \gamma(E) = 0 \). If \( E_2 < \infty \), we extend \( \gamma \) to \([E_2, \infty)\) by 0 and assume that there are constants \( k > 0 \) and \( C > 0 \) such that

\[
\gamma(E) \leq C(E_2 - E)^k \quad \text{on} \quad (\hat{E}, E_2)
\]

If \( E_2 = \infty \) we require

\[
\gamma(E) = O(E^{-5/2}) \quad \text{as} \quad E \to \infty
\]

to ensure the existence of second velocity moments

+ technical assumptions on \( \gamma \) close to \( E_2 \)
Formal asymptotics as $\epsilon \to 0$ (1)

$$f = \sum_{i=0}^{\infty} f^i \epsilon^i \quad \rho^i := \int_{\mathbb{R}^3} f^i \, dv \quad \rho = \sum_{i=1}^{\infty} \rho^i \epsilon^i$$

Let $G^i := \text{sign}(\rho^i) \gamma(|v|^2/2 - \bar{\mu}(|\rho^i|))$, $G_f \approx \sum_{i=1}^{\infty} G^i \epsilon^i$, $\mu^0 := \mu_{f0}$

$$\epsilon^0 : \quad G^0(x, v, t) = G_{f0} = \gamma(|v|^2/2 + V(x) - \mu^0(x, t)) = f^0$$

$$\epsilon^1 : \quad v \cdot \nabla_x f^0 - \nabla_x V \cdot \nabla_v f^0 = G^1 - f^1$$

$$\epsilon^2 : \quad \partial_t f^0 + v \cdot \nabla_x f^1 - \nabla_x V \cdot \nabla_v f^1 = G^2 - f^2$$

$$\partial_t \int_{\mathbb{R}^3} f^0 \, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f^1 \, dv = O(\epsilon)$$

$$f^1 = v \cdot \nabla_x \mu^0 \gamma' \left( \frac{1}{2} v^2 + V(x) - \mu^0(x, t) \right) + G^1$$
Formal asymptotics as $\epsilon \to 0$ (2)

\[ G_{f^0} = f^0, \quad \mu^0 = \mu_{f^0} = V + \bar{\mu}(\rho^0) \]

\[ f^1 = v \cdot \nabla_x \mu^0 \gamma' \left( \frac{1}{2} v^2 + V(x) - \mu^0(x, t) \right) + G^1 \]

\[ \int_{\mathbb{R}^3} v f^1 \, dv = -\rho^0 \nabla_x \mu^0 \]

\[ \partial_t \int_{\mathbb{R}^3} f^0 \, dv + \nabla_x \cdot \int_{\mathbb{R}^3} v f^1 \, dv = O(\epsilon) \]

Collecting these estimates, we get, for $\rho^0(x, t) = \int_{\mathbb{R}^3} f^0(x, v, t) \, dv$,

\[ \partial_t \rho^0 = \nabla \cdot (\rho^0 \nabla \mu^0) \]

Use: $\mu^0 = \bar{\mu}(\rho^0) + V$ to recover the expected drift-diffusion equation:

\[ \partial_t \rho^0 = \nabla \cdot \left[ \rho^0 (\nabla \bar{\mu}(\rho^0) + \nabla V(x)) \right] \quad (2) \]
An explicit example: porous media

\[ \epsilon^2 \partial_t f + \epsilon v \cdot \nabla_x f - \epsilon \nabla_x V(x) \cdot \nabla_v f = G_f - f \]

\[ G_f := \gamma \left( \frac{1}{2} |v|^2 + V(x) - \mu_{\rho_f}(x, t) \right) = \gamma \left( \frac{1}{2} |v|^2 - \bar{\mu}_{\rho_f}(x, t) \right) \]

\[ \gamma(E) = (-E)^{+} \]

\[ \bar{\mu}(\rho) = \text{Const} \cdot \rho^{\frac{1}{k+3/2}} \quad \text{where} \quad \alpha(k) = 4\pi\sqrt{2} \int_0^1 \sqrt{u(1-u)^k} \, du \]

\[ \partial_t \rho^0 = \nabla \cdot \left[ \rho^0 (\nabla \bar{\mu}(\rho^0) + \nabla V(x)) \right] \]

\[ \nu(\rho) := \int_0^\rho s \bar{\mu}'(s) \, ds = \Theta \rho^m \quad \text{with} \quad m = \frac{2k+5}{2k+3} \]

\[ \partial_t \rho = \nabla \cdot \left( \Theta \nabla (\rho^m) + \rho \nabla V \right) \]
Example 1. Power law case

\( \gamma(E) := DE^{-k} \), \( D > 0, k > 5/2 \) (existence of second velocity moments)

\[ \bar{\mu}(\rho) = -\left( \frac{\rho}{D\beta(k)} \right)^{\frac{1}{\frac{3}{2} - k}}, \quad \text{where} \quad \beta(k) := 4\pi \sqrt{2} \int_0^\infty \frac{\sqrt{s}}{(s + 1)^k} \, ds \]

Fast diffusion equations:

\[ \partial_t \rho = \nabla \cdot \left( \Theta \nabla \left( \rho^{\frac{k-5/2}{k-3/2}} \right) + \rho \nabla V \right), \quad \text{where} \quad \Theta := \frac{1}{k - \frac{5}{2}} \left( \frac{1}{D\beta(k)} \right)^{\frac{1}{3/2 - k}} \]

Outside of a finite ball, the potential has to grow faster than a power

\[ V(x) \geq C|x|^q, \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > \frac{3}{k - 5/2} \]
Exemple 2. Maxwell distribution

\[ \gamma(E) = \exp(-E) \]
\[ \bar{\nu}(\rho) = \log \rho - \frac{3}{2} \log(2\pi) \]

Linear drift-diffusion equation : \( \nu(\rho) = \rho \)

\[ \partial_t \rho = \nabla \cdot (\nabla \rho + \rho \nabla V) \]

"the linear case"

Growth assumption on the potential

\[ V(x) \geq q \log(|x|), \quad \text{a.e. for} \quad |x| > R \quad \text{with} \quad q > 3 \]
Example 3. Power law with positive exponent

Let $\gamma$ be a cut-off power with positive exponent:

$$
\gamma(E) = (E_2 - E)^k_+ := \begin{cases} 
D(E_2 - E)^k & \text{for } E < E_2, \quad D > 0, \quad k > 0 \\
0 & \text{otherwise}
\end{cases}
$$

$$
\overline{\mu}(\rho) = \left( \frac{\rho}{D\alpha(k)} \right)^{\frac{1}{k + \frac{3}{2}}} - E_2, \quad \text{where} \quad \alpha(k) = 4\pi \sqrt{2} \int_0^1 \sqrt{u(1 - u)^k} \, du
$$

Porous medium equations: $\nu(\rho) = \Theta \rho^{\frac{2k+5}{2k+3}}$

$$
\partial_t \rho = \nabla \cdot \left( \Theta \nabla \left( \rho^{\frac{k+5/2}{k+3/2}} \right) + \rho \nabla V \right), \quad \text{where} \quad \Theta := \frac{1}{k + \frac{5}{2}} \left( \frac{1}{D\alpha(k)} \right)^{\frac{1}{k+3/2}}
$$
Growth condition on the potential: if $\mu^*$ is the upper bound for the Fermi energy

\[(E_2 + \mu^* - V(x))_+ = O\left(\frac{1}{|x|^q}\right)\text{ a.e. as } |x| \to \infty, \quad q > \frac{3}{k + \frac{3}{2}}\]

This is an assumption which is compatible with the behavior of a self-consistent gravitational field.
Example 4. Fermi-Dirac distribution

\[ \gamma(E) = \frac{1}{\exp(E) + \alpha} \]

\[(\bar{\mu}^{-1})(\theta) = \frac{4\pi \sqrt{2}}{\alpha} \int_{0}^{\infty} \frac{\sqrt{p} \, dp}{\exp(p - \theta - \log \alpha) + 1} = \frac{(2\pi)^{3/2}}{\alpha} \operatorname{Li}_{3/2}(-\alpha \exp(\theta)) \]

\[ \bar{\mu}(\rho) = \log\left(-\frac{1}{\alpha} \left(\operatorname{Li}_{-1}(\frac{z}{\alpha})\right) \left(-\frac{\alpha \rho}{(2\pi)^{3/2}}\right)\right), \quad \operatorname{Li}_n(z) := \sum_{k=1}^{\infty} \frac{z^k}{k^n} \]

Macroscopic equation: \[ \partial_t \rho = \nabla \cdot \left( (D(\rho) \nabla \rho + \rho \nabla V) \right) \]

\[ D(\rho) = \nu'(\rho) = \rho \bar{\mu}'(\rho) = \frac{-\alpha}{(2\pi)^{3/2}} \frac{\rho}{\operatorname{Li}_{1/2}(\frac{z}{\alpha})} \frac{1}{((\operatorname{Li}_{-1}(\frac{z}{\alpha}))^{(2\pi)^{3/2}})} \]

Moreover the expansion of \( D(\rho) \) at \( \rho = 0 \) gives

\[ D(\rho) = 1 + \frac{\sqrt{2}}{4} \frac{\alpha \rho}{(2\pi)^{3/2}} + \left( \frac{3}{8} - \frac{2\sqrt{3}}{9} \right) \frac{\alpha^2 \rho^2}{(2\pi)^3} + O(\rho^3) \]
Example 5. Bose-Einstein distribution

\[ \gamma(E) = \frac{1}{\exp(E) - \alpha} \]

\[ (\mu^{-1})(\theta) = \frac{4\pi \sqrt{2}}{\alpha} \int_{0}^{\infty} \frac{\sqrt{p} \, dp}{\exp(p - \theta - \log \alpha) - 1} = \frac{(2\pi)^{3/2}}{\alpha} \text{Li}_{3/2}(\alpha \exp(\theta)) \]

\[ \bar{\mu}(\rho) = \log \left( \frac{1}{\alpha} (\text{Li}_{-1}^{-1}) \left( \frac{\alpha \rho}{(2\pi)^{3/2}} \right) \right) \]

Macroscopic equation: \( \partial_t \rho = \nabla \cdot \left( (D(\rho) \nabla \rho + \rho \nabla V) \right) \)

\[ D(\rho) = \nu'(\rho) = \rho \bar{\mu}'(\rho) = \frac{\alpha}{(2\pi)^{3/2}} \frac{\rho}{\text{Li}_{1/2}\left( \frac{\alpha \rho}{(2\pi)^{3/2}} \right)} \]

Observe that

\[ \lim_{\rho \to \bar{\rho}} \nu'(\rho) = 0 \quad (\bar{\rho} \text{ is the maximal density}) \text{ and} \]

\[ \lim_{\rho \to 0} \nu'(\rho) = 1 \]
Expression of $\bar{\mu}$

$$Q[f] = G_f - f, \quad G_f = \gamma \left( \frac{1}{2} |v|^2 - \bar{\mu}(\rho_f) \right)$$

$\bar{\mu}^{-1} : (-E_2, -E_1) \to (0, \infty)$ is such that

$$(\bar{\mu}^{-1})(\theta) = 4\pi \sqrt{2} \int_0^{\infty} \gamma(p - \theta) \sqrt{p} \, dp$$

We extend $\bar{\mu}^{-1}$ by the value 0 on $(-\infty, -E_2)$. Differentiation with respect to $\theta$ leads to the Abelian equation

$$\frac{(\bar{\mu}^{-1})'(\theta)}{2\pi \sqrt{2}} = \int_{-\infty}^{\theta} \frac{\gamma(-q)}{\sqrt{\theta - q}} \, dq$$

and gives an explicit expression of $\gamma$ in terms of $\bar{\mu}^{-1}$

$$\gamma(E) = \frac{1}{\sqrt{2}} \frac{d^2}{2\pi^2 \, dE^2} \int_{-\infty}^{-E} \frac{(\bar{\mu}^{-1})(\theta)}{\sqrt{-E - \theta}} \, d\theta$$
References

Formal expansions (generalized Smoluchowski equation)

[Ben Abdallah, J.D.]
[Chavanis-Laurençot, Lemou]
[Chavanis et al.]

Rigorous justification with nonlinear diffusion limits

[J.D., P. Markowich, D. Ölz and C. Schmeiser]
Assumptions on the initial data

We assume that there is a constant *Fermi level* $\mu^*$ such that

$$0 \leq f_I(x, v) \leq f^*(x, v) := \gamma \left( \frac{1}{2} |v|^2 + V(x) - \mu^* \right) \quad \forall (x, v) \in \mathbb{R}^6$$

Maximal macroscopic density:

$$\bar{\rho} := \lim_{\theta \to -E_1^+} \int_{\mathbb{R}^3} \gamma \left( \frac{1}{2} |v|^2 - \theta \right) dv$$

If $\bar{\rho} < \infty$ we assume

$$\mu^{-1}(\mu^*) \leq \bar{\rho}$$
Compatibility assumption on the potential

$\nabla_x V \in W^{1,\infty}(\mathbb{R}^3)$ and $V$ is bounded from below

$$\inf_{x \in \mathbb{R}^3} V(x) = V_{\text{min}} = 0$$

**Confinement condition**: with $f^*(x, v) := \gamma \left( \frac{1}{2} |v|^2 + V(x) - \mu^* \right)$

$$f^* \in L^1(\mathbb{R}^3 \times \mathbb{R}^3) \quad \text{and} \quad \int \int_{\mathbb{R}^6} \left( \frac{1}{2} |v|^2 + V(x) \right) f^*(x, v) \, dv \, dx < \infty$$

Observe that this implies $\| f^* \|_{L^1} \geq \| f_I \|_{L^1} = M$

A **compatibility assumption**: Given a Gibbs state (a function $\gamma$), impose some minimal growth conditions on $V$. 
Existence and uniqueness

**Proposition 1.** For any $p \in (1, \infty)$, Eq. (??) has a unique weak solution in

$$\mathcal{V} := \{ f \in C(0, \infty; (L^1 \cap L^p)(\mathbb{R}^6)) : 0 \leq f \leq f^*, \ \forall t > 0 \ \text{a.e.} \}$$

Proof: Cf. [Poupaud-Schmeiser, 1991]: define the map $f \mapsto \Gamma[f] = g$

$$\epsilon^2 \partial_t g + \epsilon v \cdot \nabla_x g - \epsilon \nabla_x V \cdot \nabla_v g = G_f - g$$

$$g(t = 0) = f_I$$

$\Gamma$ maps $\mathcal{V}$ into itself and is a contraction for sufficiently small time intervals.
Free energy

\[ \mathcal{F}[f] := \int \int_{\mathbb{R}^6} \left[ \left( \frac{1}{2} |v|^2 + V \right) f + \beta_\gamma [f] \right] dv \, dx \]

\[ \beta_\gamma [f] := \int_0^f -\gamma^{-1}(s) \, ds \]

\(-\gamma^{-1}\) is monotonically increasing \(\implies \beta_\gamma\) is a convex function

Microscopic energy associated to a distribution function \(f\):

\[ E_f(x, v, t) := \frac{1}{2} |v|^2 + V(x) - \mu_{\rho_f}(x, t) = \frac{1}{2} |v|^2 - \bar{\mu}(\rho_f(x, t)) = -\gamma^{-1}[G_f] \]

\[ \epsilon^2 \frac{d}{dt} \mathcal{F}(f(., ., t)) = \int \int_{\mathbb{R}^6} (G_f - f)(\gamma^{-1}[G_f] - \gamma^{-1}[f]) \, dv \, dx := -D[f] \leq 0 \]
Free energy and local Gibbs states (1)

\[ F_{\text{loc}}[f](x, t) := \int_{\mathbb{R}^3} \left[ \left( \frac{1}{2} |v|^2 + V(x) - \mu_f(x, t) \right) f(x, v, t) + \beta_\gamma(f(x, v, t)) \right] dv \]

is convex. Minimum if and only if \( f = G_f \)

\[ 0 = \frac{1}{2} |v|^2 + V(x) - \mu_f(x, t) + \beta'_\gamma[f] = \frac{1}{2} |v|^2 + V(x) - \mu_f(x, t) - \gamma^{-1}[f] \]

Summarizing:

\[ F_{\text{loc}}[f](x, .., t) = \int_{\mathbb{R}^3} \left[ \beta_\gamma(f) - \beta'_\gamma(G_f)(f - G_f) \right] dv + R(x, t) \]

\[ F[f](t) = \int_{\mathbb{R}^3} F_{\text{loc}}[f](x, t) \, dx \]
Free energy of a local Gibbs states (2)

\[ \mathcal{F}_{\text{loc}}[f](x, t) \geq \mathcal{F}_{\text{loc}}[G_f](x, t) \quad \text{and} \quad \mathcal{F} \geq \mathcal{F}[G_f] \]

If \( g \) is a local Gibbs state: 
\[ g(t, x, v) = \gamma \left( \frac{1}{2}|v|^2 - \mu(x, t) \right), \]
then

\[ \mathcal{F}[g] = \iint_{\mathbb{R}^6} \gamma \left( \frac{1}{2}|v|^2 + V - \mu \right) \left( \mu - \frac{|v|^2}{3} \right) \, dv \, dx = \int_{\mathbb{R}^3} (\mu \rho_g - \nu(\rho_g)) \, dx \]

with \( \rho_g = \bar{\mu}^{-1}(\mu - V) \) and \( \nu(\rho_g) := \frac{1}{3} \int_{\mathbb{R}^3} |v|^2 \gamma \left( \frac{1}{2}|v|^2 - \bar{\mu}(\rho_g) \right) \, dv \)

\[ \mathcal{F}[g] = \int_{\mathbb{R}^3} (\bar{\mu}(\rho_g) \rho_g - \nu(\rho_g) + V \rho_g) \, dx \]

Notice that if \( \partial_t \rho = \nabla \cdot \left[ \nabla \nu(\rho) + \rho \nabla V \right] \), then (Tsallis entropy)

\[ \frac{d}{dt} \int_{\mathbb{R}^3} (\bar{\mu}(\rho) \rho - \nu(\rho) + V \rho) \, dx = -\int_{\mathbb{R}^3} \rho \left| \nabla \bar{\mu}(\rho) + \nabla V \right|^2 \, dx \]
Consequences

If $\beta_\gamma$ is convex, then $\mathcal{F}$ is bounded if $V$ satisfies the compatibility assumption with $\gamma$

$$\epsilon^2 \frac{d}{dt} \mathcal{F}(f(.,.,t)) = \iint_{\mathbb{R}^6} (G_f - f)(\gamma^{-1}[G_f] - \gamma^{-1}[f]) \, dv \, dx := -D[f] \leq 0$$

$$\epsilon^2 \left[ \mathcal{F}(f(.,.,t)) - \mathcal{F}(f_I) \right] = - \int_0^T D[f](t) \, dt$$

To prove the convergence of $f^\epsilon$ to $G_{f^\epsilon} = \gamma(E_{f^\epsilon})$ consider the partition of the support of $f^*$ according to

$$\Omega_+ := \left\{ (x, v, t) \in \text{supp } f^* \subset \mathbb{R}^6 \times (0, T) : E_f = \frac{1}{2}|v|^2 - \bar{\mu}(\rho_f(x,t)) < E_2 \right\}$$

$$\Omega_0 := \left\{ (x, v, t) \in \text{supp } f^* \subset \mathbb{R}^6 \times (0, T) : E_f = \frac{1}{2}|v|^2 - \bar{\mu}(\rho_f(x,t)) \geq E_2 \right\}$$
Main result

Theorem 2. For any $\varepsilon > 0$, the equation has a unique weak solution $f^\varepsilon \in C(0, \infty; L^1 \cap L^p(\mathbb{R}^6))$ for all $p < \infty$. As $\varepsilon \to 0$, $f^\varepsilon$ weakly converges to a local Gibbs state $f^0$ given by

$$f^0(x,v,t) = \gamma \left( \frac{1}{2} |v|^2 + V(x) - \bar{\mu}(\rho(x,t)) \right) \quad \forall (x,v,t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$$

where $\rho$ is a solution of the nonlinear diffusion equation

$$\partial_t \rho = \nabla_x \cdot (\nabla_x \nu(\rho) + \rho \nabla_x V(x))$$

with initial data $\rho(x,0) = \rho_I(x) := \int_{\mathbb{R}^3} f_I(x,v) \, dv$

$$\nu(\rho) = \int_0^\rho s \, \bar{\mu}'(s) \, ds$$

Moreover $\int_{\mathbb{R}^3} f^\varepsilon \, dv$ strongly converges to $\rho$ in $L^p_{\text{loc}}$ as $\varepsilon \to 0$. 
Proof (1)

\[ \Omega_{+}^{x,t} := \{ v \in \mathbb{R}^3 : (x, v, t) \in \Omega_{+} \} \quad \text{and} \quad \Omega_{0}^{x,t} := \{ v \in \mathbb{R}^3 : (x, v, t) \in \Omega_{0} \} \]

Notice that \( \Omega_{+}^{x,t} = \mathbb{R}^3 \) if \( E_2 = \infty \)

**Lemma 1.** For any nonnegative function \( f \leq f^{*} \) there exists a constant, which does not depend on \( x \) and \( t \), such that

\[
\int_{\Omega_{+}^{x,t}} v_i^{2m} \frac{G_f - f}{\gamma^{-1}[f] - E_f} \, dv \leq \mathcal{M},
\]

for any \( m = 1, 2, i = 1, 2, 3 \).
Proof (2)

Scaled perturbations of the first and second moments

\[ j^\epsilon := \int_{\mathbb{R}^3} v \frac{f^\epsilon - G f^\epsilon}{\epsilon} \, dv \quad \text{and} \quad \kappa^\epsilon := \int_{\mathbb{R}^3} v \otimes v \frac{f^\epsilon - G f^\epsilon}{\epsilon} \, dv \]

**Lemma 2.** For any bounded, open set \( U \subset \mathbb{R}^3 \times [0, T) \), there are two constants \( M_1^U \) and \( M_2^U \), which do not depend on \( \epsilon \), such that

\[ \| j^\epsilon \|_{L^2_{x,t}(U)} \leq M_1^U \quad \text{and} \quad \| \kappa^\epsilon \|_{L^2_{x,t}(U)} \leq M_2^U \quad \text{as} \quad \epsilon \to 0 . \]

If \( g(x, v, t) := \gamma(|v|^2/2 - \bar{\mu}(\rho(x, t))) \), then

\[ \int_{\mathbb{R}^3} v \otimes v \, g \, dv = \nu(\rho) \mathrm{Id}^{3 \times 3} \quad \text{where} \quad \nu(\rho) := \int_0^\rho \sigma \bar{\mu}'(\sigma) \, d\sigma \]
Proof (3)

\[
\int_U \left( \int_{\Omega_{x,t}^+} |v|^m \frac{|f^{\varepsilon} - G_f^{\varepsilon}|}{\varepsilon} \, dv \right)^2 \, dx \, dt \\
\leq \int_U \left( \int_{\Omega_{x,t}^+} |v|^{2m} \frac{G_f^{\varepsilon} - f^{\varepsilon}}{\gamma^{-1}(f^{\varepsilon}) - E_f^{\varepsilon}} \, dv \right) \\
\cdot \left( \frac{1}{\varepsilon^2} \int_{\Omega_{x,t}^+} (G_f^{\varepsilon} - f^{\varepsilon})(\gamma^{-1}(f^{\varepsilon}) - E_f^{\varepsilon}) \, dv \right) \, dx \, dt \\
\leq C_1
\]
Proof (4)

**Proposition 3.** \( \rho^\varepsilon \to \rho^0 \) in \( L^p_{\text{loc}} \) strongly for all \( p \in (1, \infty) \).

Div-Curl Lemma as in [Goudon-Poupaud, 2001]. Integrate (??) with respect to \( dv \) and \( v \, dv \)

\[
\begin{align*}
\partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon &= 0 \\
\varepsilon^2 \partial_t j^\varepsilon + \nabla_x \cdot \int_{\mathbb{R}^3} v \otimes v f^\varepsilon \, dv &= -j^\varepsilon - \rho^\varepsilon \nabla_x V
\end{align*}
\]

Split the second moments of \( f \)

\[
\int_{\mathbb{R}^3} v \otimes v f^\varepsilon \, dv = \int_{\mathbb{R}^3} v \otimes v G f^\varepsilon \, dv + \int_{\mathbb{R}^3} v \otimes v (f^\varepsilon - G f^\varepsilon) \, dv = \nu(\rho^\varepsilon) I^{3 \times 3} + \varepsilon \kappa^\varepsilon
\]

\[
\begin{align*}
\partial_t \rho^\varepsilon + \nabla_x \cdot j^\varepsilon &= 0 \\
\nabla_x \nu(\rho^\varepsilon) &= -j^\varepsilon - \rho^\varepsilon \nabla_x V - \varepsilon \nabla_x \cdot \kappa^\varepsilon - \varepsilon^2 \partial_t j^\varepsilon
\end{align*}
\]
**Proof (5)**

Apply the Div-Curl Lemma to

\[ U^\varepsilon := (\rho^\varepsilon, j^\varepsilon), \quad V^\varepsilon := (\nu(\rho^\varepsilon), 0, 0, 0) \]

With \((\text{curl } w)_{i,j} = w^i_{x,j} - w^j_{x,i}\) and

\[
\begin{cases}
\text{div}_{t,x} U^\varepsilon = 0, \\
(\text{curl}_{t,x} V^\varepsilon)_{1,2,\ldots,4} = -j^\varepsilon - \rho^\varepsilon \nabla_x V - \epsilon \nabla_x \cdot \kappa^\epsilon - \epsilon^2 \partial_t j^\varepsilon
\end{cases}
\]

we obtain the convergence of \(U^{\varepsilon_i} \cdot V^{\varepsilon_i} = \rho^{\varepsilon_i} \nu(\rho^{\varepsilon_i})\)

As in [Marcati-Milani, 1990], we deduce using Young measures that the convergence of \(\rho^{\varepsilon_i}\) is strong. The strict convexity assumption is replaced by the strict monotonicity of the function \(\nu\) in \(\rho \nu(\rho)\).
Application to a flat rotating system of gravitating particles
Preliminaries : a kinetic description

Consider the gravitational Vlasov-Poisson-Boltzmann system

\[ \partial_t F + v \cdot \nabla_x F - \nabla_x \psi \cdot \nabla_v F = Q_\omega(F) \]

where the potential \( \psi \) is given as a solution of the Poisson equation

\[ \Delta \psi = \int_{\mathbb{R}^2 \times \mathbb{R}} F \, dv \, dw \]

the distribution function is concentrated on
\[ \{((x, z), (v, w)) \in (\mathbb{R}^2 \times \mathbb{R}) \times (\mathbb{R}^2 \times \mathbb{R}) : z = 0, \ w = 0\} \]
and \( Q_\omega(F) \) is a collision kernel which depends on the angular velocity \( \omega \), to be specified later

\[ \psi(t, x) = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} F(t, x, v) \, dv \]
Rotation at constant angular speed

Reduced problem in $\mathbb{R}^2$

$$(x, v) \mapsto (xe^{i\omega t}, (v + i\omega x)e^{i\omega t}) =: \mathcal{R}_{\omega, t}(x, v)$$

$$F(t, x, v) =: f(t, xe^{i\omega t}, (v + i\omega x)e^{i\omega t})) = f \circ \mathcal{R}_{\omega, t}(x, v).$$

The equation satisfied by $f$ can be written as

$$\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2 \text{Re} \left( i \omega v \overline{\nabla_v f} \right) - \nabla_x \phi \cdot \nabla_v f = Q(f)$$

where the collision kernel $Q$ is defined by $Q(f) := Q_\omega(F) \circ \mathcal{R}_{\omega, t}^{-1}$ and the potential $\phi$ is given by

$$\phi(t, x) = -\frac{1}{4\pi|x|} \ast \int_{\mathbb{R}^2} f(t, x, v) \, dv$$
Written in cartesian coordinates, the equation satisfied by $f$ is

$$
\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2 \omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = Q(f)
$$

$$
\phi = -\frac{1}{4\pi|x|} \ast \int_{\mathbb{R}^2} f \, dv
$$

where

$$
a \wedge b := a^\perp \cdot b = (-a_2, a_1) \cdot (b_1, b_2) = a_1 b_2 - a_2 b_1 = \text{Re}(i(a_1 + i a_2)(b_1 - i b_2))
$$

Local Gibbs state and collision kernel:

$$
G_f(t, x, v) = \gamma \left( \frac{1}{2} |v|^2 + \phi(t, x) - \frac{1}{2} \omega^2 |x|^2 + \mu_f(t, x) \right)
$$

$$
\int_{\mathbb{R}^2} G_f(t, x, v) \, dv = \int_{\mathbb{R}^2} f(t, x, v) \, dv , \quad Q(f) = G_f - f
$$
Polytropes

\[
\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = G_f - f
\]

\[
\phi = -\frac{1}{4\pi|x|} \ast \int_{\mathbb{R}^2} f \, dv
\]

For simplicity: case of the polytropic gases, or polytropes:

\[
\gamma(s) := \left(\frac{-s}{k+1}\right)^k
\]

and

\[
\bar{\mu}(\rho) = -(k + 1)\left(\frac{\rho}{2\pi}\right)^{\frac{1}{k+1}}
\]

\[
G(s) := \int_{\mathbb{R}^2} \gamma\left(\frac{1}{2}|v|^2 - s\right) \, ds = 2\pi \left(\frac{-s}{k+1}\right)^{k+1}
\]
A priori estimates

Mass:

\[ M = \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f \, dx \, dv > 0 \]

Free energy functional: with \( \beta(s) = \int_s^0 \gamma^{-1}(\sigma) \, d\sigma \)

\[ \mathcal{F}[f] := \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ f \left( \frac{1}{2} |v|^2 - \frac{1}{2} \omega^2 |x|^2 + \frac{1}{2} \phi \right) + \beta(f) \right] \, dx \, dv \]

is such that

\[ \frac{d}{dt} \mathcal{F}[f(t, \cdot, \cdot)] := \int \int_{\mathbb{R}^2 \times \mathbb{R}^2} \left( G_f - f \right) \left( \gamma^{-1}(G_f) - \gamma^{-1}(f) \right) \, dx \, dv \]
Critical points

Local Lagrange multiplier

$$\mu_f(t, x) = \frac{1}{2} \phi - \frac{1}{2} \omega^2 |x|^2 - \bar{\mu}(\rho)$$

“Global” Gibbs state (on a ball)

$$f^\infty(x, v) := \gamma \left( \frac{1}{2} |v|^2 + \phi^\infty(x) - \frac{1}{2} \omega^2 |x|^2 - C \right)$$

with $$\phi^\infty(x) := -\frac{1}{4\pi|x|} \ast \int_{\mathbb{R}^2} f^\infty(x, v) \, dv$$

$$f^\infty$$ is a critical point of $$\mathcal{F}$$ under the constraint

$$\int \int_{\mathbb{R}^2 \times \mathbb{R}^2} f^\infty(x, v) \, dx \, dv = M$$
Stationary solutions

\[ \beta(f) = \frac{f^q}{q-1} \quad \text{with} \quad k = \frac{1}{q-1} \quad \iff \quad q = 1 + \frac{1}{k} \]

\[ \phi = -\frac{1}{4\pi|x|} \ast \rho \quad \text{with} \quad \rho = G\left(\phi - \frac{1}{2} \omega^2 |x|^2 - C\right) \]

\( C \) is determined by the condition: \( \int_{\mathbb{R}^2} \rho \, dx = M \)

\[ G(s) = 2\pi \left(-\frac{q-1}{q} s\right)^{\frac{q}{q-1}} \quad \iff \quad -\frac{q}{q-1} \rho^{q-1} + \phi_{\text{eff}} - C = 0 \]

on the support of \( \rho \), where the effective potential is

\[ \phi_{\text{eff}}(x) := -\frac{1}{4\pi|x|} \ast \rho - \frac{1}{2} \omega^2 |x|^2. \]
Reduced variational problem and diffusion limit

Free energy of a local Gibbs state

\[ \mathcal{F}[G_\rho] =: G[\rho] \quad \text{with} \quad G_\rho(x, v) := \gamma \left( \frac{1}{2} |v|^2 + \bar{\mu}(\rho) \right) \]

Reduced variational problem takes the form

\[
G[\rho] = \int_{\mathbb{R}^2} \left[ h(\rho) + \left( \phi(x) - \frac{1}{2} \omega^2 |x|^2 \right) \rho \right] \, dx
\]

\[
h(\rho) := \int_{\mathbb{R}^2} \left[ (\beta \circ \gamma) \left( \frac{1}{2} |v|^2 + \bar{\mu}(\rho) \right) + \frac{1}{2} |v|^2 \gamma \left( \frac{1}{2} |v|^2 + \bar{\mu}(\rho) \right) \right] \, dv
\]

\[
= 2\pi \int_0^{\infty} \left[ (\beta \circ \gamma)(s + \bar{\mu}(\rho)) + s \gamma(s + \bar{\mu}(\rho)) \right] \, ds
\]

Polytropes: \[ h(\rho) = \frac{\kappa}{m-1} \rho^m \quad \text{with} \quad m = 2 - \frac{1}{q} \]
Results

$\omega = 0$ : [Rein] Under the mass constraint, both functionals $F$ and $G$ have a radial minimizer

[Schaeffer] : the radial minimizer is unique

[J.D., Ben Abdallah,...], [J.D., J. Fernández] : dynamical stability holds for both models

$\omega \neq 0$ : [J.D., J. Fernández] (work in progress)

**Theorem 1.** For any $M > 0$, there exists an angular velocity $\tilde{\omega}(M)$ such that for any $\omega \in (0, \tilde{\omega}(M))$, there is a stationary solution, which is a minimizer of the localized energy. This solution is never radially symmetric

Schwarz foliated symmetry
For any $M > 0$, there exists an angular velocity $\hat{\omega}(M)$ such that for any $\omega \in (0, \hat{\omega}(M))$, there is a radial stationary solution
Open questions

Dynamical stability for $\omega \neq 0$?

Systematic construction of stationary solutions with higher Morse indices?

Dynamical stability of these solutions (at the diffusion level and at the kinetic level)?