

# Nonlinear diffusions, entropy methods and stability

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*Diffusion, geometry, probability and free boundaries*

Workshop on  
*Degenerate and Singular Diffusion*

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# Outline

- 1 Interpolation inequalities on the sphere
  - Stability results on the sphere
  - Results based on a spectral analysis
  - Results based on the carré du champ method
- 2 Gaussian measure and log-Sobolev inequalities
  - Interpolation and log-Sobolev inequalities: Gaussian measure
  - More results on logarithmic Sobolev inequalities
  - Sobolev and LSI on  $\mathbb{R}^d$ : optimal dimensional dependence
- 3 Constructive stability results and entropy methods on  $\mathbb{R}^d$ 
  - Rényi entropy powers & Stability for Gagliardo-Nirenberg-Sobolev inequalities
  - Symmetry in (CKN): strategy of the proof in the critical case
  - Stability in Caffarelli-Kohn-Nirenberg inequalities ?

*Proving inequalities  
by the carré du champ method  
using a fast diffusion flow  
or a porous medium flow  
Reduction of nonlinear flows  
to linear flows*

# Introducing the flow

$$\frac{\partial u}{\partial t} = u^{-\rho(1-m)} \left( \Delta u + (m\rho - 1) \frac{|\nabla u|^2}{u} \right)$$

Check: if  $m = 1 + \frac{2}{\rho} \left( \frac{1}{\beta} - 1 \right)$ ,  $v = u^\beta$ , then  $\rho = v^\rho$ , solves  $\frac{\partial \rho}{\partial t} = \Delta \rho^m$

$$\frac{d}{dt} \|u\|_{L^p(\mathbb{S}^d)}^2 = 0, \quad \frac{d}{dt} \|u\|_{L^2(\mathbb{S}^d)}^2 = 2(\rho - 2) \int_{\mathbb{S}^d} u^{-\rho(1-m)} |\nabla u|^2 d\mu_d,$$

$$\frac{d}{dt} \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 = -2 \int_{\mathbb{S}^d} \left( \beta v^{\beta-1} \frac{\partial v}{\partial t} \right) (\Delta v^\beta) d\mu_d = -2\beta^2 \mathcal{K}[v]$$

## Lemma

Assume that  $p \in (1, 2^*)$  and  $m \in [m_-(d, p), m_+(d, p)]$ . Then

$$\frac{1}{2\beta^2} \frac{d}{dt} \left( \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[u] \right) \leq -\gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2} d\mu_d \leq 0$$

# The *carré du champ* strategy

In the linear case ( $m = 1$ ), the method goes back to [Bakry, Emery, 1985], but it applies also with  $m \neq 1$

$$\frac{d}{dt} \left( \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[u] \right) \leq 0$$

$\lim_{t \rightarrow +\infty} \left( \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[u] \right) = 0$  proves the

🟢 *Gagliardo-Nirenberg-Sobolev inequality*

$$\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 \geq d \mathcal{E}_p[u] := \frac{d}{p-2} \left( \|u\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^2(\mathbb{S}^d)}^2 \right)$$

for any  $p \in [1, 2) \cup (2, 2^*)$

with  $2^* := \frac{2d}{d-2}$  if  $d \geq 3$  and  $2^* = +\infty$  if  $d = 1$  or  $2$

🟢 Limit  $p \rightarrow 2$ : the *logarithmic Sobolev inequality*

$$\int_{\mathbb{S}^d} |\nabla u|^2 d\mu_d \geq \frac{d}{2} \int_{\mathbb{S}^d} u^2 \log \left( \frac{u^2}{\|u\|_{L^2(\mathbb{S}^d)}^2} \right) d\mu_d \quad \forall u \in H^1(\mathbb{S}^d, d\mu_d)$$

# Algebraic preliminaries

$$L\nu := H\nu - \frac{1}{d} (\Delta\nu) g_d \quad \text{and} \quad M\nu := \frac{\nabla\nu \otimes \nabla\nu}{\nu} - \frac{1}{d} \frac{|\nabla\nu|^2}{\nu} g_d$$

With  $a : b = a^{ij} b_{ij}$  and  $\|a\|^2 := a : a$ , we have

$$\|L\nu\|^2 = \|H\nu\|^2 - \frac{1}{d} (\Delta\nu)^2, \quad \|M\nu\|^2 = \left\| \frac{\nabla\nu \otimes \nabla\nu}{\nu} \right\|^2 - \frac{1}{d} \frac{|\nabla\nu|^4}{\nu^2} = \frac{d-1}{d} \frac{|\nabla\nu|^4}{\nu^2}$$

• A first identity

$$\int_{\mathbb{S}^d} \Delta\nu \frac{|\nabla\nu|^2}{\nu} d\mu_d = \frac{d}{d+2} \left( \frac{d}{d-1} \int_{\mathbb{S}^d} \|M\nu\|^2 d\mu_d - 2 \int_{\mathbb{S}^d} L\nu : \frac{\nabla\nu \otimes \nabla\nu}{\nu} d\mu_d \right)$$

• Second identity (Bochner-Lichnerowicz-Weitzenböck formula)

$$\int_{\mathbb{S}^d} (\Delta\nu)^2 d\mu_d = \frac{d}{d-1} \int_{\mathbb{S}^d} \|L\nu\|^2 d\mu_d + d \int_{\mathbb{S}^d} |\nabla\nu|^2 d\mu_d$$

# Constructing the estimate

With  $b = (\kappa + \beta - 1) \frac{d-1}{d+2}$  and  $c = \frac{d}{d+2} (\kappa + \beta - 1) + \kappa (\beta - 1)$

$$\begin{aligned} \mathcal{H}[v] &:= \int_{\mathbb{S}^d} \left( \Delta v + \kappa \frac{|\nabla v|^2}{v} \right) \left( \Delta v + (\beta - 1) \frac{|\nabla v|^2}{v} \right) d\mu_d \\ &= \frac{d}{d-1} \|L v - b M v\|^2 + (c - b^2) \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2} d\mu_d + d \int_{\mathbb{S}^d} |\nabla v|^2 d\mu_d \end{aligned}$$

Let  $\kappa = \beta(p-2) + 1$ . The condition  $\gamma := c - b^2 \geq 0$  amounts to

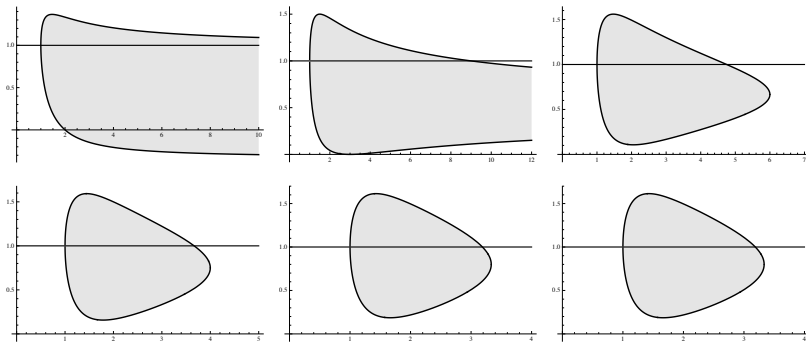
$$\gamma = \frac{d}{d+2} \beta(p-1) + (1 + \beta(p-2))(\beta-1) - \left( \frac{d-1}{d+2} \beta(p-1) \right)^2$$

## Lemma

$$\mathcal{H}[v] \geq \gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2} d\mu_d + d \int_{\mathbb{S}^d} |\nabla v|^2 d\mu_d$$

Hence  $\mathcal{H}[v] \geq d \int_{\mathbb{S}^d} |\nabla v|^2 d\mu_d$  if  $\gamma \geq 0$ , a condition on  $\beta$ , *i.e.*, on  $m$

# Admissible parameters



**Figure:**  $d = 1, 2, 3$  (first line) and  $d = 4, 5$  and  $10$  (second line): the curves  $p \mapsto m_{\pm}(p)$  determine the admissible parameters  $(p, m)$  [JD, Esteban, Kowalczyk, Loss] [JD, Esteban, 2019]

$$m_{\pm}(d, p) := \frac{1}{(d+2)p} \left( d p + 2 \pm \sqrt{d(p-1)(2d - (d-2)p)} \right)$$



# From inequalities to *improved* inequalities

## Summary

From  $\frac{1}{2\beta^2} \frac{d}{dt} \left( \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[u] \right) \leq -\gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2} d\mu_d \leq 0$  and  $\lim_{t \rightarrow +\infty} \left( \|\nabla u\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[u] \right) = 0$ , we deduce the inequality

$$\|\nabla u\|_{L^2(\mathbb{S}^d)}^2 \geq d \mathcal{E}_p[u]$$

[Bakry-Emery, 1984], [Bidaud-Véron, Véron, 1991], [Beckner, 1993]

... but we can do better

[Demange, 2008], [JD, Esteban, Kowalczyk, Loss]

# Logarithmic Sobolev and Gagliardo-Nirenberg inequalities on the sphere

*A joint work with G. Brigati and N. Simonov*  
***Logarithmic Sobolev and interpolation inequalities on the  
sphere: constructive stability results***

[arXiv:2211.13180](https://arxiv.org/abs/2211.13180),

to appear in Annales Inst. Henri Poincaré C, Analyse non linéaire

▷ *Carré du champ methods combined with spectral information*

## (Improved) logarithmic Sobolev inequality: stability (1)

$$\int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d \geq \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left( \frac{F^2}{\|F\|_{L^2(\mathbb{S}^d)}^2} \right) d\mu_d \quad \forall F \in H^1(\mathbb{S}^d, d\mu)$$

(LSI)

$d\mu_d$ : uniform probability measure; equality case: constant functions

Optimal constant: test functions  $F_\varepsilon(x) = 1 + \varepsilon x \cdot \nu$ ,  $\nu \in \mathbb{S}^d$ ,  $\varepsilon \rightarrow 0$

▷ *improved inequality* under an appropriate *orthogonality condition*

## Theorem

Let  $d \geq 1$ . For any  $F \in H^1(\mathbb{S}^d, d\mu)$  such that  $\int_{\mathbb{S}^d} x F d\mu_d = 0$ , we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left( \frac{F^2}{\|F\|_{L^2(\mathbb{S}^d)}^2} \right) d\mu_d \geq \frac{2}{d+2} \int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d$$

Improved ineq.  $\int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d \geq \left(\frac{d}{2} + 1\right) \int_{\mathbb{S}^d} F^2 \log \left( F^2 / \|F\|_{L^2(\mathbb{S}^d)}^2 \right) d\mu_d$

Earlier/weaker results in [JD, Esteban, Loss, 2015]

# Logarithmic Sobolev inequality: stability (2)

What if  $\int_{\mathbb{S}^d} x F d\mu_d \neq 0$  ? Take  $F_\varepsilon(x) = 1 + \varepsilon x \cdot \nu$  and let  $\varepsilon \rightarrow 0$

$$\|\nabla F_\varepsilon\|_{L^2(\mathbb{S}^d)}^2 - \frac{d}{2} \int_{\mathbb{S}^d} F_\varepsilon^2 \log \left( \frac{F_\varepsilon^2}{\|F_\varepsilon\|_{L^2(\mathbb{S}^d)}^2} \right) d\mu_d = O(\varepsilon^4) = O\left(\|\nabla F_\varepsilon\|_{L^2(\mathbb{S}^d)}^4\right)$$

Such a behaviour is in fact optimal: *carré du champ* method

## Proposition

Let  $d \geq 1$ ,  $\gamma = 1/3$  if  $d = 1$  and  $\gamma = (4d - 1)(d - 1)^2 / (d + 2)^2$  if  $d \geq 2$ . Then, for any  $F \in H^1(\mathbb{S}^d, d\mu)$  with  $\|F\|_{L^2(\mathbb{S}^d)}^2 = 1$  we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log F^2 d\mu_d \geq \frac{1}{2} \frac{\gamma \|\nabla F\|_{L^2(\mathbb{S}^d)}^4}{\gamma \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + d}$$

In other words, if  $\|\nabla F\|_{L^2(\mathbb{S}^d)}$  is small

$$\int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log F^2 d\mu_d \geq \frac{\gamma}{2d} \|\nabla F\|_{L^2(\mathbb{S}^d)}^4 + o\left(\|\nabla F\|_{L^2(\mathbb{S}^d)}^4\right)$$

# Logarithmic Sobolev inequality: stability (3)

Let  $\Pi_1 F$  denote the orthogonal projection of a function  $F \in L^2(\mathbb{S}^d)$  on the spherical harmonics corresponding to the first eigenvalue on  $\mathbb{S}^d$

$$\Pi_1 F(x) = \frac{x}{d+1} \cdot \int_{\mathbb{S}^d} y F(y) d\mu(y) \quad \forall x \in \mathbb{S}^d$$

▷ a global (and detailed) stability result

## Theorem

Let  $d \geq 1$ . For any  $F \in H^1(\mathbb{S}^d, d\mu)$ , we have

$$\begin{aligned} \int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left( \frac{F^2}{\|F\|_{L^2(\mathbb{S}^d)}^2} \right) d\mu_d \\ \geq \mathcal{S}_d \left( \frac{\|\nabla \Pi_1 F\|_{L^2(\mathbb{S}^d)}^4}{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{2} \|F\|_{L^2(\mathbb{S}^d)}^2} + \|\nabla(\text{Id} - \Pi_1) F\|_{L^2(\mathbb{S}^d)}^2 \right) \end{aligned}$$

for some explicit stability constant  $\mathcal{S}_d > 0$

# Gagliardo-Nirenberg inequalities: a result by R. Frank

[Frank, 2022]: if  $p \in (2, 2^*)$ , there is  $c(d, p) > 0$  such that

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[F] \geq c(d, p) \frac{\left( \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \|F - \bar{F}\|_{L^2(\mathbb{S}^d)}^2 \right)^2}{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{p-2} \|F\|_{L^2(\mathbb{S}^d)}^2}$$

where  $\bar{F} := \int_{\mathbb{S}^d} F d\mu_d$

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[F] \geq c(d, p) \frac{\|\nabla F\|_{L^2(\mathbb{S}^d)}^4}{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \frac{d}{p-2} \|F\|_{L^2(\mathbb{S}^d)}^2}$$

- a compactness method,
- the exponent 4 in the r.h.s. is optimal
- the (generalized) entropy is

$$\mathcal{E}_p[u] := \frac{d}{p-2} \left( \|u\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^2(\mathbb{S}^d)}^2 \right)$$

# Gagliardo-Nirenberg inequalities: stability

As in the case of the logarithmic Sobolev inequality, an improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined

## Theorem

Let  $d \geq 1$  and  $p \in (1, 2) \cup (2, 2^*)$ . For any  $F \in H^1(\mathbb{S}^d, d\mu)$ , we have

$$\begin{aligned} \int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d - d \mathcal{E}_p[F] \\ \geq \mathcal{S}_{d,p} \left( \frac{\|\nabla \Pi_1 F\|_{L^2(\mathbb{S}^d)}^4}{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 + \|F\|_{L^2(\mathbb{S}^d)}^2} + \|\nabla(\text{Id} - \Pi_1) F\|_{L^2(\mathbb{S}^d)}^2 \right) \end{aligned}$$

for some explicit stability constant  $\mathcal{S}_{d,p} > 0$

*A first stability result based on  
an improved inequality  
under an orthogonality constraint:  
a spectral analysis*



# Improved interpolation inequalities under orthogonality

Decomposition of  $L^2(\mathbb{S}^d, d\mu) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_{\ell}$  into spherical harmonics

Let  $\Pi_k$  be the orthogonal projection onto  $\bigoplus_{\ell=1}^k \mathcal{H}_{\ell}$

## Theorem

Assume that  $d \geq 1$ ,  $p \in (1, 2^*)$  and  $k \in \mathbb{N} \setminus \{0\}$  be an integer. For some  $\mathcal{C}_{d,p,k} \in (0, 1)$  with  $\mathcal{C}_{d,p,k} \leq \mathcal{C}_{d,p,1} = \frac{2d-p(d-2)}{2(d+p)}$

$$\int_{\mathbb{S}^d} |\nabla F|^2 d\mu_d - d \mathcal{E}_p[F] \geq \mathcal{C}_{d,p,k} \int_{\mathbb{S}^d} |\nabla(\text{Id} - \Pi_k) F|^2 d\mu_d$$

- $\mathcal{H}_1$  is generated by the coordinate functions  $x_i$ ,  $i = 1, 2, \dots, d+1$
- ▷ **Funk-Hecke formula** as in [Lieb, 1983] and [Beckner, 1993]
- ▷ Use convexity estimates and monotonicity properties of the coefficients

# *The convex improvement based on the carré du champ method*

# Improved inequalities: flow estimates

With  $\|u\|_{L^p(\mathbb{S}^d)} = 1$ , consider the *entropy* and the *Fisher information*

$$e := \frac{1}{p-2} \left( \|u\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^2(\mathbb{S}^d)}^2 \right) \quad \text{and} \quad i := \|\nabla u\|_{L^2(\mathbb{S}^d)}^2$$

## Lemma

With  $\delta := \frac{2-(4-p)\beta}{2\beta(p-2)}$  if  $p > 2$ ,  $\delta := 1$  if  $p \in [1, 2]$

$$(i - d e)' \leq -\gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2} d\mu_d \leq \frac{\gamma i e'}{(1 - (p-2)e)^\delta}$$

If  $F \in H^1(\mathbb{S}^d)$  is such that  $\|F\|_{L^p(\mathbb{S}^d)} = 1$ , then

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \geq d \varphi(\mathcal{E}_p[F])$$

# Some global stability estimates

[JD, Esteban, Kowalczyk, Loss], [JD, Esteban 2020]

[Brigati, JD, Simonov]

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \geq d \varphi(\mathcal{E}_p[F]) \quad \forall F \in H^1(\mathbb{S}^d) \text{ s.t. } \|F\|_{L^p(\mathbb{S}^d)}^2 = 1$$

Since  $\varphi(0) = 0$ ,  $\varphi'(0) = 1$ ,  $\varphi'' > 0$ , we know that  $\varphi : [0, s_*) \rightarrow \mathbb{R}^+$  is invertible and  $\psi : \mathbb{R}^+ \rightarrow [0, s_*)$ ,  $s \mapsto \psi(s) := s - \varphi^{-1}(s)$ , is convex increasing:  $\psi'' > 0$ , with  $\psi(0) = \psi'(0) = 0$ ,  $\lim_{t \rightarrow +\infty} (t - \psi(t)) = s_*$

## Proposition

If  $d \geq 1$  and  $p \in (1, 2^\#)$

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[F] \geq d \|F\|_{L^p(\mathbb{S}^d)}^2 \psi \left( \frac{1}{d} \frac{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2}{\|F\|_{L^p(\mathbb{S}^d)}^2} \right) \quad \forall F \in H^1(\mathbb{S}^d)$$

► If  $p = 2$ , notice that  $\psi(t) = t - \frac{1}{\gamma} \log(1 + \gamma t)$

# *Stability: the general result*

It remains to combine the *improved entropy – entropy production inequality* (carré du champ method) and the *improved interpolation inequalities under orthogonality constraints*

# The “far away” regime and the “neighborhood” of $\mathcal{M}$

▷ If  $\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 / \|F\|_{L^p(\mathbb{S}^d)}^2 \geq \vartheta_0 > 0$ , by the convexity of  $\psi$

$$\begin{aligned} \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - d \mathcal{E}_p[F] &\geq d \|F\|_{L^p(\mathbb{S}^d)}^2 \psi \left( \frac{1}{d} \frac{\|\nabla F\|_{L^2(\mathbb{S}^d)}^2}{\|F\|_{L^p(\mathbb{S}^d)}^2} \right) \\ &\geq \frac{d}{\vartheta_0} \psi \left( \frac{\vartheta_0}{d} \right) \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 \end{aligned}$$

▷ From now on, we assume that  $\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 < \vartheta_0 \|F\|_{L^p(\mathbb{S}^d)}^2$ , take  $\|F\|_{L^p(\mathbb{S}^d)} = 1$ , learn that

$$\|\nabla F\|_{L^2(\mathbb{S}^d)}^2 < \vartheta := \frac{d \vartheta_0}{d - (p-2) \vartheta_0} > 0$$

from the standard interpolation inequality and deduce from the Poincaré inequality that

$$\frac{d - \vartheta}{d} < \left( \int_{\mathbb{S}^d} F d\mu_d \right)^2 \leq 1$$

## Partial decomposition on spherical harmonics

$$\mathcal{M} = \Pi_0 F \text{ and } \Pi_1 F = \varepsilon \mathcal{Y} \text{ where } \mathcal{Y}(x) = \sqrt{\frac{d+1}{d}} x \cdot \nu, \nu \in \mathbb{S}^d$$

$$F = \mathcal{M} (1 + \varepsilon \mathcal{Y} + \eta G)$$

Apply  $c_{p,d}^{(-)} \varepsilon^6 \leq \|1 + \varepsilon \mathcal{Y}\|_{L^p(\mathbb{S}^d)}^p - (1 + a_{p,d} \varepsilon^2 + b_{p,d} \varepsilon^4) \leq c_{p,d}^{(+)} \varepsilon^6$   
 (with explicit constants) to  $u = 1 + \varepsilon \mathcal{Y}$  and  $r = \eta G$  the estimate

$$\begin{aligned} & \|u + r\|_{L^p(\mathbb{S}^d)}^2 - \|u\|_{L^p(\mathbb{S}^d)}^2 \\ & \leq \frac{2}{p} \|u\|_{L^p(\mathbb{S}^d)}^{2-p} \left( p \int_{\mathbb{S}^d} u^{p-1} r \, d\mu_d + \frac{p}{2} (p-1) \int_{\mathbb{S}^d} u^{p-2} r^2 \, d\mu_d \right. \\ & \quad \left. + \sum_{2 < k < p} C_k^p \int_{\mathbb{S}^d} u^{p-k} |r|^k \, d\mu_d + K_p \int_{\mathbb{S}^d} |r|^p \, d\mu_d \right) \end{aligned}$$

Estimate  $\int_{\mathbb{S}^d} (1 + \varepsilon \mathcal{Y})^{p-1} G \, d\mu_d$ ,  $\int_{\mathbb{S}^d} (1 + \varepsilon \mathcal{Y})^{p-k} |G|^k \, d\mu_d$ , etc. to  
 obtain (under the condition that  $\varepsilon^2 + \eta^2 \sim \vartheta$ )

$$\begin{aligned} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d - d \mathcal{E}_p[F] & \geq \mathcal{M}^2 (A \varepsilon^4 - B \varepsilon^2 \eta + C \eta^2 - \mathcal{R}_{p,d} (\vartheta^p + \vartheta^{5/2})) \\ & \geq \mathcal{C} \left( \frac{\varepsilon^4}{\varepsilon^2 + \eta^2 + 1} + \eta^2 \right) \end{aligned}$$

# Gaussian interpolation inequalities

*Joint work with G. Brigati and N. Simonov*  
***Gaussian interpolation inequalities***  
[arXiv:2302.03926](#)

▷ *The large dimensional limit of the sphere*



# Large dimensional limit

*Gagliardo-Nirenberg-Sobolev inequalities on  $\mathbb{S}^d$ ,  $p \in [1, 2)$*

$$\|\nabla u\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 \geq \frac{d}{p-2} \left( \|u\|_{L^p(\mathbb{S}^d, d\mu_d)}^2 - \|u\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 \right)$$

## Theorem

Let  $v \in H^1(\mathbb{R}^n, dx)$  with compact support,  $d \geq n$  and

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \dots, \omega_n/\sqrt{d}\right)$$

where  $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$ . With  $d\gamma(y) := (2\pi)^{-n/2} e^{-\frac{1}{2}|y|^2} dy$ ,

$$\begin{aligned} \lim_{d \rightarrow +\infty} d \left( \|\nabla u_d\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 - \frac{d}{2-p} \left( \|u_d\|_{L^2(\mathbb{S}^d, d\mu_d)}^2 - \|u_d\|_{L^p(\mathbb{S}^d, d\mu_d)}^2 \right) \right) \\ = \|\nabla v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \frac{1}{2-p} \left( \|v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \|v\|_{L^p(\mathbb{R}^n, d\gamma)}^2 \right) \end{aligned}$$

*Gaussian interpolation inequalities on  $\mathbb{R}^n$*

$$\|\nabla v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 \geq \frac{1}{2-p} \left( \|v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \|v\|_{L^p(\mathbb{R}^n, d\gamma)}^2 \right) \quad (1)$$

- 1  $\leq p < 2$  [Beckner, 1989], [Bakry, Emery, 1984]
- Poincaré inequality corresponding:  $p = 1$
- Gaussian logarithmic Sobolev inequality*  $p \rightarrow 2$

$$\|\nabla v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 \geq \frac{1}{2} \int_{\mathbb{R}^n} |v|^2 \log \left( \frac{|v|^2}{\|v\|_{L^2(\mathbb{R}^n, d\gamma)}^2} \right) d\gamma$$

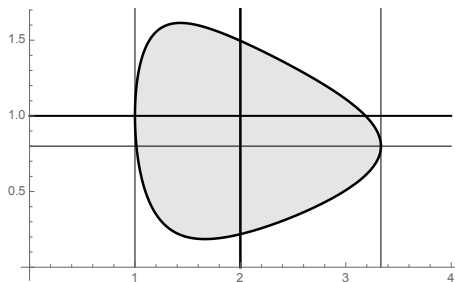
$$d\gamma(y) := (2\pi)^{-n/2} e^{-\frac{1}{2}|y|^2} dy$$

# Admissible parameters on $\mathbb{S}^d$

Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p(1-m)} \left( \Delta u + (mp - 1) \frac{|\nabla u|^2}{u} \right)$$

$$m_{\pm}(d, p) := \frac{1}{(d+2)^p} \left( dp + 2 \pm \sqrt{d(p-1)(2d - (d-2)p)} \right)$$



**Figure:** Case  $d = 5$ : admissible parameters  $1 \leq p \leq 2^* = 10/3$  and  $m$   
(horizontal axis:  $p$ , vertical axis:  $m$ )

# Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p(1-m)} \left( \mathcal{L}v + (mp-1) \frac{|\nabla v|^2}{v} \right) \quad \text{on } \mathbb{R}^n$$

Ornstein-Uhlenbeck operator:  $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_{\pm}(p) := \lim_{d \rightarrow +\infty} m_{\pm}(d, p) = 1 \pm \frac{1}{p} \sqrt{(p-1)(2-p)}$$

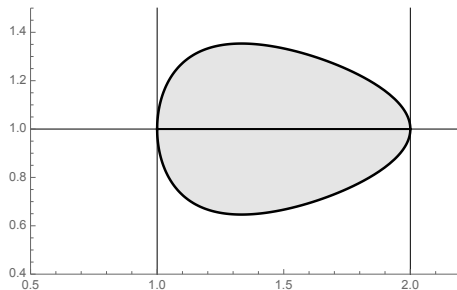


Figure: The admissible parameters  $1 \leq p \leq 2$  and  $m$  are independent of  $n$

# A stability result for Gaussian interpolation inequalities

## Theorem

For all  $n \geq 1$ , and all  $p \in (1, 2)$ , there is an explicit constant  $c_{n,p} > 0$  such that, for all  $v \in H^1(d\gamma)$ ,

$$\begin{aligned} & \|\nabla v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \frac{1}{p-2} \left( \|v\|_{L^p(\mathbb{R}^n, d\gamma)}^2 - \|v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 \right) \\ & \geq c_{n,p} \left( \|\nabla(\text{Id} - \Pi_1)v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 + \frac{\|\nabla \Pi_1 v\|_{L^2(\mathbb{R}^n, d\gamma)}^4}{\|\nabla v\|_{L^2(\mathbb{R}^n, d\gamma)}^2 + \|v\|_{L^2(\mathbb{R}^n, d\gamma)}^2} \right) \end{aligned}$$

# More results on logarithmic Sobolev inequalities

*Joint work with G. Brigati and N. Simonov*  
*Stability for the logarithmic Sobolev inequality*  
[arXiv:2303.12926](https://arxiv.org/abs/2303.12926)

▷ *Entropy methods, with constraints*

# Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon x$  in the limit as  $\varepsilon \rightarrow 0$

$$d(u_\varepsilon, 1)^2 = \|u'_\varepsilon\|_{L^2(\mathbb{R}, d\gamma)}^2 = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathcal{M}} d(u_\varepsilon, w)^\alpha \leq \frac{1}{2} \varepsilon^4 + O(\varepsilon^6).$$

$\mathcal{M} := \{w_{a,c} : (a, c) \in \mathbb{R}^d \times \mathbb{R}\}$  where  $w_{a,c}(x) = c e^{-a \cdot x}$

## Proposition

For all  $u \in H^1(\mathbb{R}^d, d\gamma)$  such that  $\|u\|_{L^2(\mathbb{R}^d)} = 1$  and  $\|xu\|_{L^2(\mathbb{R}^d)}^2 \leq d$ , we have

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \geq \frac{1}{2d} \left( \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \right)^2$$

and, with  $\psi(s) := s - \frac{d}{4} \log(1 + \frac{4}{d}s)$ ,

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \geq \psi \left( \|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 \right)$$

# Stability under log-concavity

$$\mathcal{C}_\star = 1 + \frac{1}{1728} \approx 1.0005787$$

## Theorem

For all  $u \in H^1(\mathbb{R}^d, d\gamma)$  such that  $u^2 \gamma$  is log-concave and such that

$$\int_{\mathbb{R}^d} (1, x) |u|^2 d\gamma = (1, 0) \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 |u|^2 d\gamma \leq d$$

we have

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 - \frac{\mathcal{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma \geq 0$$



## Theorem

Let  $d \geq 1$ . For any  $\varepsilon > 0$ , there is some explicit  $\mathcal{C} > 1$  depending only on  $\varepsilon$  such that, for any  $u \in H^1(\mathbb{R}^d, d\gamma)$  with

$$\int_{\mathbb{R}^d} (1, x) |u|^2 d\gamma = (1, 0), \quad \int_{\mathbb{R}^d} |x|^2 |u|^2 d\gamma \leq d, \quad \int_{\mathbb{R}^d} |u|^2 e^{\varepsilon |x|^2} d\gamma < \infty$$

for some  $\varepsilon > 0$ , then we have

$$\|\nabla u\|_{L^2(\mathbb{R}^d, d\gamma)}^2 \geq \frac{\mathcal{C}}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2 d\gamma$$

Additionally, if  $u$  is compactly supported in a ball of radius  $R > 0$ , then

$$\mathcal{C} = 1 + \frac{\mathcal{C}_\star - 1}{1 + \mathcal{C}_\star R^2}$$

compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

# Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

*A joint work with JD, M.J. Esteban, A. Figalli, R. Frank, M. Loss*  
*Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence*  
[arXiv: 2209.08651](https://arxiv.org/abs/2209.08651)

# A stability results for the Sobolev inequality

Sobolev inequality on  $\mathbb{R}^d$  with  $d \geq 3$

$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 \geq S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2 \quad \forall f \in \dot{H}^1(\mathbb{R}^d)$$

with equality on the manifold  $\mathcal{M}$  of the Aubin–Talenti functions

$$g(x) = c (a + |x - b|^2)^{-\frac{d-2}{2}}, \quad a \in (0, \infty), \quad b \in \mathbb{R}^d, \quad c \in \mathbb{R}$$

## Theorem

*There is a constant  $\beta > 0$  with an explicit lower estimate which does not depend on  $d$  such that for all  $d \geq 3$  and all  $f \in H^1(\mathbb{R}^d) \setminus \mathcal{M}$  we have*

$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d \|f\|_{L^{2^*}(\mathbb{R}^d)}^2 \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_{L^2(\mathbb{R}^d)}^2$$

[JD, Esteban, Figalli, Frank, Loss]

- No compactness argument
- The (estimate of the) constant  $\beta$  is explicit
- The decay rate  $\beta/d$  is optimal as  $d \rightarrow +\infty$

# A stability results for the logarithmic Sobolev inequality

Use the inverse stereographic projection to rewrite the result on  $\mathbb{S}^d$

$$\begin{aligned} \|\nabla F\|_{L^2(\mathbb{S}^d)}^2 - \frac{1}{4} d(d-2) \left( \|F\|_{L^{2^*}(\mathbb{S}^d)}^2 - \|F\|_{L^2(\mathbb{S}^d)}^2 \right) \\ \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \left( \|\nabla F - \nabla G\|_{L^2(\mathbb{S}^d)}^2 + \frac{1}{4} d(d-2) \|F - G\|_{L^2(\mathbb{S}^d)}^2 \right) \end{aligned}$$

## Corollary

With  $\beta > 0$  as above

$$\begin{aligned} \|\nabla F\|_{L^2(\mathbb{R}^n, d\gamma)}^2 - \pi \int_{\mathbb{R}^d} F^2 \ln \left( \frac{|F|^2}{\|F\|_{L^2(\mathbb{R}^n, d\gamma)}^2} \right) d\gamma \\ \geq \frac{\beta \pi}{2} \inf_{a \in \mathbb{R}^d, c \in \mathbb{R}} \int_{\mathbb{R}^d} |F - c e^{a \cdot x}|^2 d\gamma \end{aligned}$$

# Stability for Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

# Rényi entropy powers, inequalities and flow, a formal approach

[Toscani, Savaré, 2014]

[JD, Toscani, 2016]

[JD, Esteban, Loss, 2016]

▷ *How do we relate Gagliardo-Nirenberg-Sobolev inequalities on  $\mathbb{R}^d$*

$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^\theta \|f\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\text{GNS}} \|f\|_{L^{2p}(\mathbb{R}^d)} \quad (\text{GNS})$$

*and the fast diffusion equation*

$$\frac{\partial u}{\partial t} = \Delta u^m \quad (\text{FDE})$$

# Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With  $m \geq m_c := (d-2)/d$  and  $u_0 \in L^1_+(\mathbb{R}^d)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} u(t, x) dx = 0$$

(ii) *Second moment.* With  $m > d/(d+2)$  and  $u_0 \in L^1_+(\mathbb{R}^d, (1+|x|^2) dx)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} |x|^2 u(t, x) dx = 2d \int_{\mathbb{R}^d} u^m(t, x) dx$$

(iii) *Entropy estimate.* With  $m \geq m_1 := (d-1)/d$ ,  $u_0^m \in L^1(\mathbb{R}^d)$  and  $u_0 \in L^1_+(\mathbb{R}^d, (1+|x|^2) dx)$

$$\frac{d}{dt} \int_{\mathbb{R}^d} u^m(t, x) dx = \frac{m^2}{1-m} \int_{\mathbb{R}^d} u |\nabla u^{m-1}|^2 dx$$

*Entropy functional* and *Fisher information functional*

$$E[u] := \int_{\mathbb{R}^d} u^m dx \quad \text{and} \quad I[u] := \frac{m^2}{(1-m)^2} \int_{\mathbb{R}^d} u |\nabla u^{m-1}|^2 dx$$

# Entropy growth rate as a consequence of (GNS)

*Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^\theta \|f\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq C_{\text{GNS}} \|f\|_{L^{2p}(\mathbb{R}^d)} \quad (\text{GNS})$$

$$p = \frac{1}{2m-1} \iff m = \frac{p+1}{2p} \in [m_1, 1)$$

$$u = f^{2p} \text{ so that } u^m = f^{p+1} \text{ and } u |\nabla u^{m-1}|^2 = (p-1)^2 |\nabla f|^2$$

$$\mathcal{M} = \|f\|_{L^{2p}(\mathbb{R}^d)}^{2p}, \quad \mathbf{E}[u] = \|f\|_{L^{p+1}(\mathbb{R}^d)}^{p+1}, \quad \mathbf{I}[u] = (p+1)^2 \|\nabla f\|_{L^2(\mathbb{R}^d)}^2$$

If  $u$  solves (FDE)  $\frac{\partial u}{\partial t} = \Delta u^m$ , then  $\mathbf{E}' = m \mathbf{I}$

$$\mathbf{E}' \geq \frac{p-1}{2p} (p+1)^2 C_{\text{GNS}}^{\frac{2}{\theta}} \|f\|_{L^{2p}(\mathbb{R}^d)}^{\frac{2}{\theta}} \|f\|_{L^{p+1}(\mathbb{R}^d)}^{-\frac{2(1-\theta)}{\theta}} = C_0 \mathbf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d} u^m(t, x) dx \geq \left( \int_{\mathbb{R}^d} u_0^m dx + \frac{(1-m) C_0}{m-m_c} t \right)^{\frac{1-m}{m-m_c}} \quad \forall t \geq 0$$



# Self-similar solutions

$$\int_{\mathbb{R}^d} u^m(t, x) dx \geq \left( \int_{\mathbb{R}^d} u_0^m dx + \frac{(1-m) C_0}{m-m_c} t \right)^{\frac{1-m}{m-m_c}} \quad \forall t \geq 0$$

Equality case is achieved if and only if, up to a normalisation and a translation

$$u(t, x) = \frac{c_1}{R(t)^d} \mathcal{B} \left( \frac{c_2 x}{R(t)} \right)$$

where  $\mathcal{B}$  is the *Barenblatt self-similar solution*

$$\mathcal{B}(x) := (1 + |x|^2)^{\frac{1}{m-1}}$$

Notice that  $\mathcal{B} = \varphi^{2p}$  means that

$$\varphi(x) = (1 + |x|^2)^{-\frac{1}{p-2}}$$

is an *Aubin-Talenti profile*

# Pressure variable and decay of the Fisher information

The derivative of the *Rényi entropy power*  $E^{\frac{2}{d} \frac{1}{1-m} - 1}$  is proportional to

$$I^\theta E^{2 \frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$P := \frac{m}{1-m} u^{m-1}$$

▷ *Fisher information*

$$I[u] = \int_{\mathbb{R}^d} u |\nabla P|^2 dx$$

If  $u$  solves (FDE), then

$$\begin{aligned} I' &= \int_{\mathbb{R}^d} \Delta(u^m) |\nabla P|^2 dx + 2 \int_{\mathbb{R}^d} u \nabla P \cdot \nabla \left( (m-1) P \Delta P + |\nabla P|^2 \right) dx \\ &= -2 \int_{\mathbb{R}^d} u^m \left( \|D^2 P\|^2 - (1-m) (\Delta P)^2 \right) dx \end{aligned}$$

# Rényi entropy powers and interpolation inequalities

► Integrations by parts and completion of squares: with  $m_1 = \frac{d-1}{d}$

$$\begin{aligned} & -\frac{1}{2\theta} \frac{d}{dt} \log \left( I^\theta E^{2\frac{1-\theta}{p+1}} \right) \\ & = \int_{\mathbb{R}^d} u^m \left\| D^2 P - \frac{1}{d} \Delta P \text{Id} \right\|^2 dx + (m - m_1) \int_{\mathbb{R}^d} u^m \left| \Delta P + \frac{1}{E} \right|^2 dx \end{aligned}$$

► Analysis of the asymptotic regime as  $t \rightarrow +\infty$

$$\lim_{t \rightarrow +\infty} \frac{I[u(t, \cdot)]^\theta E[u(t, \cdot)]^{2\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\theta}{p}}} = \frac{I[\mathcal{B}]^\theta E[\mathcal{B}]^{2\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_{L^1(\mathbb{R}^d)}^{\frac{2\theta}{p}}} = (p+1)^{2\theta} \mathcal{C}_{\text{GNS}}^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$I[u]^\theta E[u]^{2\frac{1-\theta}{p+1}} \geq (p+1)^{2\theta} (\mathcal{C}_{\text{GNS}})^{2\theta} \mathcal{M}^{\frac{2\theta}{p}}$$

# *Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$*

*in collaboration with M. Bonforte, B. Nazaret and N. Simonov*

*Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows,  
regularity and the entropy method*

[arXiv:2007.03674](https://arxiv.org/abs/2007.03674), to appear in *Memoirs of the AMS*

*Constructive stability results in interpolation inequalities  
and explicit improvements of decay rates of fast diffusion eq.  
DCDS, 43 (3 & 4): 1070-1089, 2023*

# Entropy – entropy production inequality

*Fast diffusion equation (written in self-similar variables)*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot (v (\nabla v^{m-1} - 2x)) = 0 \quad (r\text{FDE})$$

*Generalized entropy (free energy) and Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} (v^m - \mathcal{B}^m - m \mathcal{B}^{m-1} (v - \mathcal{B})) \, dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v |\nabla v^{m-1} + 2x|^2 \, dx$$

satisfy an *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4 \mathcal{F}[v]$$

[del Pino, JD, 2002] so that

$$\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0] e^{-4t}$$

The *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4 \mathcal{F}[v]$$

is equivalent to the *Gagliardo-Nirenberg-Sobolev inequalities*

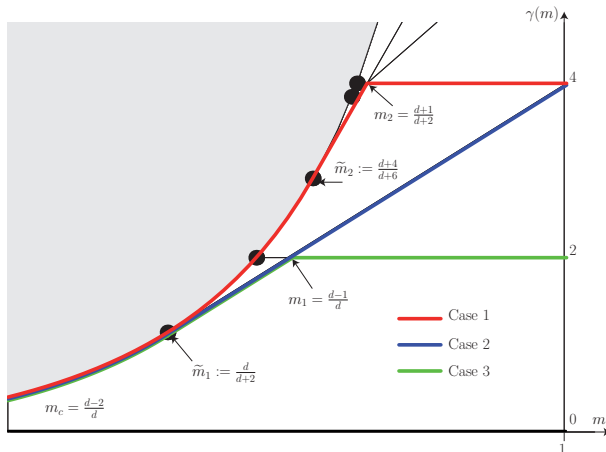
$$\|\nabla f\|_{L^2(\mathbb{R}^d)}^\theta \|f\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\text{GNS}} \|f\|_{L^{2p}(\mathbb{R}^d)} \quad (\text{GNS})$$

with equality if and only if  $|f|^{2p}$  is the *Barenblatt profile* such that

$$|f(x)|^{2p} = \mathcal{B}(x) = (1 + |x|^2)^{\frac{1}{m-1}}$$

$$v = f^{2p} \text{ so that } v^m = f^{p+1} \text{ and } v |\nabla v^{m-1}|^2 = (p-1)^2 |\nabla f|^2$$

# Spectral gap and Taylor expansion around $\mathcal{B}$

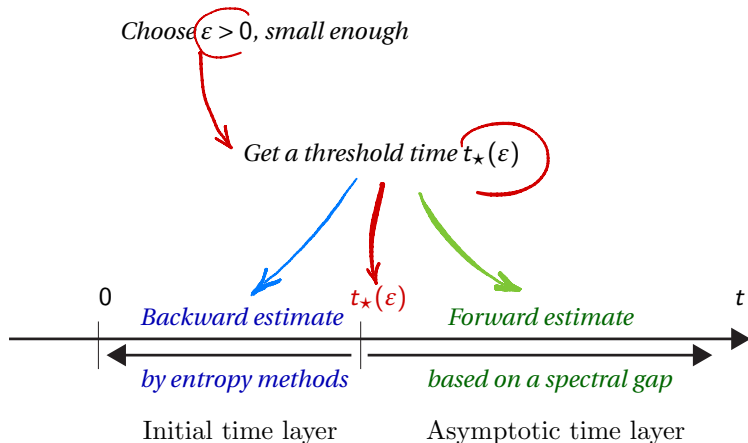


[Denzler, McCann, 2005]

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]

Much more is known, *e.g.*, [Denzler, Koch, McCann, 2015]

# Strategy of the method





# A constructive stability result (subcritical case)

*Stability in the entropy - entropy production estimate*

$\mathcal{I}[v] - 4\mathcal{F}[v] \geq \zeta \mathcal{F}[v]$  also holds in a stronger sense

$$\mathcal{I}[v] - 4\mathcal{F}[v] \geq \frac{\zeta}{4 + \zeta} \mathcal{I}[v]$$

if  $\int_{\mathbb{R}^d} x \cdot v \, dx = 0$  and  $A[v] = \sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} v \, dx < \infty$

## Theorem

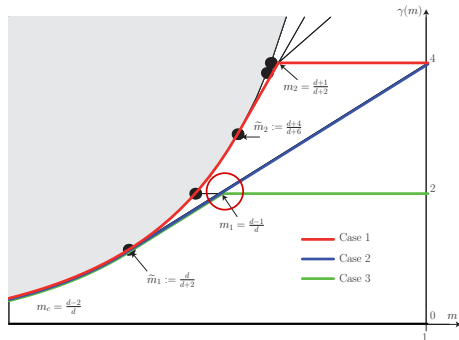
Let  $d \geq 1$  and  $p \in (1, p^*)$ . There is an explicit  $\mathcal{C} = \mathcal{C}[f] > 0$  such that, for any  $f \in L^{2p}(\mathbb{R}^d, (1 + |x|^2) \, dx)$  s.t.  $\nabla f \in L^2(\mathbb{R}^d)$  and  $A[f^{2p}] < \infty$

$$\begin{aligned} (p-1)^2 \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 + 4 \frac{d-p(d-2)}{p+1} \|f\|_{L^{p+1}(\mathbb{R}^d)}^{p+1} - \mathcal{K}_{\text{GNS}} \|f\|_{L^{2p}(\mathbb{R}^d)}^{2p\gamma} \\ \geq \mathcal{C}[f] \inf_{\varphi \in \mathfrak{M}} \int_{\mathbb{R}^d} |(p-1)\nabla f + f^p \nabla \varphi^{1-p}|^2 \, dx \end{aligned}$$

# Extending the subcritical result to the critical case

To improve the spectral gap for  $m = m_1$ , we need to adjust the Barenblatt function  $\mathcal{B}_\lambda(x) = \lambda^{-d/2} \mathcal{B}(x/\sqrt{\lambda})$  in order to match  $\int_{\mathbb{R}^d} |x|^2 v dx$  where the function  $v$  solves (rFDE) or to further rescale  $v$  according to

$$v(t, x) = \frac{1}{\mathfrak{R}(t)^d} w\left(t + \tau(t), \frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left( \frac{1}{\mathcal{K}_*} \int_{\mathbb{R}^d} |x|^2 v dx \right)^{-\frac{d}{2}(m-m_c)} - 1, \quad \tau(0) = 0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\tau(t)}$$

## Lemma

$t \mapsto \lambda(t)$  and  $t \mapsto \tau(t)$  are bounded on  $\mathbb{R}^+$

# A constructive stability result (critical case)

Let  $2p^* = 2d/(d-2) = 2^*$ ,  $d \geq 3$  and

$$\mathcal{W}_{p^*}(\mathbb{R}^d) = \{f \in L^{p^*+1}(\mathbb{R}^d) : \nabla f \in L^2(\mathbb{R}^d), |x| f^{p^*} \in L^2(\mathbb{R}^d)\}$$

## Theorem

Let  $d \geq 3$  and  $A > 0$ . For any nonnegative  $f \in \mathcal{W}_{p^*}(\mathbb{R}^d)$  such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2) f^{2^*} dx = \int_{\mathbb{R}^d} (1, x, |x|^2) g dx \text{ and } \sup_{r>0} r^d \int_{|x|>r} f^{2^*} dx \leq A$$

we have

$$\begin{aligned} \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 - S_d^2 \|f\|_{L^{2^*}(\mathbb{R}^d)}^2 \\ \geq \frac{C_*(A)}{4 + C_*(A)} \int_{\mathbb{R}^d} \left| \nabla f + \frac{d-2}{2} f^{\frac{d}{d-2}} \nabla g^{-\frac{2}{d-2}} \right|^2 dx \end{aligned}$$

$$C_*(A) = C_*(0) (1 + A^{1/(2d)})^{-1} \text{ and } C_*(0) > 0 \text{ depends only on } d$$

# *Symmetry in the “critical” case of the Caffarelli-Kohn-Nirenberg inequalities*

*in collaboration with M.J. Esteban and M. Loss  
and M.J. Esteban, M. Loss and M. Muratori*

▷ A formal proof based on a fast diffusion flow

# Critical Caffarelli-Kohn-Nirenberg inequalities

Let  $\mathcal{D}_{a,b} := \left\{ v \in L^p(\mathbb{R}^d, |x|^{-b} dx) : |x|^{-a} |\nabla v| \in L^2(\mathbb{R}^d, dx) \right\}$

$$\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} dx \quad \forall v \in \mathcal{D}_{a,b}$$

hold under the conditions that  $a \leq b \leq a+1$  if  $d \geq 3$ ,  $a < b \leq a+1$  if  $d = 2$ ,  $a + 1/2 < b \leq a+1$  if  $d = 1$ , and  $a < a_c := (d-2)/2$

$$p = \frac{2d}{d-2+2(b-a)}$$

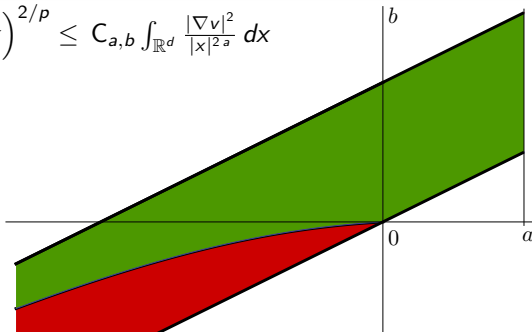
▷ An optimal function among radial functions:

$$v_*(x) = \left( 1 + |x|^{(p-2)(a_c-a)} \right)^{-\frac{2}{p-2}} \quad \text{and} \quad C_{a,b}^* = \frac{\| |x|^{-b} v_* \|_p^2}{\| |x|^{-a} \nabla v_* \|_2^2}$$

▷ Is  $v_*$  optimal without symmetry assumption ?

# Symmetry *versus* symmetry breaking: the sharp result in the critical case

$$\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} dx$$



## Theorem

Let  $d \geq 2$  and  $p < 2^*$ .  $C_{a,b} = C_{a,b}^*$  (symmetry) if and only if either  $a \in [0, a_c)$  and  $b > 0$ , or  $a < 0$  and  $b \geq b_{FS}(a)$

[JD, Esteban, Loss, 2016]

## Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension*  $n > d$  (here  $n$  is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight  $|x|^{n-d}$  which is the same in all norms. With

$$v(|x|^{\alpha-1} x) = w(x), \quad \alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2 \frac{d - \gamma}{\beta + 2 - \gamma},$$

we claim that Inequality (CKN) can be rewritten for a function  $v(|x|^{\alpha-1} x) = w(x)$  as

$$\|v\|_{L^{2p, d-n}(\mathbb{R}^d)} \leq K_{\alpha, n, p} \|D_{\alpha} v\|_{L^{2, d-n}(\mathbb{R}^d)}^{\vartheta} \|v\|_{L^{p+1, d-n}(\mathbb{R}^d)}^{1-\vartheta} \quad \forall v \in H_{d-n, d-n}^p(\mathbb{R}^d)$$

with the notations  $s = |x|$ ,  $D_{\alpha} v = (\alpha \frac{\partial v}{\partial s}, \frac{1}{s} \nabla_{\omega} v)$  and

$$d \geq 2, \quad \alpha > 0, \quad n > d \quad \text{and} \quad p \in (1, p_{\star}] .$$

By our change of variables,  $w_{\star}$  is changed into

$$v_{\star}(x) := (1 + |x|^2)^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d$$

## The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left( \int_{\mathbb{R}^d} u^m d\mu_d \right)^{\sigma-1} \int_{\mathbb{R}^d} u |D_\alpha P|^2 d\mu_d$$

where  $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$ . Here  $d\mu = |x|^{n-d} dx$  and the pressure is

$$P := \frac{m}{1-m} u^{m-1}$$

*Looking for an optimal function in (CKN) is equivalent to minimize  $\mathcal{G}$  under a mass constraint*



With  $L_\alpha = -D_\alpha^* D_\alpha = \alpha^2 \left( u'' + \frac{n-1}{s} u' \right) + \frac{1}{s^2} \Delta_\omega u$ , we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = L_\alpha u^m$$

in the subcritical range  $1 - 1/n < m < 1$ . The key computation is the proof that

$$\begin{aligned} & - \frac{d}{dt} \mathcal{G}[u(t, \cdot)] \left( \int_{\mathbb{R}^d} u^m d\mu_d \right)^{1-\sigma} \\ & \geq (1-m)(\sigma-1) \int_{\mathbb{R}^d} u^m \left| L_\alpha P - \frac{\int_{\mathbb{R}^d} u |D_\alpha P|^2 d\mu_d}{\int_{\mathbb{R}^d} u^m d\mu_d} \right|^2 d\mu_d \\ & + 2 \int_{\mathbb{R}^d} \left( \alpha^4 \left( 1 - \frac{1}{n} \right) \left| P'' - \frac{P'}{s} - \frac{\Delta_\omega P}{\alpha^2 (n-1) s^2} \right|^2 + \frac{2\alpha^2}{s^2} \left| \nabla_\omega P' - \frac{\nabla_\omega P}{s} \right|^2 \right) u^m d\mu_d \\ & + 2 \int_{\mathbb{R}^d} \left( (n-2) (\alpha_{\text{FS}}^2 - \alpha^2) |\nabla_\omega P|^2 + c(n, m, d) \frac{|\nabla_\omega P|^4}{P^2} \right) u^m d\mu_d =: \mathcal{H}[u] \end{aligned}$$

for some numerical constant  $c(n, m, d) > 0$ . Hence if  $\alpha \leq \alpha_{\text{FS}}$ , the r.h.s.  $\mathcal{H}[u]$  vanishes if and only if  $P$  is an affine function of  $|x|^2$ , which proves the symmetry result. *A quantifier elimination problem [Tarski, 1951] ?*

## (3/3: elliptic regularity, boundary terms)

This method has a hidden difficulty: integrations by parts ! Hints:

🟢 use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

🟢 use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if  $u$  solves the Euler-Lagrange equation, we test by  $L_\alpha u^m$

$$0 = \int_{\mathbb{R}^d} d\mathcal{G}[u] \cdot L_\alpha u^m d\mu_d \geq \mathcal{H}[u] \geq 0$$

$\mathcal{H}[u]$  is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by  $|x|^\gamma \operatorname{div}(|x|^{-\beta} \nabla w^{1+p})$  the equation

$$\frac{(p-1)^2}{p(p+1)} w^{1-3p} \operatorname{div}(|x|^{-\beta} w^{2p} \nabla w^{1-p}) + |\nabla w^{1-p}|^2 + |x|^{-\gamma} (c_1 w^{1-p} - c_2) = 0$$

# *Stability in Caffarelli-Kohn-Nirenberg inequalities ?*

*in collaboration with M. Bonforte, B. Nazaret and N. Simonov*

*Constructive stability results in interpolation inequalities  
and explicit improvements of decay rates of fast diffusion eq.  
DCDS, 43 (3 & 4): 1070-1089, 2023*

# Subcritical Caffarelli-Kohn-Nirenberg inequalities

On  $\mathbb{R}^d$  with  $d \geq 1$ , let us consider the *Caffarelli-Kohn-Nirenberg interpolation inequalities*

$$\|f\|_{L^{2p,\gamma}(\mathbb{R}^d)} \leq C_{\beta,\gamma,p} \|\nabla f\|_{L^{2,\beta}(\mathbb{R}^d)}^\theta \|f\|_{L^{p+1,\gamma}(\mathbb{R}^d)}^{1-\theta}$$

$$\gamma - 2 < \beta < \frac{d-2}{d} \gamma, \quad \gamma \in (-\infty, d), \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2},$$

$$\text{with } \theta = \frac{(d-\gamma)(p-1)}{p(d+\beta+2-2\gamma-p(d-\beta-2))}$$

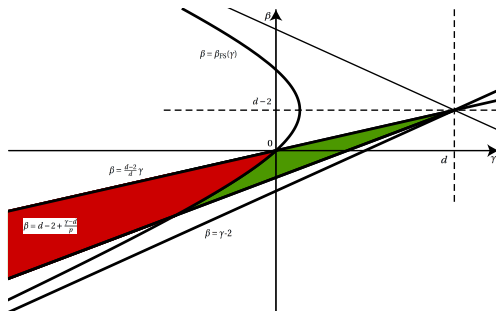
$$\text{and } \|f\|_{L^{q,\gamma}(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} |f|^q |x|^{-\gamma} dx \right)^{1/q}$$

Symmetry: equality is achieved by the *Aubin-Talenti type functions*

$$g(x) = (1 + |x|^{2+\beta-\gamma})^{-\frac{1}{p-1}}$$

[JD, Esteban, Loss, Muratori, 2017] Symmetry holds if and only if

$$\gamma < d, \quad \text{and} \quad \gamma - 2 < \beta < \frac{d-2}{d} \gamma \quad \text{and} \quad \beta \leq \beta_{\text{FS}}(\gamma)$$



$d = 4$  and  $p = 6/5$ :  $(\gamma, \beta)$  admissible region

V

# An improved decay rate along the flow

In self-similar variables, with  $m = (p+1)/(2p)$

$$|x|^{-\gamma} \frac{\partial v}{\partial t} + \nabla \cdot (|x|^{-\beta} v \nabla v^{m-1}) = \sigma \nabla \cdot (x |x|^{-\gamma} v)$$

$$\mathcal{F}[v] = \frac{2p}{1-p} \int_{\mathbb{R}^d} \left( v^{\frac{p+1}{2p}} - g^{p+1} - \frac{p+1}{2p} g^{1-p} (v - g^{2p}) \right) |x|^{-\gamma} dx$$

## Theorem

*In the symmetry region, if  $v \geq 0$  is a solution with a initial datum  $v_0$  s.t.*

$$A[v_0] := \sup_{R>0} R^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)} \int_{|x|>R} v_0(x) |x|^{-\gamma} dx < \infty$$

*then there are some  $\zeta > 0$  and some  $T > 0$  such that*

$$\mathcal{F}[v(t, \cdot)] \leq \mathcal{F}[v_0] e^{-(4\alpha^2 + \zeta)t} \quad \forall t \geq 2T$$

[Bonforte, JD, Nazaret, Simonov, 2022]

# Entropy methods and stability: some basic references

- Model inequalities: [Gagliardo, 1958], [Nirenberg, 1958]  
*Carré du champ*: [Bakry, Emery, 1985]
- Motivated by asymptotic rates of convergence in kinetic equations:
  - ▷ linear diffusions: [Toscani, 1998], [Arnold, Markowich, Toscani, Unterreiter, 2001]
  - ▷ Nonlinear diffusion for the carré du champ [Carrillo, Toscani], [Carrillo, Vázquez], [Carrillo, Jüngel, Markowich, Toscani, Unterreiter]
  - ▷ Sharp global decay rates, nonlinear diffusions: [del Pino, JD, 2001] (variational methods), [Carrillo, Jüngel, Markowich, Toscani, Unterreiter] (carré du champ), [Jüngel], [Demange] (manifolds)
- Refinements and stability [Arnold, Dolbeault], [Blanchet, Bonforte, JD, Grillo, Vázquez], [JD, Toscani], [JD, Esteban, Loss]
- Detailed stability results by entropy methods
  - on  $\mathbb{R}^d$ : [Bonforte, JD, Nazaret, Simonov]
  - on  $\mathbb{S}^d$ : [Brigati, JD, Simonov]
  - ▷ Side results: hypocoercivity; symmetry in CKN inequalities
  - ▷ Angle of attack: *entropy methods and diffusion flows as a tool*

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Thank you for your attention !