Interpolation between logarithmic Sobolev and Poincaré inequalities

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*A work in collaboration with

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Gaussian measures

[W. Beckner, 1989]: a family of generalized Poincaré inequalities (GPI)

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\mu-\left(\int_{\mathbb{R}^d}|f|^p\,d\mu\right)^{2/p}\right]\le\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu\quad\forall f\in H^1(d\mu)\qquad(1)$$

where $\mu(x)$ denotes the normal centered Gaussian distribution on $\mathbb{R}^d$

$$\mu(x):=(2\pi)^{-d/2}\,e^{-\frac{1}{2}\,|x|^2}$$

For $p=1$: the Poincaré inequality

$$\int_{\mathbb{R}^d} f^2\,d\mu-\left(\int_{\mathbb{R}^d} f\,d\mu\right)^2\le\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu\quad\forall f\in H^1(d\mu)$$

In the limit $p\to 2$: the logarithmic Sobolev inequality (LSI) [L. Gross 1975]

$$\int_{\mathbb{R}^d} f^2\log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\mu}\right)\,d\mu\int_{\mathbb{R}^d}|\nabla f|^2\,d\mu\quad\forall\ f\in H^1(d\mu)$$

Generalizations of (1) to other probability measures...
Quest for "sharpest" constants in such inequalities...
... best (worst ?) decay rates

[AMTU]: for strictly log-concave distribution functions $\nu(x)$

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall f \in H^1(d\nu)$$

where κ is the uniform convexity bound of $-\log \nu(x)$...

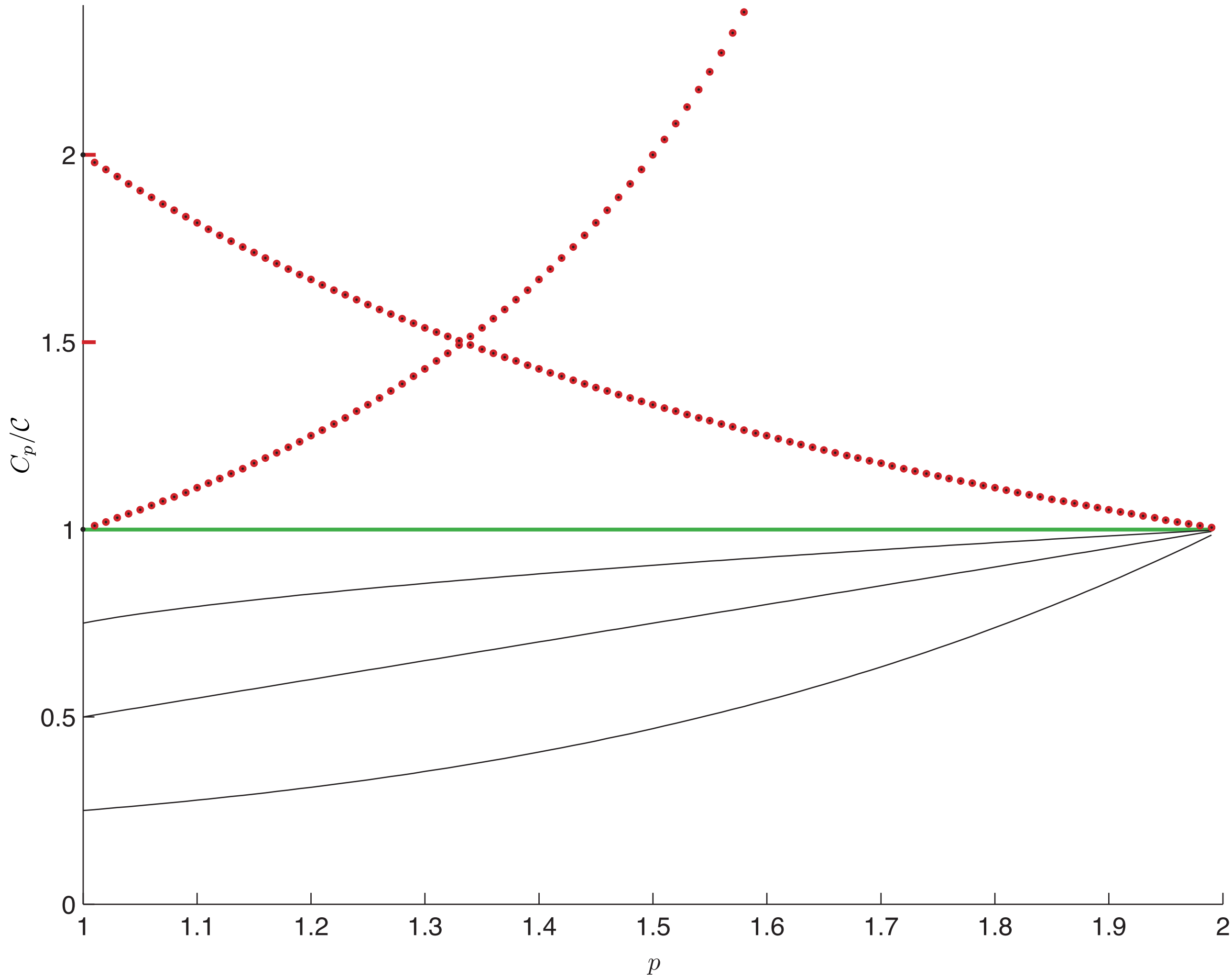
...the Bakry-Emery criterion

[Latała and Oleszkiewicz]: under the weaker assumption that $\nu(x)$ satisfies a LSI with constant $0 < \mathcal{C} < \infty$

$$\int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\int_{\mathbb{R}^d} f^2\,d\nu}\right) d\nu \le 2\,\mathcal{C}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \quad \forall\, f \in H^1(d\nu) \qquad (2)$$

they proved for $1 \le p < 2$:

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p}\right] \le \mathcal{C}\min\left\{\frac{2}{p}, \frac{1}{2-p}\right\}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu$$



$C_p/\mathcal{C}$
0
0.5
1
1.5
2
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
p

Proof. 1) The function $q \mapsto \alpha(q) := q \log\left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)$ is convex since

$$\alpha''(q) = \frac{4}{q^3}\,\frac{\left(\int_{\mathbb{R}^d} |f|^{2/q}\,(\log|f|)^2\, d\nu\right)\left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right) - \left(\int_{\mathbb{R}^d} |f|^{2/q} \log|f|\, d\nu\right)^2}{\left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^2}$$

is nonnegative. Thus $q \mapsto e^{\alpha(q)}$ is also convex and

$$q \mapsto \varphi(q) := \frac{e^{\alpha(1)} - e^{\alpha(q)}}{q-1} \quad \searrow$$

$$\varphi(q) \le \lim_{q_1 \to 1} \varphi(q_1) = \int_{\mathbb{R}^d} f^2 \log\left(\frac{f^2}{\|f\|^2_{L^2(d\mu)}}\right) d\nu$$

This proves that

$$\frac{1}{q-1}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^q\right] \le 2\mathcal{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

$\frac{1}{q-1}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^q\right] = \frac{p}{2-p}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^p\, d\nu\right)^{2/p}\right]$
if $p = 2/q$: $C_p \le 2\mathcal{C}/p$.

2) Linearization $f = 1 + \varepsilon\, g$ with $\int_{\mathbb{R}^d} g\, d\nu = 0$, limit $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} f\, d\nu\right)^2 \le \mathcal{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

Hölder's inequality, $\left(\int_{\mathbb{R}^d} f\, d\nu\right)^2 \le \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^q$

$$\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^{2/q}\, d\nu\right)^q \le \int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} f\, d\nu\right)^2 \le \mathcal{C} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

1) proof of (1)
2) improve upon (1) for functions f that are in the orthogonal of the first eigenspaces of N
3) generalize to other measures

The spectrum of the Ornstein-Uhlenbeck operator $\mathsf{N} := -\Delta + x \cdot \nabla$ is made of all nonnegative integers $k \in \mathbb{N}$, the corresponding eigenfunctions are the Hermite polynomials. Observe that

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu = \int_{\mathbb{R}^d} f \cdot \mathsf{N} f \, d\mu \quad \forall\, f \in H^1(d\mu)$$

Strategy of Beckner (improved): consider the $L^2(d\mu)$-orthogonal decomposition of f on the eigenspaces of N, i.e.

$$f = \sum_{k \in \mathbb{N}} f_k,$$

where $\mathsf{N}\, f_k = k\, f_k$. If we denote by π_k the orthogonal projection on the eigenspace of N associated to the eigenvalue $k \in \mathbb{N}$, then $f_k = \pi_k[f]$.

$$a_k := \|f_k\|^2_{L^2(d\mu)}\,, \quad \|f\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} a_k \quad \text{and} \quad \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu = \sum_{k\in\mathbb{N}} k\,a_k$$

The solution of the evolution equation associated to N

$$u_t = -\mathsf{N}\,u = \Delta u - x\cdot\nabla u$$

with initial data f is given by

$$u(x,t) = (e^{-t\,\mathsf{N}}\,f)(x) = \sum_{k\in\mathbb{N}} e^{-k\,t}\,f_k(x)$$

$$\left\|e^{-t\,\mathsf{N}}\,f\right\|^2_{L^2(d\mu)} = \sum_{k\in\mathbb{N}} e^{-2\,k\,t} a_k$$

Lemma 1 *Let $f\in H^1(d\mu)$. If $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0\geq 1$, then*

$$\int_{\mathbb{R}^d} |f|^2\,d\mu - \int_{\mathbb{R}^d} \left|e^{-t\,N} f\right|^2\,d\mu \leq \frac{1-e^{-2k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu$$

The component f_0 of f does not contribute to the inequality.

Proof. We use the decomposition on the eigenspaces of N

$$\int_{\mathbb{R}^d} |f_k|^2 \, d\mu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f_k \right|^2 \, d\mu = (1 - e^{-2\,k\,t})\, a_k$$

For any fixed $t > 0$, the function

$$k \mapsto \frac{1 - e^{-2\,k\,t}}{k}$$

is monotone decreasing: if $k \geq k_0$, then

$$1 - e^{-2\,k\,t} \leq \frac{1 - e^{-2\,k_0\,t}}{k_0}\, k$$

Thus we get

$$\int_{\mathbb{R}^d} |f_k|^2 \, d\mu - \int_{\mathbb{R}^d} \left| e^{-t\,\mathsf{N}} f_k \right|^2 \, d\mu \leq \frac{1 - e^{-2\,k_0\,t}}{k_0} \int_{\mathbb{R}^d} |\nabla f_k|^2 \, d\mu$$

which proves the result by summation □

The second preliminary result is Nelson's hypercontractive estimates, equivalent to the logarithmic Sobolev estimates, [Gross 1975]

Lemma 2 *For any $f \in L^p(d\mu)$, $p \in (1,2)$, it holds*

$$\left\| e^{-t\mathsf{N}} f \right\|_{L^2(d\mu)} \le \|f\|_{L^p(d\mu)} \quad \forall\, t \ge -\frac{1}{2}\log(p-1)$$

Proof. We set

$$F(t) := \left(\int_{\mathbb{R}^d} |u(t)|^{q(t)}\, d\mu \right)^{1/q(t)}$$

with $q(t)$ to be chosen later and $u(x,t) := \left(e^{-t\mathsf{N}} f\right)(x)$. A direct computation gives

$$\frac{F'(t)}{F(t)} = \frac{q'(t)}{q^2(t)} \int_{\mathbb{R}^d} \frac{|u|^q}{F^q} \log\left(\frac{|u|^q}{F^q}\right) d\mu - \frac{4}{F^q}\frac{q-1}{q^2} \int_{\mathbb{R}^d} \left|\nabla\left(|u|^{q/2}\right)\right|^2 d\mu$$

We set $v := |u|^{q/2}$, use the LSI (2) with $\nu = \mu$ and $\mathcal{C} = 1$, and choose q such that $4\,(q-1) = 2q'$, $q(0) = p$ and $q(t) = 2$. This implies $F'(t) \le 0$ and ends the proof with $2 = q(t) = 1 + (p-1)\,e^{2t}$ □

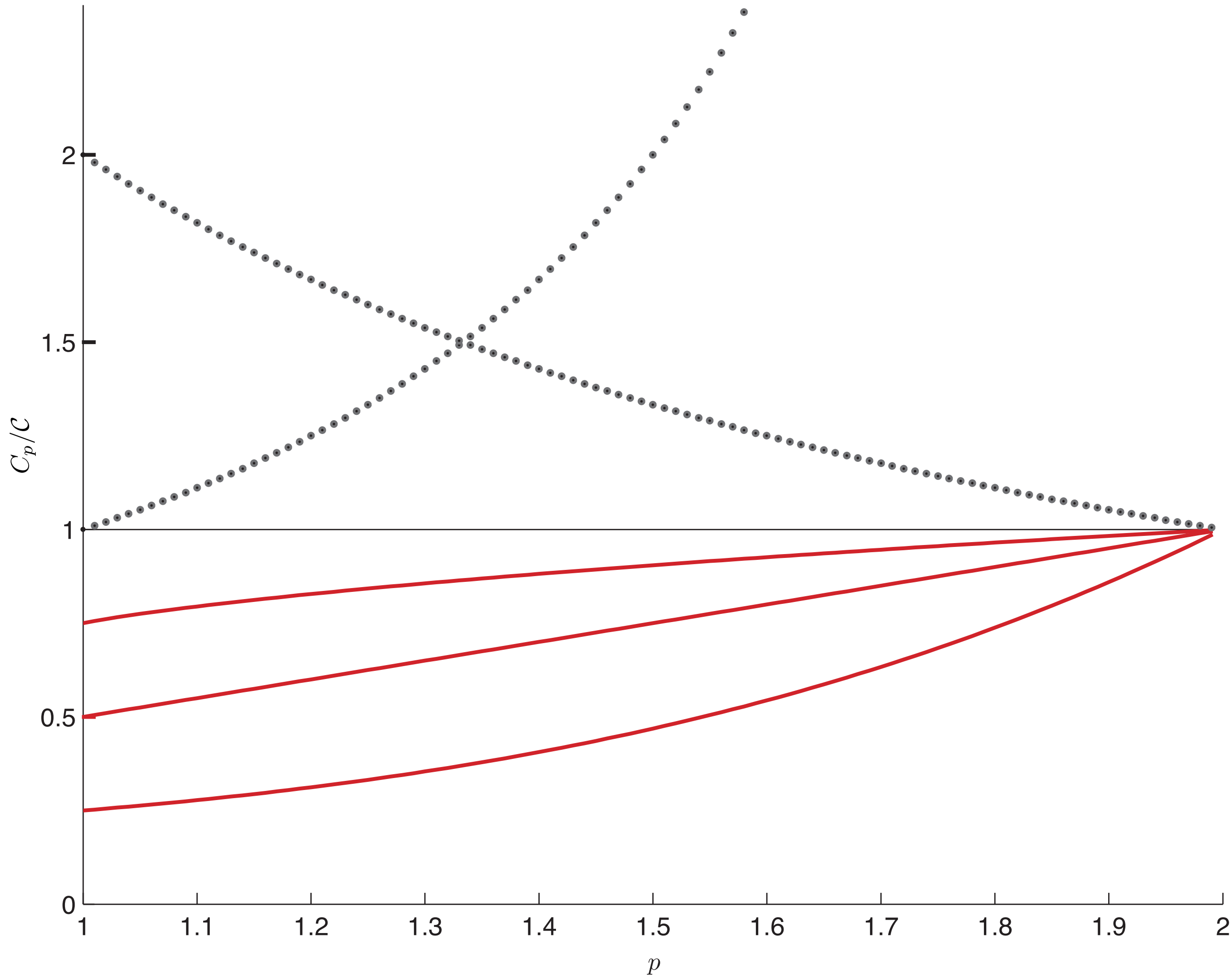
[Arnold, Bartier, J.D.] First result, for the Gaussian distribution $\mu(x)$: a generalization of Beckner's estimates.

Theorem 3 *Let* $f \in H^1(d\mu)$*. If* $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ *for some* $k_0 \geq 1$*, then*

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} |f|^2\,d\mu - \left(\int_{\mathbb{R}^d} |f|^p\,d\mu\right)^{2/p}\right] \leq \frac{1-(p-1)^{k_0}}{k_0\,(2-p)} \int_{\mathbb{R}^d} |\nabla f|^2\,d\mu$$

holds for $1 \leq p < 2$*.*

In the special case $k_0 = 1$ this is exactly the GPI (1) due to Beckner, and for $k_0 > 1$ it is a strict improvement for any $p \in [1,2)$.



2
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$C_p/\mathcal{C}$
1
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1.5
1.6
1.7
1.8
1.9
2
p

Other measures: generalization

Generalization to probability measures with densities with respect to Lebesgue's measure given by

$$\nu(x) := e^{-V(x)}$$

on $\mathbb{R}^d$, that give rise to a LSI (2) with a positive constant $\mathcal{C}$. The operator $\mathsf{N} := -\Delta + \nabla V \cdot \nabla$, considered on $L^2(\mathbb{R}^d, d\nu)$, has a pure point spectrum made of nonnegative eigenvalues by λ_k, $k \in \mathbb{N}$.

$\lambda_0 = 0$ is non-degenerate. The spectral gap λ_1 yields the sharp Poincaré constant $1/\lambda_1$, and it satisfies

$$\frac{1}{\lambda_1} \leq \mathcal{C}$$

This is easily recovered by taking $f = 1 + \varepsilon g$ in (2) and letting $\varepsilon \to 0$.

Same as in the Gaussian case: with $a_k := \|f_k\|^2_{L^2(d\nu)}$,

$$\|f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}} a_k\,,\ \|\nabla f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}} \lambda_k\, a_k\,,\ \|e^{-t\,\mathsf{N}} f\|^2_{L^2(d\nu)} = \sum_{k\in\mathbb{N}} e^{-2\,\lambda_k t} a_k$$

Using the monotonicity of λ_k, we get

$$\int_{\mathbb{R}^d} |f|^2\, d\nu - \int_{\mathbb{R}^d} \left|e^{-t\,\mathsf{N}} f\right|^2 d\nu \le \frac{1 - e^{-2\lambda_{k_0} t}}{\lambda_{k_0}} \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

if $f \in H^1(d\nu)$ is such that $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \ge 1$

$$\left\|e^{-t\mathsf{N}} f\right\|_{L^2(d\nu)} \le \|f\|_{L^p(d\nu)} \quad \forall\, t \ge -\frac{\mathcal{C}}{2} \log(p-1) \quad \forall\, p \in (1,2)$$

Theorem 4 [Arnold, Bartier, J.D.] *Let ν satisfy the LSI (2). If $f \in H^1(d\nu)$ is such that $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \ge 1$, then*

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^p\, d\nu\right)^{2/p}\right] \le C_p \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu \qquad (3)$$

holds for $1 \le p < 2$, with $C_p := \frac{1-(p-1)^\alpha}{\lambda_{k_0}\,(2-p)}$, $\quad \alpha := \lambda_{k_0}\, \mathcal{C} \ge 1$

1) "α large" ? Even in the special case $k_0 = 1$, the measure $d\nu$ satisfies in many cases $\alpha = \lambda_1 \mathcal{C} > 1$: $\nu(x) := c_\varepsilon \exp(-|x| - \varepsilon x^2)$ with $\varepsilon \to 0$.

2) Optimal case: $\mathcal{C} = 1/\lambda_1$ (i.e. $\alpha = 1$), $k_0 = 1$: $C_p = \mathcal{C}$, for any $p \in [1, 2]$ is optimal [generalizes the situation for gaussian measures].

3) For $k_0 > 1$, $\alpha > 1$ is always true.

4) For fixed $\alpha \geq 1$, C_p takes the sharp limiting values for the Poincaré inequality ($p = 1$) and the LSI ($p = 2$): $C_1 = 1/\lambda_1$ and $\lim_{p \to 2} C_p = \mathcal{C}$.

5) For $\alpha > 1$, C_p is monotone increasing in p. Hence, $C_p < \mathcal{C}$ for $p < 2$ and $\alpha > 1$, and Theorem 4 strictly improves upon known constants.

A REFINED INTERPOLATION INEQUALITY

Theorem 5 [Arnold, J.D.] *for all* $p \in [1,2)$

$$\frac{1}{(2-p)^2}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2\left(\frac{2}{p}-1\right)}\left(\int_{\mathbb{R}^d} f^2\,d\nu\right)^{p-1}\right] \le \frac{1}{\kappa}\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu \tag{4}$$

for any $f \in H^1(d\nu)$, *where* κ *is the uniform convexity bound of* $-\log\nu(x)$

(GPI) is a consequence of (4): use Hölder's inequality,

$$\left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{2/p} \le \int_{\mathbb{R}^d} f^2\,d\nu$$

and the inequality $(1-t^{2-p})/(2-p) \ge 1-t$ for any $t \in [0,1]$, $p \in (1,2)$

Entropy-entropy production method [Bakry, Emery, 1984]
[Toscani 1996], [Arnold, Markowich, Toscani, Unterreiter, 2001]

Relative entropy of $u = u(x)$ w.r.t. $u_\infty(x)$

$$\Sigma[u|u_\infty] := \int_{\mathbb{R}^d} \psi\left(\frac{u}{u_\infty}\right) u_\infty \, dx \geq 0$$

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \end{aligned}$$

Admissibility condition $(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$

Examples

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(u|u_\infty) = \int u \ln\left(\frac{u}{u_\infty}\right) dx \ldots$ "physical entropy"

$\psi_p = w^p - p(w-1) - 1$, $1 < p \leq 2$, $\Sigma_2(u|u_\infty) = \int_{\mathbb{R}^d} (u - u_\infty)^2 u_\infty^{-1} dx$

Exponential decay of entropy production

$$u_t = \Delta u + \nabla \cdot (u \nabla V)$$

$$-I(u(t)|u_\infty) := \frac{d}{dt}\Sigma[u(t)|u_\infty] = -\int \psi''\left(\frac{u}{u_\infty}\right) \underbrace{\left|\nabla\left(\frac{u}{u_\infty}\right)\right|^2}_{=:v} u_\infty \, dx \leq 0$$

$$V(x) = -\log u_\infty \ldots \text{ unif. convex: } \underbrace{\frac{\partial^2 V}{\partial x^2}}_{\text{Hessian}} \geq \lambda_1 Id, \ \lambda_1 > 0$$

Entropy production rate

$$-I' \;=\; 2\int \psi''\left(\frac{u}{u_\infty}\right) v^T \cdot \frac{\partial^2 V}{\partial x^2} \cdot v \, u_\infty \, dx + \underbrace{2\int \mathrm{Tr}\,(XY)\, u_\infty \, dx}_{\geq 0}$$

$$\geq \; +2\,\lambda_1\, I$$

Positivity of $\mathrm{Tr}\,(XY)$?

$$X = \begin{pmatrix} \psi''\left(\frac{u}{u_\infty}\right) & \psi'''\left(\frac{u}{u_\infty}\right) \\ \psi'''\left(\frac{u}{u_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{u}{u_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij}(\frac{\partial v_i}{\partial x_j})^2 & v^T \cdot \frac{\partial v}{\partial x} \cdot v \\ v^T \cdot \frac{\partial v}{\partial x} \cdot v & |v|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow \quad I(t) \leq e^{-2\lambda_1 t} I(t=0) \qquad \forall\, t > 0$$

$$\forall\, u_0 \text{ with } I(t=0) = I(u_0|u_\infty) < \infty$$

Exponential decay of relative entropy

Known: $\quad -I' \geq \; 2\,\lambda_1 \underbrace{I}_{=\Sigma'}$

$\int_t^\infty \ldots dt \qquad \Rightarrow \Sigma' = I \leq -2\lambda_1 \Sigma$

Theorem 6 [Bakry, Emery] [Arnold, Markowich, Toscani, Unterreiter]
Under the "Bakry–Emery condition"

$$\frac{\partial^2 V}{\partial x^2} \geq \lambda_1 Id$$

if $\Sigma[u_0|u_\infty] < \infty$, *then*

$$\Sigma[u(t)|u_\infty] \leq \Sigma[u_0|u_\infty]\, e^{-2\lambda_1 t} \qquad \forall\, t > 0$$

Convex Sobolev inequalities

Entropy–entropy production estimate for $V(x) = -\ln u_\infty$

$$\Sigma[u|u_\infty] \leq \frac{1}{2\lambda_1} |I(u|u_\infty)| \tag{5}$$

Example 1 logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int u \ln\left(\frac{u}{u_\infty}\right) dx \leq \frac{1}{2\lambda_1} \int u \left|\nabla \ln\left(\frac{u}{u_\infty}\right)\right|^2 dx$$

$$\forall u,\, u_\infty \in L^1_+(\mathbb{R}^d),\ \int u\, dx = \int u_\infty\, dx = 1$$

Example 2 power law entropies

$\psi_p(w) = w^p - p(w-1) - 1, \qquad 1 < p \leq 2$

$$\frac{p}{p-1}\left[\int f^2\, du_\infty - \left(\int |f|^{\frac{2}{p}}\, du_\infty\right)^p\right] \leq \frac{2}{\lambda_1} \int |\nabla f|^2 du_\infty$$

from (5) with $\frac{u}{u_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} du_\infty}$, $f \in L^{\frac{2}{p}}(\mathbb{R}^d, u_\infty dx)$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production

$$\begin{aligned} I' &= 2\int \psi''\left(\frac{u}{u_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u u_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) u_\infty dx}_{\geq 0} \\ &\geq -2\lambda_1 I \end{aligned}$$

[Arnold, J.D.] Observe that $\psi_p(w) = w^p - p(w-1) - 1$, $1 < p < 2$

$$X = \begin{pmatrix} \psi''\left(\frac{u}{u_\infty}\right) & \psi'''\left(\frac{u}{u_\infty}\right) \\ \psi'''\left(\frac{u}{u_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{u}{u_\infty}\right) \end{pmatrix} > 0$$

• Assume $\frac{\partial V^2}{\partial x^2} \geq \lambda_1 Id \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$, $\qquad \kappa = \frac{2-p}{p} < 1$

$\Rightarrow \boxed{k(\Sigma[u|u_\infty]) \leq \frac{1}{2\lambda_1}|\Sigma'|} = \frac{1}{2\lambda_1}\int \psi''\left(\frac{u}{u_\infty}\right)|\nabla \frac{u}{u_\infty}|^2 du_\infty$

Refined convex Sobolev inequality with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

• Set $u/u_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} du_\infty \Rightarrow$ *Refined Beckner inequality* [Arnold, J.D.]

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 du_\infty - \left(\int |f|^{\frac{2}{p}} du_\infty\right)^{2(p-1)} \left(\int f^2 du_\infty\right)^{\frac{2-p}{p}}\right] \leq \frac{2}{\lambda_1}\int |\nabla f|^2 du_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^d, du_\infty)$$

$\square$

Back to the method of Beckner... First extension: for all $\gamma \in (0,2)$

$$\frac{1}{(2-p)^2}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d}|f|^p\,d\nu\right)^{\frac{\gamma}{p}}\left(\int_{\mathbb{R}^d} f^2\,d\nu\right)^{\frac{2-\gamma}{2}}\right] \le K_p(\gamma)\int_{\mathbb{R}^d}|\nabla f|^2\,d\nu$$

with $K_p(\gamma) := \frac{1-(p-1)^{\alpha\gamma/2}}{\lambda_{k_0}\,(2-p)^2}$

$$\mathcal{N} := \|f\|^2_{L^2(d\nu)} - \left\|e^{-t\mathsf{N}} f\right\|^{\gamma}_{L^2(d\nu)}\,\|f\|^{2-\gamma}_{L^2(d\nu)} = \sum_{k\ge k_0} a_k - \left(\sum_{k\ge k_0} a_k\,e^{-2\lambda_k t}\right)^{\frac{\gamma}{2}}\left(\sum_{k\ge k_0} a_k\right)$$

for any $t \ge -\frac{\mathcal{C}}{2}\log(p-1)$. By Hölder's inequality

$$\sum_{k\ge k_0} a_k\,e^{-\gamma\lambda_k t} = \sum_{k\ge k_0}\left(a_k\,e^{-2\lambda_k t}\right)^{\frac{\gamma}{2}}\cdot a_k^{\frac{2-\gamma}{2}} \le \left(\sum_{k\ge k_0} a_k\,e^{-2\lambda_k t}\right)^{\frac{\gamma}{2}}\left(\sum_{k\ge k_0} a_k\right)^{\frac{2-\gamma}{2}}$$

$$\mathcal{N} \le \sum_{k\ge k_0} a_k\left(1-e^{-\gamma\lambda_k t}\right) \le \frac{1-e^{-\gamma\lambda_{k_0} t}}{\lambda_{k_0}}\sum_{k\ge k_0}\lambda_k\,a_k = \frac{1-e^{-\gamma\lambda_{k_0} t}}{\lambda_{k_0}}\int_{\mathbb{R}^d}|\nabla f|^2\,d\nu$$

$$\frac{1}{(2-p)^2}\left[\int_{\mathbb{R}^d} f^2\,d\nu - \left(\int_{\mathbb{R}^d} |f|^p\,d\nu\right)^{\frac{\gamma}{p}} \left(\int_{\mathbb{R}^d} f^2\,d\nu\right)^{\frac{2-\gamma}{2}}\right] \le K_p(\gamma)\int_{\mathbb{R}^d} |\nabla f|^2\,d\nu$$

Optimize w.r.t. $\gamma \in (0,2)$. After dividing the l.h.s. by $K_p(\gamma)$ we have to find the maximum of the function

$$\gamma \mapsto h(\gamma) := \frac{1-a^\gamma}{1-b^\gamma}\,, \qquad \text{with} \quad a = \frac{\|f\|_{L^p(d\nu)}}{\|f\|_{L^2(d\nu)}} \le 1\,, \quad b = (p-1)^{\alpha/2} \le 1$$

on $\gamma \in [0,2]$.

Theorem 7 *Let ν satisfy the* LSI *(2) with the positive constant $\mathcal{C}$. If $f \in H^1(d\nu)$ is such that $f_1 = f_2 = \ldots = f_{k_0-1} = 0$ for some $k_0 \ge 1$, then*

$$\lambda_{k_0} \max\left\{ \frac{\|f\|^2_{L^2(d\nu)} - \|f\|^2_{L^p(d\nu)}}{1-(p-1)^\alpha},\ \frac{\|f\|^2_{L^2(d\nu)}}{\log(p-1)^\alpha} \log\left(\frac{\|f\|^2_{L^p(d\nu)}}{\|f\|^2_{L^2(d\nu)}}\right) \right\} \le \|\nabla f\|^2_{L^2(d\nu)} \tag{6}$$

holds for $1 \le p < 2$, with $\alpha := \lambda_{k_0}\,\mathcal{C} \ge 1$.

1) limiting cases of (6): the sharp Poincaré inequality ($p = 1$) and the LSI ($p = 2$). The previous result corresponds to the refined convex Sobolev inequality (4) for $\gamma = 2(2-p)$, but with a different constant $K_p(\gamma)$

2) the refined convex Sobolev inequality (4) holds under the Bakry-Emery condition, while the new estimate (6) holds under the weaker assumption that $\nu(x)$ satisfies the (LSI) inequality

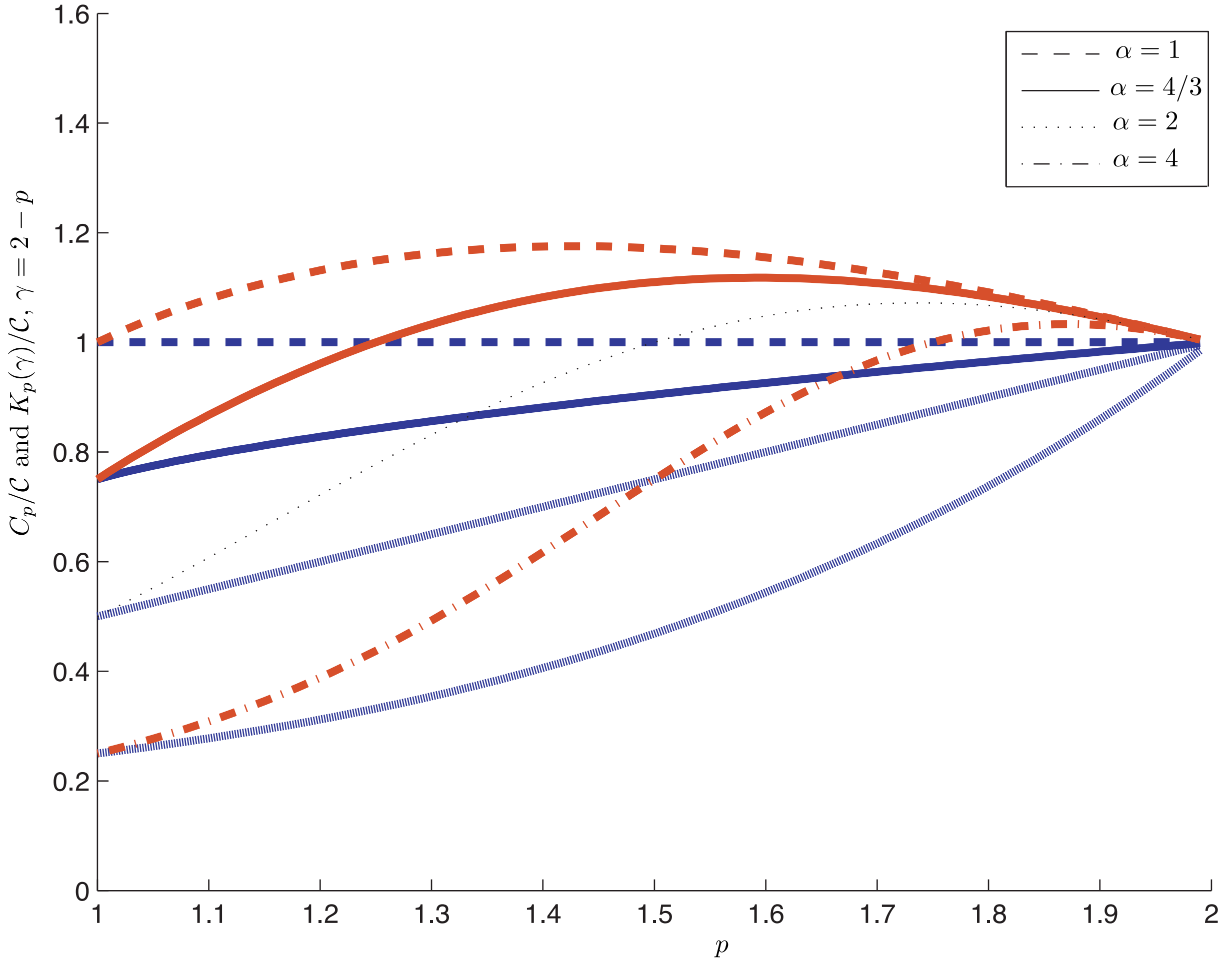
3) $1 - x^{\gamma/2} \geq \frac{\gamma}{2}(1-x)$ with $x = \|f\|^2_{L^p(d\nu)}/\|f\|^2_{L^2(d\nu)} \leq 1$

$$\|f\|^2_{L^2(d\nu)} - \|f\|^{\gamma}_{L^p(d\nu)}\|f\|^{2-\gamma}_{L^2(d\nu)} \geq \frac{\gamma}{2}\left[\|f\|^2_{L^2(d\nu)} - \|f\|^2_{L^p(d\nu)}\right] \quad \forall\, \gamma \in (0,2)$$

(BPI) is a consequence with $1/\kappa = 2(2-p)\, K_p(\gamma)/\gamma$:

$$\frac{1}{2-p}\left[\int_{\mathbb{R}^d} f^2\, d\nu - \left(\int_{\mathbb{R}^d} |f|^p\, d\nu\right)^{2/p}\right] \leq \frac{2(2-p)}{\gamma}\, K_p(\gamma) \int_{\mathbb{R}^d} |\nabla f|^2\, d\nu$$

Notice that $C_p = \frac{1-(p-1)^{\alpha}}{\lambda_{k_0}\,(2-p)} \leq \frac{2(2-p)}{\gamma}\, K_p(\gamma) = \frac{2}{\gamma}\, \frac{1-(p-1)^{\alpha\gamma/2}}{\lambda_{k_0}\,(2-p)}$



1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
1
1.1
1.2
1.3
1.4
1.5
1.6
1.7
1.8
1.9
2
p
$C_p/\mathcal{C}$ and $K_p(\gamma)/\mathcal{C}$, $\gamma = 2 - p$
$\alpha = 1$
$\alpha = 4/3$
$\alpha = 2$
$\alpha = 4$

4) Concavity of the map $x \mapsto x^{\gamma/2}$... if $x = \|f\|^2_{L^p(d\nu)}/\|f\|^2_{L^2(d\nu)} \leq (p-1)^\alpha$

$$\frac{1}{(2-p)\,C_p}\left[\|f\|^2_{L^2(d\nu)} - \|f\|^2_{L^p(d\nu)}\right] \leq \frac{1}{(2-p)^2\,K_p(\gamma)}\left[\|f\|^2_{L^2(d\nu)} - \|f\|^{\gamma}_{L^p(d\nu)}\|f\|^{2-\gamma}_{L^2(d\nu)}\right]$$

The new inequality is stronger than Inequality (3)...

Assume $\alpha = 1$ and $\mathcal{C} = 1/\kappa$, where κ is the uniform convexity bound on $-\log \nu$ and define

$$\mathsf{e}_p[f] := \frac{\|f\|^2_{L^2(d\nu)}}{\|f\|^2_{L^p(d\nu)}} - 1$$

which is related to the entropy Σ by

$$d\nu = u_\infty\,dx\,, \quad \mathsf{e}_p[f] = (2-p)\Sigma[u|u_\infty]\,, \quad \frac{u}{u_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} du_\infty}$$

The corresponding *entropy production* is

$$\mathrm{I}_p[f] := \frac{2(2-p)}{\|f\|^2_{L^p(d\nu)}} \|\nabla f\|^2_{L^2(d\nu)}$$

(GPI) gives a lower bound for $\mathrm{I}_p[f]$:

$$\mathrm{e}_p[f] \le \frac{1}{2\kappa} \mathrm{I}_p[f]$$

while (4) and (6) are *nonlinear refinements*:

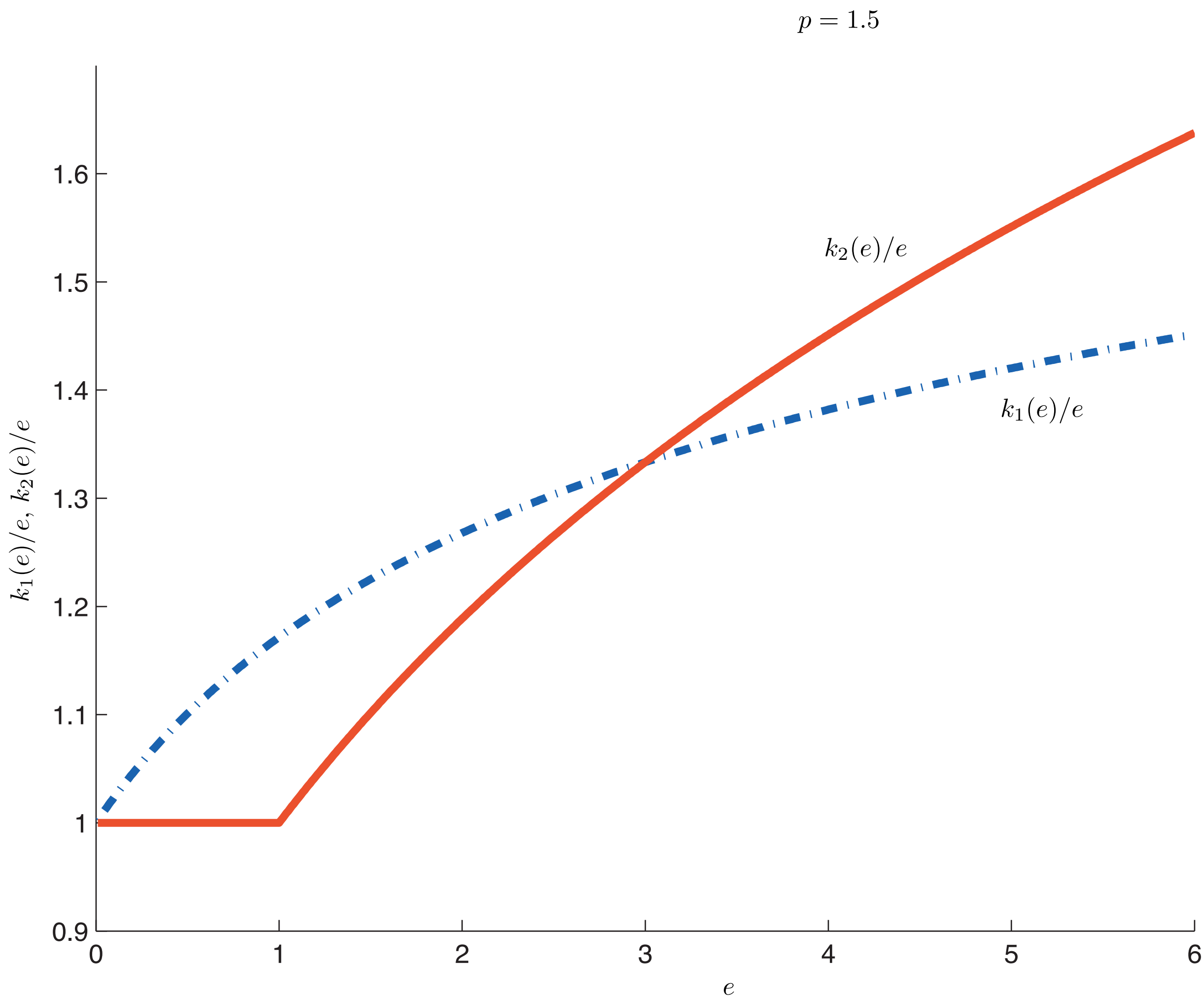
$$k_1(\mathrm{e}_p[f]) \le \frac{1}{2\kappa} \mathrm{I}_p[f], \quad k_1(e) := \frac{1}{2-p}[e+1-(e+1)^{p-1}] \ge e$$

and

$$k_2(\mathrm{e}_p[f]) \le \frac{1}{2\kappa} \mathrm{I}_p[f],$$

$$k_2(e) := \max \left\{ e,\ \frac{2-p}{|\log(p-1)|}(e+1)\log(e+1) \right\} \ge e$$

We remark that for the logarithmic entropy similar nonlinear estimates are discussed in §§1.3, 4.3 of [L].


$p = 1.5$
$k_2(e)/e$
$k_1(e)/e$
$k_1(e)/e,\ k_2(e)/e$
e
1.6
1.5
1.4
1.3
1.2
1.1
1
0.9
0
1
2
3
4
5
6

Holley-Stroock type perturbations results

Assume that the inequality

$$k\left(\int_{\mathbb{R}^d}\psi\left(f^2\right)\,d\rho_\infty\right)\leq\frac{2}{\lambda_1}\int_{\mathbb{R}^d}f^2\,\psi''\left(f^2\right)D|\nabla f|^2\,d\rho_\infty \tag{7}$$

holds with $\|f\|^2_{L^2(du_\infty)}=1$, $\psi(w)=w^p-1-p(w-1)$, $1<p<2$

Theorem 8 [Arnold, J.D.] *Let* $u_\infty(x)=e^{-V(x)}$, $\widetilde{u_\infty}(x)=e^{-\widetilde{V}(x)}\in L^1_+(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d}u_\infty\,dx=\int_{\mathbb{R}^d}\widetilde{u_\infty}\,dx=M$ *and*

$$\begin{aligned}\widetilde{V}(x)&=V(x)+v(x)\\ 0<a&\leq e^{-v(x)}\leq b<\infty\end{aligned}$$

Then a convex Sobolev inequality also holds for $d\widetilde{u_\infty}$:

$$\frac{1}{a^{p-1}}\,k_1\left(\frac{a^p}{b}\int_{\mathbb{R}^d}\psi\left(\frac{f^2}{\|f\|^2_{L^2}}\right)d\widetilde{u_\infty}\right)\leq\frac{2}{\lambda_1}\int_{\mathbb{R}^d}\frac{f^2}{\|f\|^4_{L^2}}\,\psi''\left(\frac{f^2}{\|f\|^2_{L^2}}\right)D|\nabla f|^2\,d\widetilde{u_\infty}$$

Here $k_1(e):=\frac{1}{2-p}[e+1-(e+1)^{p-1}]$

Another perturbation result

"μ is a operturbation of ν..."

$$C_p(\mu) := \sup_{u\in H^1(\mu)} \frac{\int |u|^2\,d\mu - (\int |u|^{2/p}\,d\mu)^p}{(p-1)\int |\nabla u|^2\,d\mu} \tag{8}$$

Theorem 9 [Bartier, J.D.] *Let $p \in [1,2)$ and $p' = (1-1/p)^{-1}$ if $p>1$, $p' = \infty$ if $p=1$.*
Let $d\mu = e^{-V}dx$ and $\nu = e^{-W}dx$ be two probability measures such that $C_p(\nu)$ and $C_2(\mu)$ are finite. Let $Z := \frac{1}{2}(V-W)$ and assume that $Z_+ \in L^{p'}(\nu)$,

$$m := \inf_{\mathbb{R}^d}(|\nabla Z|^2 - \Delta Z + \nabla Z\cdot\nabla W) > -\infty$$

Then we have

$$C_p(\mu) \le \mathcal{C}_p := \frac{2}{p}C_2(\mu) + (\frac{2}{p}-1)\,\mathcal{C}_p^*$$

with $\mathcal{C}_p^* := \Big[C_p(\nu) + C_2(\mu)\,(2\,\|Z_+\|_{L^{p'}(d\nu)} - m\,C_p(\nu))_+\Big]$

Lemma 10

$$\sup_{v\in H^1(\mu),\,\bar v=0} \frac{\int |v|^2\,d\mu - \left(\int |v|^{2/p}\,d\mu\right)^p}{(p-1)\int |\nabla v|^2\,d\mu} \le \mathcal{C}_p^*$$

Proof. $p>1$. Take v in $H^1(\mu)$ with $\bar v = 0$,

$$\mathcal{A}(t) := \|\nabla v\|^2_{L^2(d\mu)} - \frac{t}{(p-1)\,C_p(\nu)}\left[\int |v|^2\,d\mu - \left(\int |v|^{2/p}\,d\mu\right)^p\right]$$

$\mathcal{A}(t) = (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III})$ with
$(\mathrm{I}) = (1-t)\int |\nabla v|^2\,d\mu$
$(\mathrm{II}) = t\int |\nabla v|^2\,d\mu$
$(\mathrm{III}) = \frac{-t}{(p-1)\,C_p(\nu)}\left[\int |v|^2\,d\mu - \left(\int |v|^{2/p}\,d\mu\right)^p\right]$.

Define g such that $v = g\,e^Z$:
$\int |v|^2\,d\mu = \int |g|^2\,d\nu$
$\int |\nabla v|^2\,d\mu = \int |\nabla g|^2\,d\nu + \int \delta\,|g|^2\,d\nu$, where $\delta := |\nabla Z|^2 - \Delta Z + \nabla Z\cdot\nabla W$

Poincaré for μ

$$(\mathrm{I}) \geq \frac{1-t}{C_2(\mu)} \int |v|^2 \, d\mu = \frac{1-t}{C_2(\mu)} \int |g|^2 \, d\nu$$

$C_p(\nu) < \infty$

$$(\mathrm{II}) \geq \frac{t}{(p-1)\, C_p(\nu)} \Big(\int |g|^2 \, d\nu - (\int |g|^{2/p} \, d\nu)^p \Big) + t \int \delta \, |g|^2 \, d\nu$$

and

$$(\mathrm{III}) = \frac{\mathcal{B}\, t}{(p-1)\, C_p(\nu)}$$

with $\mathcal{B} := (\int |v|^{2/p} \, d\mu)^p - (\int |g|^{2/p} \, d\nu)^p$. Collecting these estimates, we have

$$\mathcal{A}(t) \geq \int (\frac{(1-t)}{C_2(\mu)} + t\, \delta) |g|^2 \, d\mu + \frac{\mathcal{B}\, t}{(p-1)\, C_p(\nu)}$$

Let $d\pi := |g|^{2/p} / \int |g|^{2/p} \, d\nu$. By Jensen's inequality applied to the convex function $t \mapsto e^{-t}$, we get

$$\frac{\int |v|^{2/p} \, d\mu}{\int |g|^{2/p} \, d\nu} = \frac{\int |g|^{2/p} e^{-2(1-\frac{1}{p})Z} \, d\nu}{\int |g|^{2/p} \, d\nu} = \int e^{-2(1-\frac{1}{p})Z} \, d\pi \geq \exp\big[-2(1-\frac{1}{p}) \int Z \, d\pi\big]$$

(...) $\mathcal{B} \geq -2\,(p-1)\,\|Z_+\|_{L^{p'}(d\mu)} \int |g|^2\,d\nu$. Altogether, we get

$$\mathcal{A}(t) \geq \left[\frac{1-t}{C_2(\mu)} + t\left(m - \frac{2\,\|Z_+\|_{L^{p'}(d\nu)}}{C_p(\nu)}\right)\right] \int |g|^2\,d\nu$$

This proves that $\mathcal{A}(t) \geq 0$ for any $t \in (0, t^*]$
with $t^* := [1 + \frac{C_2(\mu)}{C_p(\nu)}\,(2\,\|Z_+\|_{L^{p'}(d\nu)} - m\,C_p(\nu))_+]^{-1}$
Conclusion: $\mathcal{C}_p = C_p(\nu)/t^*$.

Case $p = 1$: take the limit $\square$

The *unrestricted case* follows from the *restricted case*

Lemma 11 [Wang, Barthe-Roberto] *Let $q \in [1,2]$. For any function $u \in L^1 \cap L^q(\mu)$, if $\bar{u} := \int u\, d\mu$, then*

$$\Big(\int |u|^q\, d\mu \Big)^{2/q} \geq |\bar{u}|^2 + (q-1) \Big(\int |u - \bar{u}|^q\, d\mu \Big)^{2/q}.$$

Proof. Let $v := u - \bar{u}$, $\phi(t) := \big(\int |\bar{u} + t\, v|^q\, d\mu \big)^{2/q}$, so that $\phi(0) = |\bar{u}|^2$, $\phi'(0) = 0$, $\phi(1) = \big(\int |u|^q\, d\mu \big)^{2/q}$ and $\frac{1}{2}\, \phi''(t) \geq (q-1)\, (\int |v|^q\, d\mu)^{2/q}$. This proves that $\phi(1) \geq \phi(0) + (q-1)\, \big(\int |v|^q\, d\mu \big)^{2/q}$. □

Proof of Theorem 9. Let $v := u - \bar{u}$ and apply Lemma 11 with $q = 2/p \in [1,2)$. Since $\int |u|^2\, d\mu - |\bar{u}|^2 = \int |u - \bar{u}|^2\, d\mu = \int |v|^2\, d\mu$, we can write

$$\int |u|^2\, d\mu - \Big(\int |u|^{2/p}\, d\mu \Big)^p \leq 2\, \frac{p-1}{p} \int |v|^2\, d\mu + \frac{2-p}{p} \Big[\int |v|^2\, d\mu - \Big(\int |v|^{2/p}\, d\mu \Big)^p \Big]$$

$\mathcal{C}_p = \frac{2}{p}\, C_2(\mu) + (\frac{2}{p} - 1)\, \mathcal{C}_p^*$. □