Functional inequalities: nonlinear flows and entropy methods as a tool for obtaining sharp and constructive results

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Outline

- A brief historical perspective
  - ▷ Sobolev and some other interpolation inequalities
  - ▷ Branches of solutions
  - ▷ Entropies and *carré du champ* methods
- Entropy methods and the fast diffusion equation on $\mathbb{R}^d$
  - ▷ Rényi entropy powers
  - ▷ Relative entropy and relative Fisher information
  - ▷ Linearized entropy methods and Hardy-Poincaré inequalities
  - ▷ A stability result for Gagliardo-Nirenberg-Sobolev inequalities
- Symmetry and symmetry breaking
  - ▷ Caffarelli-Kohn-Nirenberg inequalities
  - ▷ A proof in 4 steps
  - ▷ Optimality by entropy methods

with Matteo Bonforte, Bruno Nazaret, Nikita Simonov
with Maria J. Esteban, Michael Loss, Matteo Muratori

Inequalities, entropies, flows

A little bit of history

- Sobolev's inequality

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

▷ [Lane, 1870], [Emden, Fowler], [Bliss, 1930]
▷ [Sobolev, 1938]
▷ [Aubin, 1976], [Talenti, 1976], [Rodemich, 1966], [Lieb, 1983]
▷ [Brézis, Nirenberg, 1983], [Brézis, Lieb, 1985], [Bianchi, Egnell, 1991]

- Entropy

$$S = \int_{\mathbb{R}^d} f \log f \, dx$$

▷ [Clausius, 1865], [Boltzmann, 1872]
▷ [Shannon, 1948], [Blackman, Stam, 1959]
▷ [Gross, 1975]
▷ [Bakry, Emery, 1985], [Jordan, Kinderlehrer, Otto, 1998]

Two (apocryphal ?) jokes

The Arnold Principle. If a notion bears a personal name, then this name is not the name of the discoverer.

The Berry Principle. The Arnold Principle is applicable to itself.

About entropy. When Shannon first derived his famous formula for information, he asked von Neumann what he should call it and von Neumann replied: *You should call it entropy for two reasons: first because that is what the formula is in statistical mechanics but second and more important, as nobody knows what entropy is, whenever you use the term you will always be at an advantage !*

The stability result of G. Bianchi and H. Egnell

In Sobolev's inequality (with optimal constant S_d),

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge 0$$

is there a natural way to bound the l.h.s. from below in terms of a "distance" to the set of optimal [Aubin-Talenti] functions when $d \ge 3$?
A question raised in [Brezis, Lieb (1985)]

▷ [Bianchi, Egnell (1991)] There is a positive constant α such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \alpha \inf_{\varphi\in\mathscr{M}} \|\nabla f - \nabla\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Various improvements, *e.g.*, [Cianchi, Fusco, Maggi, Pratelli (2009)] there are constants α and κ and $f \mapsto \lambda(f)$ such that

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \big(1 + \kappa\,\lambda(f)^\alpha\big)\,\mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$$

*However, the question of **constructive** estimates is still widely open*

From the carré du champ method to stability results

Carré du champ method (adapted from D. Bakry and M. Emery)

$$\frac{\partial u}{\partial t} = \Delta u^m , \quad \frac{d\mathscr{F}}{dt} = -\mathscr{I} , \quad \frac{d\mathscr{I}}{dt} \le -\Lambda \mathscr{I}$$

deduce that $\mathscr{I} - \Lambda \mathscr{F}$ is monotone non-increasing with limit 0

$$\boxed{\mathscr{I}[u] \ge \Lambda \mathscr{F}[u]}$$

▷ ***Improved entropy – entropy production inequality*** (weaker form)

$$\mathscr{I} \ge \Lambda \psi(\mathscr{F})$$

for some ψ such that $\psi(0) = 0$, $\psi'(0) = 1$ and $\psi'' > 0$

$$\mathscr{I} - \Lambda \mathscr{F} \ge \Lambda \big(\psi(\mathscr{F}) - \mathscr{F}\big) \ge 0$$

▷ ***Improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star > \Lambda$ such that

$$\mathscr{I} - \Lambda \mathscr{F} \ge (\Lambda_\star - \Lambda) \mathscr{F} \ge 0 \quad \text{or} \quad \mathscr{I} - \Lambda \mathscr{F} \ge \left(1 - \frac{\Lambda}{\Lambda_\star}\right) \mathscr{I} \ge 0$$

Gagliardo-Nirenberg-Sobolev inequalities

We consider the inequalities

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GNS}}(p)\, \|f\|_{2p} \qquad \text{(GNS)}$$

$$\theta = \frac{d\,(p-1)}{(d+2-p\,(d-2))\,p}\,, \quad p \in (1,+\infty) \text{ if } d=1 \text{ or } 2\,, \quad p \in (1,p^*] \text{ if } d \ge 3\,, \quad p^* = \frac{d}{d-2}$$

Theorem (del Pino, JD)

Equality case in (GNS) is achieved if and only if

$$f \in \mathfrak{M} := \left\{ g_{\lambda,\mu,y} \,:\, (\lambda,\mu,y) \in (0,+\infty) \times \mathbb{R} \times \mathbb{R}^d \right\}$$

Aubin-Talenti functions: $g_{\lambda,\mu,y}(x) := \mu\, g((x-y)/\lambda)$, $g(x) = (1+|x|^2)^{-\frac{1}{p-1}}$

[del Pino, JD, 2002], [Gunson, 1987, 1991]

Related inequalities

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\rm GNS}(p)\, \|f\|_{2p} \qquad \text{(GNS)}$$

▷ *Sobolev's inequality*: $d \ge 3$, $p = p^* = d/(d-2)$

$$\|\nabla f\|_2^2 \ge \mathsf{S}_d\, \|f\|_{2p^*}^2$$

▷ *Euclidean Onofri inequality*

$$\int_{\mathbb{R}^2} e^{h-\overline{h}}\,\frac{dx}{\pi\,(1+|x|^2)^2} \le e^{\frac{1}{16\pi}\int_{\mathbb{R}^2}|\nabla h|^2\,dx}$$

$d=2$, $p \to +\infty$ with $f_p(x) := \mathsf{g}(x)\left(1+\frac{1}{2p}\,(h(x)-\overline{h})\right)$, $\overline{h} = \int_{\mathbb{R}^2} h(x)\,\frac{dx}{\pi\,(1+|x|^2)^2}$

▷ *Euclidean logarithmic Sobolev inequality in scale invariant form*

$$\frac{d}{2}\log\left(\frac{2}{\pi\,d\,e}\int_{\mathbb{R}^d}|\nabla f|^2\,dx\right) \ge \int_{\mathbb{R}^d}|f|^2\log|f|^2\,dx$$

or $\int_{\mathbb{R}^d}|\nabla f|^2\,dx \ge \frac{1}{2}\int_{\mathbb{R}^d}|f|^2\log\left(\frac{|f|^2}{\|f\|_2^2}\right)dx + \frac{d}{4}\log\left(2\pi\,e^2\right)\int_{\mathbb{R}^d}|f|^2\,dx$

Interpolation inequalities on the sphere

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \ne 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathscr{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathscr{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathscr{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ): use the heat flow $\frac{\partial \rho}{\partial t} = \Delta \rho$ where Δ denotes the Laplace-Beltrami operator on $\mathbb{S}^d$, and compute

$$\frac{d}{dt}\mathscr{E}_p[\rho] = -\mathscr{I}_p[\rho] \quad \text{and} \quad \frac{d}{dt}\mathscr{I}_p[\rho] \le -d\,\mathscr{I}_p[\rho]$$

$\frac{d}{dt}\left(\mathscr{I}_p[\rho] - d\,\mathscr{E}_p[\rho]\right) \le 0 \Longrightarrow \mathscr{I}_p[\rho] \ge d\,\mathscr{E}_p[\rho]$ with $\rho = |u|^p$, if

$$p \le 2^\# := \frac{2\,d^2+1}{(d-1)^2}$$

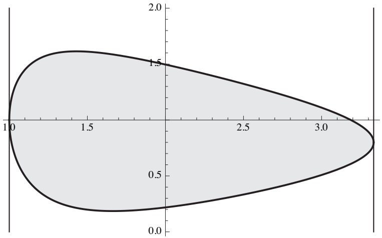
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m . \tag{1}$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathscr{K}_p[\rho] := \frac{d}{dt}\Big(\mathscr{I}_p[\rho] - d\,\mathscr{E}_p[\rho]\Big) \le 0 ,$$



(p, m) *admissible region,* $d = 5$

Sobolev's inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho\,\phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \ge 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\, v(x)$$

- $p = 2^*$, $\mathsf{S}_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\,dx \ge \mathsf{S}_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\,dx\right]^{\frac{d-2}{d}} \quad \forall\, v \in \mathscr{D}^{1,2}(\mathbb{R}^d)$$

Schwarz symmetrization and the ultraspherical setting

$(\xi_0, \xi_1, \ldots \xi_d) \in \mathbb{S}^d$, $\xi_d = z$, $\sum_{i=0}^d |\xi_i|^2 = 1$

Schwarz foliated symmetrization: [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on $\xi_d = z$

- Let $d\sigma(\theta) := \frac{(\sin\theta)^{d-1}}{Z_d}\,d\theta$, $Z_d := \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}$: $\forall\, v \in \mathrm{H}^1([0,\pi], d\sigma)$

$$\frac{p-2}{d}\int_0^\pi |v'(\theta)|^2\,d\sigma + \int_0^\pi |v(\theta)|^2\,d\sigma \ge \left(\int_0^\pi |v(\theta)|^p\,d\sigma\right)^{\frac{2}{p}}$$

- Change of variables $z = \cos\theta$, $v(\theta) = f(z)$

$$\frac{p-2}{d}\int_{-1}^1 |f'|^2\,\nu\,d\nu_d + \int_{-1}^1 |f|^2\,d\nu_d \ge \left(\int_{-1}^1 |f|^p\,d\nu_d\right)^{\frac{2}{p}}$$

where $\nu_d(z)\,dz = d\nu_d(z) := Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1-z^2$, consider the space $\mathrm{L}^2((-1,1),d\nu_d)$ with scalar product

$$\langle f_1,f_2\rangle = \int_{-1}^1 f_1\,f_2\,d\nu_d\,,\quad \|f\|_{\mathrm{L}^p(\mathbb{S}^d)} = \left(\int_{-1}^1 f^p\,d\nu_d\right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathscr{L}\,f := (1-z^2)\,f'' - d\,z\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1,\mathscr{L}\,f_2\rangle = -\int_{-1}^1 f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge 1$. *For any* $f\in\mathrm{H}^1([-1,1],d\nu_d)$,

$$-\langle f,\mathscr{L}\,f\rangle = \int_{-1}^1 |f'|^2\,\nu\,d\nu_d \ge d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

Heat flow and the Bakry-Emery method

With $g = f^p$, *i.e.* $f = g^\alpha$ with $\alpha = 1/p$

$$\text{(Ineq.)} \quad -\langle f, \mathcal{L} f \rangle = -\langle g^\alpha, \mathcal{L} g^\alpha \rangle =: \mathcal{I}[g] \geq d\, \frac{\|g\|^{2\alpha}_{\mathrm{L}^1(\mathbb{S}^d)} - \|g^{2\alpha}\|_{\mathrm{L}^1(\mathbb{S}^d)}}{p-2} =: \mathcal{F}[g]$$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

$$\frac{d}{dt}\|g\|_{\mathrm{L}^1(\mathbb{S}^d)} = 0\,, \quad \frac{d}{dt}\|g^{2\alpha}\|_{\mathrm{L}^1(\mathbb{S}^d)} = -2\,(p-2)\,\langle f, \mathcal{L} f \rangle = 2\,(p-2)\int_{-1}^1 |f'|^2\,\nu\,d\nu_d$$

which finally gives

$$\frac{d}{dt}\mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\,\frac{d}{dt}\|g^{2\alpha}\|_{\mathrm{L}^1(\mathbb{S}^d)} = -2\,d\,\mathcal{I}[g(t,\cdot)]$$

$$\text{Ineq.} \iff \frac{d}{dt}\mathcal{F}[g(t,\cdot)] \leq -2\,d\,\mathcal{F}[g(t,\cdot)] \Longleftarrow \frac{d}{dt}\mathcal{I}[g(t,\cdot)] \leq -2\,d\,\mathcal{I}[g(t,\cdot)]$$

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathscr{L} f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f,\mathscr{L} f\rangle = \langle \mathscr{L} f,\mathscr{L} f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu,\mathscr{L} f\right\rangle$$

$$\frac{d}{dt}\mathscr{I}[g(t,\cdot)] + 2\,d\,\mathscr{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\tfrac{d}{d+2}\,\tfrac{|f'|^4}{f^2} - 2\,(p-1)\,\tfrac{d-1}{d+2}\,\tfrac{|f'|^2\,f''}{f} \ge 0 \quad \text{a.e.}$$

$$\left[(p-1)\,\tfrac{d-1}{d+2}\right]^2 \le (p-1)\,\tfrac{d}{d+2} \iff p \le \tfrac{2\,d^2+1}{(d-1)^2} = 2^\# < \tfrac{2\,d}{d-2} = 2^*$$

... up to the critical exponent: a proof in two slides

$$\left[\frac{d}{dz},\mathscr{L}\right]u=(\mathscr{L}\,u)'-\mathscr{L}\,u'=-2\,z\,u''-d\,u'$$

$$\int_{-1}^1(\mathscr{L}\,u)^2\,d\nu_d=\int_{-1}^1|u''|^2\,\nu^2\,d\nu_d+d\int_{-1}^1|u'|^2\,\nu\,d\nu_d$$

$$\int_{-1}^1(\mathscr{L}\,u)\,\frac{|u'|^2}{u}\,\nu\,d\nu_d=\frac{d}{d+2}\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d-2\,\frac{d-1}{d+2}\int_{-1}^1\frac{|u'|^2\,u''}{u}\,\nu^2\,d\nu_d$$

On $(-1,1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t=u^{2-2\beta}\left(\mathscr{L}\,u+\kappa\,\frac{|u'|^2}{u}\,\nu\right),\quad\beta=\frac{1}{2-p\,(1-m)}$$

If $\kappa=\beta\,(p-2)+1$, the L^p norm is conserved

$$\frac{d}{dt}\int_{-1}^1u^{\beta p}\,d\nu_d=\beta\,p\,(\kappa-\beta\,(p-2)-1)\int_{-1}^1u^{\beta(p-2)}\,|u'|^2\,\nu\,d\nu_d=0$$

$f = u^\beta$, $\|f'\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right) \geq 0$?

$$\mathscr{A} := \int_{-1}^{1} |u''|^2\,\nu^2\,d\nu_d - 2\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\int_{-1}^{1} u''\,\frac{|u'|^2}{u}\,\nu^2\,d\nu_d + \left[\kappa\,(\beta-1) + \frac{d}{d+2}\,(\kappa+\beta-1)\right]\int_{-1}^{1}\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d$$

$\mathscr{A}$ is nonnegative for some β if

$$\frac{8\,d^2}{(d+2)^2}\,(p-1)\,(2^*-p) \geq 0$$

$\mathscr{A}$ is a sum of squares if $p \in (2, 2^*)$ for an arbitrary choice of β in a certain interval (depending on p and d)

$$\mathscr{A} = \int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\,d\nu_d \geq 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

The rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta}\left(\mathscr{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$

$$-\mathscr{L}\,u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathscr{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathscr{L}\,u\,u^\kappa\,d\nu_d = -\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\,d\nu_d$$

The two terms cancel and we are left only with the two-homogenous terms

Integral constraints

Proposition

For any $p \in (2, 2^\#)$, the inequality

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{\lambda}{p-2} \, \|f\|_2^2 \geq \frac{\lambda}{p-2} \, \|f\|_p^2$$

$$\forall \, f \in \mathrm{H}^1((-1,1), d\nu_d) \; s.t. \int_{-1}^{1} z \, |f|^p \, d\nu_d = 0$$

holds for some $\lambda^\star > d$ with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x)=u(x)\quad\forall\,x\in\mathbb{S}^d$$

Theorem

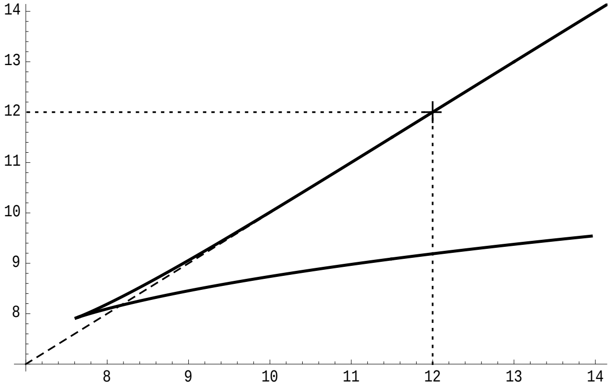
If $p\in(1,2)\cup(2,2^*)$, we have

$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu\ge\frac{d}{p-2}\left[1+\frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1}\right]\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $u\in\mathrm{H}^1(\mathbb{S}^d,d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

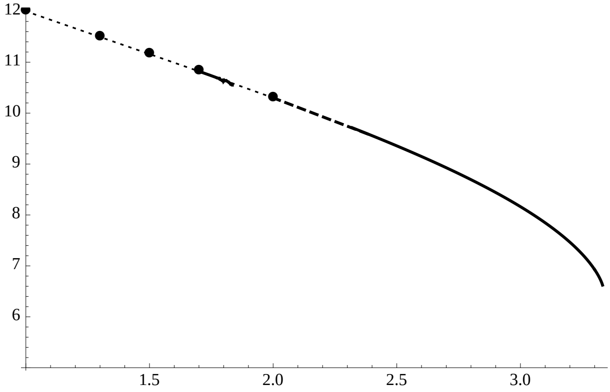
$$\int_{\mathbb{S}^d}|\nabla u|^2\,d\mu\ge\frac{d}{2}\,\frac{(d+3)^2}{(d+1)^2}\int_{\mathbb{S}^d}|u|^2\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu$$

The larger picture: branches of antipodal solutions



Case $d = 5$, $p = 3$: values of the shooting parameter a as a function of λ

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d = 5$ *and* $1 \leq p \leq 10/3 \approx 3.33$. *The limiting value of the constant is numerically found to be equal to* $\lambda_\star = 2^{1-2/p}\,d \approx 6.59754$ *with* $d = 5$ *and* $p = 10/3$

Entropy methods and the fast diffusion equation on the Euclidean space

$$\frac{\partial u}{\partial t} = \Delta u^m$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy-entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \geq 1$, $m \in [m_1, 1)$, $m_1 := (d-1)/d$

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with initial datum $u(t=0,x) = u_0(x) \geq 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \mathcal{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, \mathrm{d}x < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$, $\kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$ and $\mathscr{B}$ is the Barenblatt profile

$$\mathscr{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

Entropy growth rate and Rényi entropy powers

With $p=\frac{1}{2m-1} \iff m=\frac{p+1}{2p}$, let us consider f such that $u=f^{2p}$
$u^m=f^{p+1}$ and $u\,|\nabla^{m-1}u|^2=(p-1)^2\,|\nabla f|^2$

$$M=\|f\|_{2p}^{2p}\,,\quad \mathsf E[u]:=\int_{\mathbb R^d}u^m\,dx=\|f\|_{p+1}^{p+1}\quad\text{and}\quad \mathsf I[u]:=(p+1)^2\,\|\nabla f\|_2^2$$

By (GNS), if u solves (2), then

$$\mathsf E'=\frac{p-1}{2p}\,\mathsf I=\frac{p-1}{2p}\,(p+1)^2\int_{\mathbb R^d}|\nabla f|^2\,dx$$
$$\ge\frac{p-1}{2p}\,(p+1)^2\left(\mathscr C_{\mathrm{GNS}(p)}\right)^{\frac2\theta}\,\|f\|_{2p}^{\frac2\theta}\,\|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}}\ge C_0\,\mathsf E^{1-\frac{m-m_c}{1-m}}$$

with $C_0:=\frac{p-1}{2p}\,(p+1)^2\,\big(\mathscr C_{\mathrm{GNS}}(p)\big)^{\frac2\theta}\,M^{\frac{(d+2)\,m-d}{d\,(1-m)}}$

$$\int_{\mathbb R^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb R^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx \quad \text{with} \quad \mathsf{P} = \frac{m}{m-1} \, u^{m-1} \text{ is the } \textit{pressure variable}$$

If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m)\,\mathsf{I}$$

The *Rényi entropy power* $\mathsf{F} := \mathsf{E}^\sigma = \left(\int_{\mathbb{R}^d} u^m \, dx\right)^\sigma$ with $\sigma = \frac{2}{d}\,\frac{1}{1-m} - 1$ applied to self-similar Barenblatt solutions has a linear growth in t

[Toscani, Savaré, 2014], [JD, Toscani, 2016]

Nonlinear carré du champ method

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$

If u is a smooth and rapidly decaying function on $\mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$
$$= -2\int_{\mathbb{R}^d} u^m\,\Big\|\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id}\Big\|^2\,dx - 2\,(m-m_1)\int_{\mathbb{R}^d} u^m\,(\Delta \mathsf{P})^2\,dx$$

Lemma

Let $d\ge1$ and assume that $m\in(m_1,1)$. If u solves (2) with initial datum $u_0\in\mathrm{L}^1_+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}|x|^2\,u_0\,dx<+\infty$ and if, for any $t\ge0$, $u(t,\cdot)$ is a smooth and rapidly decaying function on $\mathbb{R}^d$, then for any $t\ge0$ we have

$$-\frac{d}{dt}\log\left(\mathsf{I}^{\frac12}\,\mathsf{E}^{\frac{1-\theta}{\theta(p+1)}}\right) = \int_{\mathbb{R}^d} u^m\Big\|\mathrm{D}^2\mathsf{P} - \tfrac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id}\Big\|^2 dx + (m-m_1)\int_{\mathbb{R}^d} u^m\Big|\Delta \mathsf{P} + \frac{\mathsf{I}}{\mathsf{E}}\Big|^2 dx$$

Self-similar variables: entropy-entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\Big(\tau,\frac{x}{\kappa\,R}\Big)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu}\,,\quad\tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial\tau}+\nabla\cdot\Big[v\Big(\nabla u^{m-1}-2\,x\Big)\Big]=0 \tag{3}$$

Generalized entropy (free energy) and*Fisher information*

$$\mathscr F[v]:=-\frac1m\int_{\mathbb R^d}\Big(v^m-\mathscr B^m-m\,\mathscr B^{m-1}\,(v-\mathscr B)\Big)\,\mathrm dx$$

$$\mathscr I[v]:=\int_{\mathbb R^d}v\,\Big|\nabla v^{m-1}+2\,x\Big|^2\,\mathrm dx$$

are such that $\boxed{\mathscr I[v]\ge4\,\mathscr F[v]}$ by (GNS) [del Pino, JD, 2002] so that

$$\mathscr F[v(t,\cdot)]\le\mathscr F[v_0]\,e^{-4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

$$\mathcal{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t\ge 0\,, \quad \gamma(m) := (1-m)\,\Lambda_{\alpha,d}\,, \quad \alpha := \frac{1}{m-1} < 0$$

where $\Lambda_{\alpha,d}>0$ is the best constant in the Hardy–Poincaré inequality

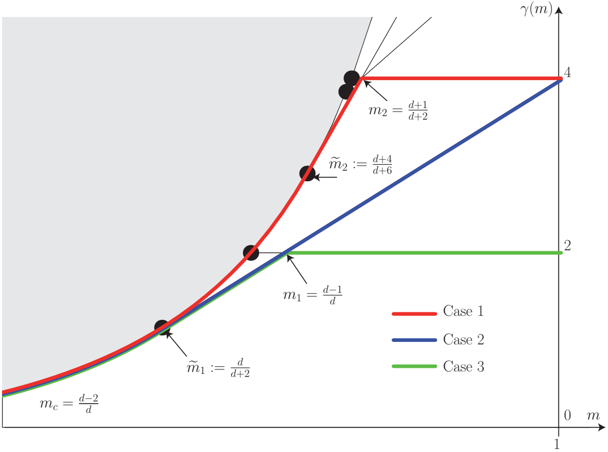
$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d}\frac{f^2}{1+|x|^2}\,(1+|x|^2)^\alpha\,dx \le \int_{\mathbb{R}^d}|\nabla f|^2\,(1+|x|^2)^\alpha\,dx\,, \quad \int_{\mathbb{R}^d} f\,(1+|x|^2)^{\alpha-1}\,dx = 0$$

Lemma

Under assumption (H), $\mathcal{I}[v] \ge (4-\eta)\,\mathcal{F}[v]$ *for some* $\eta\in(0,2(\gamma(m)-2))$

Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Spectral gap and the asymptotic time layer



$$\mathscr{F}[v(t,\cdot)] \le C\, e^{-2\gamma(m)\,t} \quad \forall\, t \ge 0$$

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,\mathrm{d}x \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,\mathrm{d}x$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,\mathrm{d}x = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,\mathrm{d}x = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition (already a stability result)

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,d\,(m - m_1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,\mathrm{d}x = 0$ and

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathscr{Q}[v] := \frac{\mathscr{I}[v]}{\mathscr{F}[v]} \ge 4 + \eta$$

The initial time layer improvement: backward estimate

[Bonforte, JD, Nazaret, Simonov, 2020]
Rephrasing the *carré du champ* method, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q}-4)$$

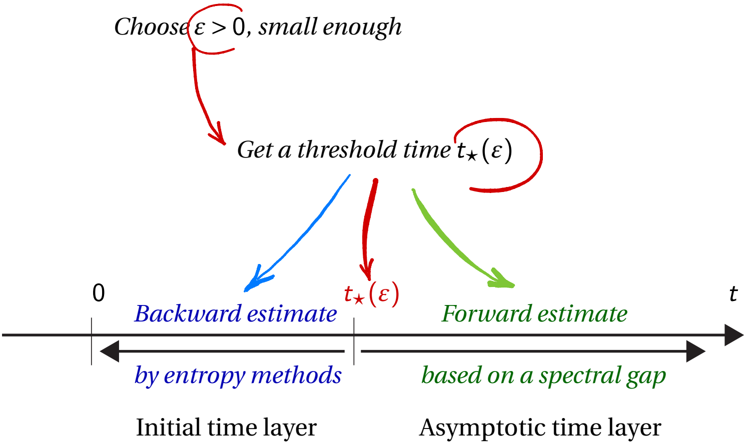
Lemma

Assume that $m > m_1$ and v is a solution to (3) with nonnegative initial datum v_0. If for some $\eta > 0$ and $T > 0$, we have $\mathcal{Q}[v(T,\cdot)] \ge 4+\eta$, then

$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}} \quad \forall\, t \in [0,T]$$

Regularity and stability

Our strategy



Uniform convergence in relative error

Theorem

Assume that $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$ *and let* $\varepsilon \in (0, 1/2)$, *small enough,* $A > 0$, *and* $G > 0$ *be given. There exists an explicit time* $t_\star \geq 0$ *such that, if* u *is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *satisfying*

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u_0 \, dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x = \mathscr{M}$ *and* $\mathscr{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq t_\star$$

The threshold time

Proposition

Let $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ *and* $G > 0$

$$t_\star = \mathsf{c}_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a} = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$ *and* $\vartheta = \nu/(d+\nu)$

$$\mathsf{c}_\star = \mathsf{c}_\star(m,d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max\left\{\varepsilon\,\kappa_1(\varepsilon, m), \varepsilon^{\mathsf{a}} \kappa_2(\varepsilon, m), \varepsilon\,\kappa_3(\varepsilon, m)\right\}$$

$$\kappa_1(\varepsilon, m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m}\,\kappa_\star}{1 - (1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon, m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon, m) := \frac{8\,\alpha^{-1}}{1 - (1-\varepsilon)^{1-m}}$$

Improved entropy-entropy production inequality

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathcal{I}[v] \geq (4 + \zeta)\, \mathcal{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v] = G$, $\int_{\mathbb{R}^d} v \, \mathrm{d}x = \mathcal{M}$, $\int_{\mathbb{R}^d} x \, v \, \mathrm{d}x = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0, 2\varepsilon_\star)\,, \quad \varepsilon_\star := \frac{1}{2} \min\{\varepsilon_{m,d}, \chi\,\eta\} \quad \text{with} \quad T = \frac{1}{2} \log R(t_\star)$$

$$(1 - \varepsilon)\,\mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon)\,\mathcal{B} \quad \forall\, t \geq T$$

and, as a consequence, the *initial time layer estimate*

$$\mathcal{I}[v(t, .)] \geq (4 + \zeta)\, \mathcal{F}[v(t, .)] \quad \forall\, t \in [0, T]\,, \quad \text{where} \quad \zeta = \frac{4\,\eta\, e^{-4\,T}}{4 + \eta - \eta\, e^{-4\,T}}$$

Two consequences

$$\zeta = \mathsf{Z}(A, \mathscr{F}[u_0])\,, \quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\,\frac{2}{\alpha}} + G}\,, \quad \zeta_\star := \frac{4\,\eta}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf{a}}}{2\,\alpha\,\mathsf{c}_\star}\right)^{\frac{2}{\alpha}} c_\alpha$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (3) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,\mathrm{d}x = 0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)] \le \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \ge 0$$

▷ The stability in the entropy - entropy production estimate $\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

An abstract stability result

Relative entropy

$$\mathscr{F}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(f^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p} \, \mathrm{g}^{1-p} \left(f^{2p} - \mathrm{g}^{2p} \right) \right) dx$$

Deficit functional

$$\delta[f] := a \, \|\nabla f\|_2^2 + b \, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}} \, \|f\|_{2p}^{2p\gamma} \ge 0$$

Theorem

Let $d \ge 1$ and $p \in (1, p^*)$. There is a $\mathscr{C} > 0$ such that

$$\delta[f] \ge \mathscr{C} \, \mathscr{F}[f]$$

for any $f \in \mathscr{W} := \{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2 \, dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \}$ such that

$$\int_{\mathbb{R}^d} f^{2p}(1,x) \, dx = \int_{\mathbb{R}^d} |\mathrm{g}|^{2p}(1,x) \, dx$$

A constructive result

The *relative entropy*

$$\mathscr{F}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(f^{p+1}-\mathrm{g}^{p+1}-\frac{1+p}{2p}\,\mathrm{g}^{1-p}\left(f^{2p}-\mathrm{g}^{2p}\right)\right)\mathrm{d}x$$

The *deficit functional*

$$\delta[f] := a\,\|\nabla f\|_2^2+b\,\|f\|_{p+1}^{p+1}-\mathscr{K}_{\mathrm{GN}}\,\|f\|_{2p}^{2p\gamma}\ge 0$$

Theorem

Let $d\ge 1$, $p\in(1,p^*)$, $A>0$ and $G>0$. There is a $\mathscr{C}>0$ such that

$$\delta[f]\ge\mathscr{C}\,\mathscr{F}[f]$$

for any $f\in\mathscr{W}:=\left\{f\in\mathrm{L}^1(\mathbb{R}^d,(1+|x|)^2\,\mathrm{d}x)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d,\mathrm{d}x)\right\}$ such that

$$\int_{\mathbb{R}^d} f^{2p}\,\mathrm{d}x=\int_{\mathbb{R}^d}|\mathrm{g}|^{2p}\,\mathrm{d}x\,,\quad\int_{\mathbb{R}^d}x\,f^{2p}\,\mathrm{d}x=0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r} f^{2p}\,dx\le A\quad\text{and}\quad\mathscr{F}[f]\le G$$

Symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

Caffarelli-Kohn-Nirenberg inequalities

Let $\mathscr{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) : |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,\mathrm{d}x\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,\mathrm{d}x \quad \forall\, v \in \mathscr{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d=2$, $a+1/2 < b \le a+1$ if $d=1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

▷ *An optimal function among radial functions:*

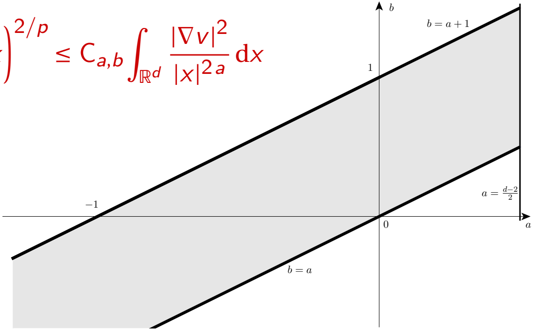
$$v_\star(x) = \left(1+|x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,\mathrm{d}x\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,\mathrm{d}x$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d=2$, $a+1/2 < b \le a+1$ if $d=1$

and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]

[F. Catrina, Z.-Q. Wang (2001)]

▷ Proving symmetry breaking
[F. Catrina, Z.-Q. Wang], [V. Felli, M. Schneider (2003)]
[J.D., Esteban, Loss, Tarantello, 2009] There is a curve...

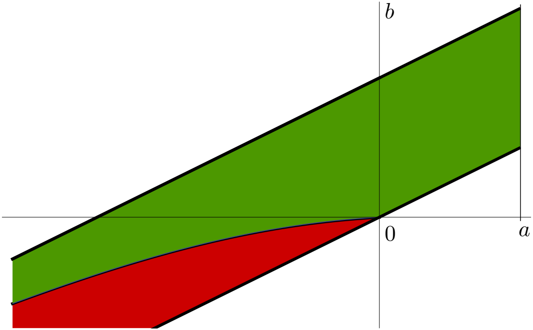
▷ Moving planes and symmetrization techniques
[Chou, Chu], [Horiuchi]
[Betta, Brock, Mercaldo, Posteraro]
+ Perturbation results: [CS Lin, ZQ Wang], [Smets, Willem], [JD, Esteban, Tarantello 2007], [J.D., Esteban, Loss, Tarantello, 2009]

▷ Linear instability of radial minimizers: the Felli-Schneider curve
[Catrina, Wang], [Felli, Schneider]

▷ Direct spectral estimates
[J.D., Esteban, Loss, 2011]: sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

Symmetry *versus* symmetry breaking

A result based on entropies and nonlinear flows



[J.D., Esteban, Loss, 2015]

The symmetry result

The Felli & Schneider curve is defined by

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

A change of variables (1/4)

With $(r=|x|, \omega=x/r) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$, the Caffarelli-Kohn-Nirenberg inequality is

$$\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |v|^p \, r^{d-b\,p} \, \frac{dr}{r} \, d\omega\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b} \int_0^\infty \int_{\mathbb{S}^{d-1}} |\nabla v|^2 \, r^{d-2\,a} \, \frac{dr}{r} \, d\omega$$

Change of variables $r \mapsto r^\alpha$, $v(r,\omega) = w(r^\alpha,\omega)$

$$\alpha^{1-\frac{2}{p}} \left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |w|^p \, r^{\frac{d-b\,p}{\alpha}} \, \frac{dr}{r} \, d\omega\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left(\alpha^2 \left|\frac{\partial w}{\partial r}\right|^2 + \frac{1}{r^2} |\nabla_\omega w|^2\right) r^{\frac{d-2\,a-2}{\alpha}+2} \, \frac{dr}{r} \, d\omega$$

Choice of α

$$n = \frac{d-b\,p}{\alpha} = \frac{d-2\,a-2}{\alpha} + 2$$

Then $p = \frac{2\,n}{n-2}$ is the critical Sobolev exponent associated with n

A Sobolev type inequality

The parameters α and n vary in the ranges $0<\alpha<\infty$ and $d<n<\infty$ and the *Felli-Schneider curve* in the (α,n) variables is given by

$$\alpha=\sqrt{\tfrac{d-1}{n-1}}=:\alpha_{\rm FS}$$

With

$$\mathscr{D}w=\left(\alpha\,\tfrac{\partial w}{\partial r},\tfrac{1}{r}\,\nabla_\omega w\right),\quad d\mu:=r^{n-1}\,dr\,d\omega$$

the inequality becomes

$$\alpha^{1-\frac{2}{p}}\left(\int_{\mathbb{R}^d}|w|^p\,d\mu\right)^{\frac{2}{p}}\le \mathsf{C}_{a,b}\int_{\mathbb{R}^d}|\mathscr{D}w|^2\,d\mu$$

Proposition

Let $d\ge2$. Optimality is achieved by radial functions and $\mathsf{C}_{a,b}=\mathsf{C}^\star_{a,b}$ if $\alpha\le\alpha_{\rm FS}$

Gagliardo-Nirenberg inequalities on general cylinders: similar results [J.D., Esteban, Loss, Muratori, 2016]

Notations

When there is no ambiguity, we will omit the index $_\omega$ and from now on write that $\nabla = \nabla_\omega$ denotes the gradient with respect to the angular variable $\omega \in \mathbb{S}^{d-1}$ and that Δ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. We define the self-adjoint operator $\mathscr{L}$ by

$$\mathscr{L}\, w := -\mathscr{D}^* \mathscr{D}\, w = \alpha^2\, w'' + \alpha^2\, \frac{n-1}{r}\, w' + \frac{\Delta\, w}{r^2}$$

The fundamental property of $\mathscr{L}$ is the fact that

$$\int_{\mathbb{R}^d} w_1\, \mathscr{L}\, w_2\, d\mu = -\int_{\mathbb{R}^d} \mathscr{D} w_1 \cdot \mathscr{D} w_2\, d\mu \quad \forall\, w_1, w_2 \in \mathscr{D}(\mathbb{R}^d)$$

▷ Heuristics: we look for a monotonicity formula along a well chosen nonlinear flow, based on the analogy with the decay of the Fisher information along the fast diffusion flow in $\mathbb{R}^d$

Fisher information (2/4)

Let $u^{\frac{1}{2}-\frac{1}{n}}=|w| \iff u=|w|^p,\ p=\frac{2n}{n-2}$

$$\mathscr{I}[u]:=\int_{\mathbb{R}^d} u\,|\mathscr{D}\mathsf{P}|^2\,d\mu\,,\quad \mathsf{P}=\frac{m}{1-m}\,u^{m-1}\quad \text{and}\quad m=1-\frac{1}{n}$$

Here $\mathscr{I}$ is the *Fisher information* and P is the *pressure function*

Proposition

With $\Lambda=4\,\alpha^2/(p-2)^2$ and for some explicit numerical constant κ, we have

$$\kappa\,\mu(\Lambda)=\inf\left\{\mathscr{I}[u]\,:\,\|u\|_{\mathrm{L}^1(\mathbb{R}^d,d\mu)}=1\right\}$$

▷ Optimal solutions solutions of the elliptic PDE) are (constrained) critical point of $\mathscr{I}$

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathscr{L}\, u^m\,,\quad m = 1 - \frac{1}{n}$$

Barenblatt self-similar solutions

$$u_\star(t,r,\omega) = t^{-n}\left(\mathsf{c}_\star + \frac{r^2}{2\,(n-1)\,\alpha^2\,t^2}\right)^{-n}$$

Lemma

Barenblatt solutions realize the minimum of $\mathscr{I}$ among radial functions:

$$\kappa\,\mu_\star(\Lambda) = \mathscr{I}[u_\star(t,\cdot)]\quad\forall\,t>0$$

▷ Strategy:
1) prove that $\frac{d}{dt}\mathscr{I}[u(t,\cdot)] \le 0$,
2) prove that $\frac{d}{dt}\mathscr{I}[u(t,\cdot)] = 0$ means that $u = u_\star$ up to a time shift

Decay of the Fisher information along the flow ? (3/4)

The *pressure function* $\mathsf{P} = \frac{m}{1-m}\,u^{m-1}$ satisfies

$$\frac{\partial \mathsf{P}}{\partial t} = \frac{1}{n}\,\mathsf{P}\,\mathscr{L}\,\mathsf{P} - |\mathscr{D}\mathsf{P}|^2$$

$$\mathscr{Q}[\mathsf{P}] := \frac{1}{2}\,\mathscr{L}\,|\mathscr{D}\mathsf{P}|^2 - \mathscr{D}\mathsf{P}\cdot\mathscr{D}\mathscr{L}\,\mathsf{P}$$

$$\mathscr{K}[\mathsf{P}] := \int_{\mathbb{R}^d} \mathsf{k}[\mathsf{P}]\,\mathsf{P}^{1-n}\,d\mu = \int_{\mathbb{R}^d}\left(\mathscr{Q}[\mathsf{P}] - \frac{1}{n}\,(\mathscr{L}\,\mathsf{P})^2\right)\mathsf{P}^{1-n}\,d\mu$$

Lemma

If u solves the weighted fast diffusion equation, then

$$\frac{d}{dt}\mathscr{I}[u(t,\cdot)] = -2\,(n-1)^{n-1}\,\mathscr{K}[\mathsf{P}]$$

If u is a critical point, then $\mathscr{K}[\mathsf{P}] = 0$

▷ Boundary terms ! Regularity !

Proving decay 1

$$\mathsf{k}[\mathsf{P}] := \mathscr{Q}(\mathsf{P}) - \frac{1}{n}(\mathscr{L}\,\mathsf{P})^2 = \frac{1}{2}\,\mathscr{L}\,|\mathscr{D}\mathsf{P}|^2 - \mathscr{D}\mathsf{P}\cdot\mathscr{D}\,\mathscr{L}\,\mathsf{P} - \frac{1}{n}(\mathscr{L}\,\mathsf{P})^2$$

$$\mathsf{k}_{\mathfrak{M}}[\mathsf{P}] := \frac{1}{2}\,\Delta\,|\nabla\mathsf{P}|^2 - \nabla\mathsf{P}\cdot\nabla\Delta\,\mathsf{P} - \tfrac{1}{n-1}\,(\Delta\,\mathsf{P})^2 - (n-2)\,\alpha^2\,|\nabla\mathsf{P}|^2$$

Lemma

Let $n \neq 1$ be any real number, $d \in \mathbb{N}$, $d \geq 2$, and consider a function $\mathsf{P} \in C^3((0,\infty)\times\mathfrak{M})$, where $(\mathfrak{M}, g)$ is a smooth, compact Riemannian manifold. Then we have

$$\mathsf{k}[\mathsf{P}] = \alpha^4\left(1-\frac{1}{n}\right)\left[\mathsf{P}'' - \frac{\mathsf{P}'}{r} - \frac{\Delta\,\mathsf{P}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla\mathsf{P}' - \frac{\nabla\mathsf{P}}{r}\right|^2 + \frac{1}{r^4}\,\mathsf{k}_{\mathfrak{M}}[\mathsf{P}]$$

Proving decay 2

Lemma

Assume that $d \geq 3$, $n > d$ *and* $\mathfrak{M} = \mathbb{S}^{d-1}$. *For some* $\zeta_\star > 0$ *we have*

$$\int_{\mathbb{S}^{d-1}} \mathrm{k}_{\mathfrak{M}}[\mathrm{P}]\,\mathrm{P}^{1-n}\,d\omega \geq \left(\lambda_\star - (n-2)\,\alpha^2\right) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{P}|^2\,\mathrm{P}^{1-n}\,d\omega + \zeta_\star\,(n-d) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{P}|^4\,\mathrm{P}^{1-n}\,d\omega$$

Proof based on the Bochner-Lichnerowicz-Weitzenböck formula

Corollary

Let $d \geq 2$ *and assume that* $\alpha \leq \alpha_{\mathrm{FS}}$. *Then for any nonnegative function* $u \in \mathrm{L}^1(\mathbb{R}^d)$ *with* $\mathscr{I}[u] < +\infty$ *and* $\int_{\mathbb{R}^d} u\,d\mu = 1$, *we have*

$$\mathscr{I}[u] \geq \mathscr{I}_\star$$

When $\mathfrak{M} = \mathbb{S}^{d-1}$, $\lambda_\star = (n-2)\,\frac{d-1}{n-1}$

A perturbation argument, regularity issues (4/4)

- If u is a critical point of $\mathscr{I}$ under the mass constraint $\int_{\mathbb{R}^d} u\,d\mu = 1$, then

$$o(\varepsilon) = \mathscr{I}[u + \varepsilon\,\mathscr{L}\,u^m] - \mathscr{I}[u] = -2\,(n-1)^{n-1}\,\varepsilon\,\mathscr{K}[\mathrm{P}] + o(\varepsilon)$$

because $\varepsilon\,\mathscr{L}\,u^m$ is an admissible perturbation (formal). Indeed, we know that

$$\int_{\mathbb{R}^d} (u + \varepsilon\,\mathscr{L}\,u^m)\,d\mu = \int_{\mathbb{R}^d} u\,d\mu = 1$$

but positivity of $u + \varepsilon\,\mathscr{L}\,u^m$ is an issue: compute

$$0 = D\mathscr{I}[u]\cdot\mathscr{L}\,u^m = -\mathscr{K}[\mathrm{P}]$$

- Regularity issues (uniform decay of various derivatives up to order 3) and boundary terms

- If $\alpha \le \alpha_{\mathrm{FS}}$, then $\mathscr{K}[\mathrm{P}] = 0$ implies that $u = u_\star$

Some concluding remarks

Some concluding remarks

- Entropy methods provide a framework

▷ for interpreting the terms
▷ for computing
▷ for understanding optimality cases

- Flows

▷ parabolic equations provide extra-regularity
▷ relate nonlinear problems with asymptotic (linear) problems
▷ bypass symmetrization techniques (useful for some problems with magnetic fields, open for systems)

- Extensions

▷ non-homogeneous non-linearity or weights
▷ nonlinear non-local equations (Poisson couplings)
▷ kinetic equations and defective parabolic equations

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
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Thank you for your attention !