Best matching self-similar profiles and improved rates of convergence in nonlinear diffusion equations

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October 8, 2012

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Fast diffusion equations

1. entropy methods
2. linearization of the entropy
3. improved Gagliardo-Nirenberg inequalities

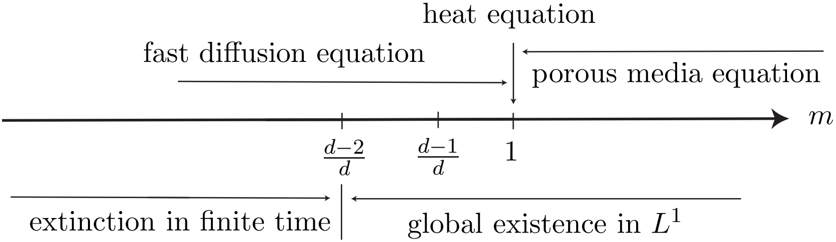
Fast diffusion equations: entropy methods

Existence, classical results

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Time-dependent rescaling, Free energy

- *Time-dependent rescaling*: Take $u(\tau, y) = R^{-d}(\tau)\, v\,(t, y/R(\tau))$ where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}\;, \quad R(0) = 1\;, \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\;, \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\,|x|^2 v \right)\, dx - \mathcal{F}_0$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]\;, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Relative entropy and entropy production

- *Stationary solution:* choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \tfrac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[v_\infty] = 0$

- *Entropy – entropy production inequality*

Theorem

$d \ge 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \ge 2\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[u_0]\,e^{-\,2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\mathcal{F}[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \mathcal{F}_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} \mathcal{I}[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \geq 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} v \, dx = \int_{\mathbb{R}^d} w^{2p} \, dx \right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With* $1 < p \leq \frac{d}{d-2}$ *(fast diffusion case) and* $d \geq 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq \mathcal{C}^{\mathrm{GN}}_{p,d} \, \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)} \, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$\mathcal{C}^{\mathrm{GN}}_{p,d} = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}, \; \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \; y = \frac{p+1}{p-1}$$

... a proof by the Bakry-Emery method

Consider the generalized Fisher information

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v\,|z|^2\,dx \quad \text{with} \quad z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt}\mathcal{I}[v(t,\cdot)] + 2\,\mathcal{I}[v(t,\cdot)] = -2\,(m-1)\int_{\mathbb{R}^d} u^m\,(\mathrm{div} z)^2\,dx - 2\sum_{i,j=1}^{d}\int_{\mathbb{R}^d} u^m\,(\partial_i z^j)^2\,dx$$

- the Fisher information decays exponentially:

$$\mathcal{I}[v(t,\cdot)] \le \mathcal{I}[u_0]\,e^{-2t}$$

- $\lim_{t\to\infty}\mathcal{I}[v(t,\cdot)] = 0$ and $\lim_{t\to\infty}\mathcal{F}[v(t,\cdot)] = 0$
- $\frac{d}{dt}\Big(\mathcal{I}[v(t,\cdot)] - 2\,\mathcal{F}[v(t,\cdot)]\Big) \le 0$ means $\mathcal{I}[v] \ge 2\,\mathcal{F}[v]$

[Carrillo, Toscani], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

The Bakry-Emery method: details (1/2)

With $z(x,t) := \eta\,\nabla u^{m-1} - 2\,x$, the equation can be rewritten as

$$\frac{\partial u}{\partial t} + \nabla\cdot(u\,z) = 0$$

(up to a time rescaling, which introduces a factor 2) and we have

$$\frac{\partial z}{\partial t} = \eta\,(1-m)\nabla\big(u^{m-2}\,\nabla\cdot(u\,z)\big) \quad \text{and} \quad \nabla\otimes z = \eta\,\nabla\otimes\nabla u^{m-1} - 2\,\mathrm{Id}$$

$$\frac{d}{dt}\int_{\mathbb{R}^d} u\,|z|^2\,dx = \underbrace{\int_{\mathbb{R}^d}\frac{\partial u}{\partial t}\,|z|^2\,dx}_{\text{(I)}} + \underbrace{2\int_{\mathbb{R}^d} u\,z\cdot\frac{\partial z}{\partial t}\,dx}_{\text{(II)}}$$

$$\begin{aligned}\text{(I)} &= \int_{\mathbb{R}^d}\frac{\partial u}{\partial t}\,|z|^2\,dx = \int_{\mathbb{R}^d}\nabla\cdot(u\,z)\,|z|^2 dx\\ &= 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\,(\nabla u\cdot z)^2 dx + 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-1}\,(\nabla u\cdot z)\,(\nabla\cdot z)\,dx\\ &\quad + 2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-1}\,(z\otimes\nabla u):(\nabla\otimes z)\,dx - 4\int_{\mathbb{R}^d} u\,|z|^2 dx\end{aligned}$$

The Bakry-Emery method: details (2/2)

$$
\begin{aligned}
\text{(II)} &= 2\int_{\mathbb{R}^d} u\, z \cdot \frac{\partial z}{\partial t}\, dx \\
&= -2\,\eta\,(1-m)\int_{\mathbb{R}^d} \left[u^m(\nabla\cdot z)^2 + 2\,u^{m-1}(\nabla u\cdot z)\,(\nabla\cdot z) + u^{m-2}(\nabla u\cdot z)^2\right] dx
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^d} \frac{\partial u}{\partial t}\,|z|^2 dx + 4\int_{\mathbb{R}^d} u\,|z|^2 dx
&= -2\,\eta\,(1-m)\int_{\mathbb{R}^d} u^{m-2}\left[u^2(\nabla\cdot z)^2 + u\,(\nabla u\cdot z)\,(\nabla\cdot z)\right. \\
&= -2\,\eta\,\frac{1-m}{m}\int_{\mathbb{R}^d} u^m\left(|\nabla z|^2 - (1-m)\,(\nabla\cdot z)^2\right) dx
\end{aligned}
$$

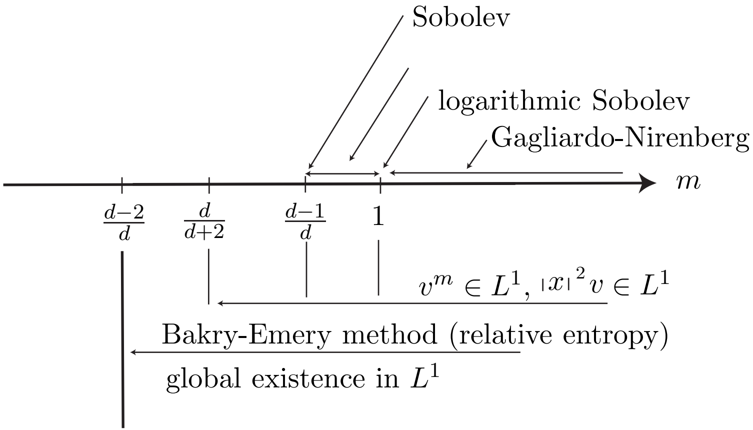
By the arithmetic geometric inequality, we know that

$$
|\nabla z|^2 - (1-m)\,(\nabla\cdot z)^2 \geq 0
$$

if $1-m \leq 1/d$, that is, if $m \geq m_1 = 1 - 1/d$

Fast diffusion: finite mass regime

Inequalities...



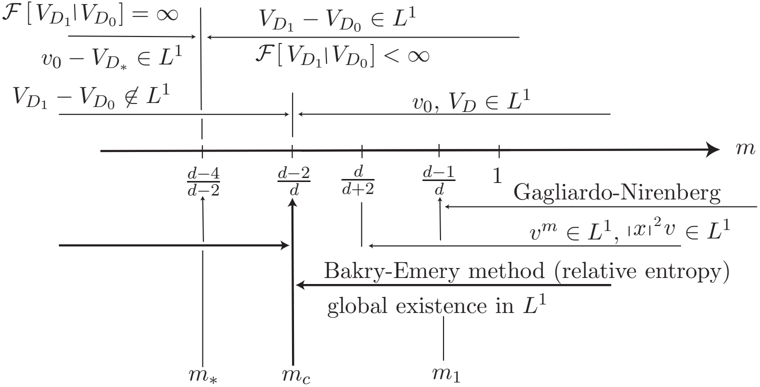
... existence of solutions of $u_t = \Delta u^m$

Fast diffusion equations: the infinite mass regime by linearization of the entropy

Extension to the infinite mass regime, finite time vanishing

- If $m > m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in $L^1(\mathbb{R}^d)$ and the Barenblatt self-similar solution has finite mass
- For $m \leq m_c$, the Barenblatt self-similar solution has infinite mass

Extension to $m \leq m_c$? Work in relative variables !



Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez]

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u \nabla u^{m-1}) = \frac{1-m}{m} \Delta u^m \tag{1}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$

$$R(\tau) := (T + \tau)^{\frac{1}{d\,(m-m_c)}}$$

- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \nearrow T} u(\tau, y) = 0$

$$R(\tau) := (T - \tau)^{-\frac{1}{d\,(m_c-m)}}$$

Self-similar *Barenblatt type solutions* exists for any m

$$t := \tfrac{1-m}{2} \log\left(\tfrac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\tfrac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

Generalized Barenblatt profiles: $V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}}$

Sharp rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \leq v_0 \leq V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \geq 3$ *and* $m \leq m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_* := \frac{d-4}{d-2}$, *the entropy decays according to*

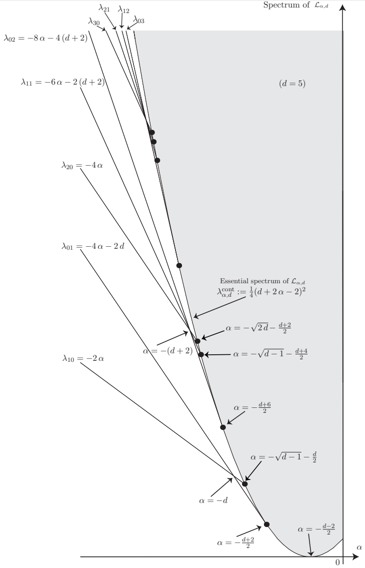
$$\mathcal{F}[v(t,\cdot)] \leq C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\; t \geq 0$$

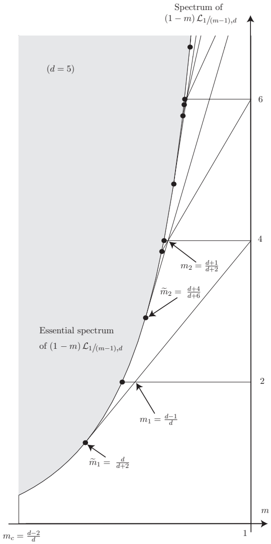
where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\; f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$

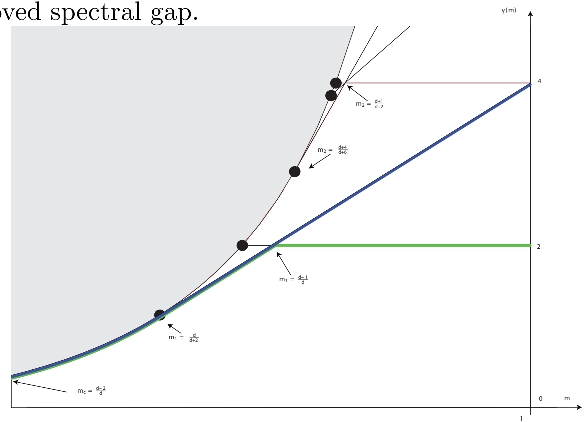
Plots ($d = 5$)


Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d = 5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d+2\,\alpha-2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d+2}{2}$
$\alpha = -\frac{d-2}{2}$
α
0


Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d = 5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
$m_c = \frac{d-2}{d}$
1

Improved asymptotic rates

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of the fast diffusion equation with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$, then the asymptotic convergence holds with an improved rate corresponding to the improved spectral gap.



Higher order matching asymptotics

[J.D., G. Toscani] For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

Without choosing R, we may define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \;, \quad R = R(\tau) \;, \quad t = \tfrac{1}{2} \log R \;, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \;, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma} |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{2}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution (it plays the role of a *local Gibbs state*) but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\, B_\sigma^{m-1} \left(v - B_\sigma \right) \right] dx$$

The time derivative of this relative entropy is

$$\frac{d}{dt} \mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt} \left(\frac{d}{d\sigma} \mathcal{F}_\sigma[v] \right)_{|\sigma=\sigma(t)}}_{\text{choose it } = 0} + \frac{m}{m-1} \int_{\mathbb{R}^d} \left(v^{m-1} - B_{\sigma(t)}^{m-1} \right) \frac{\partial v}{\partial t}\, dx$$

$\Longleftrightarrow$ Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\Longleftrightarrow$ $\int_{\mathbb{R}^d} |x|^2\, B_\sigma\, dx = \int_{\mathbb{R}^d} |x|^2\, v\, dx$

The entropy / entropy production estimate

Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right]B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$

When linearizing, one more mode is killed and $\sigma(t)$ *scales out*

Improved rates of convergence

Theorem (J.D., G. Toscani)

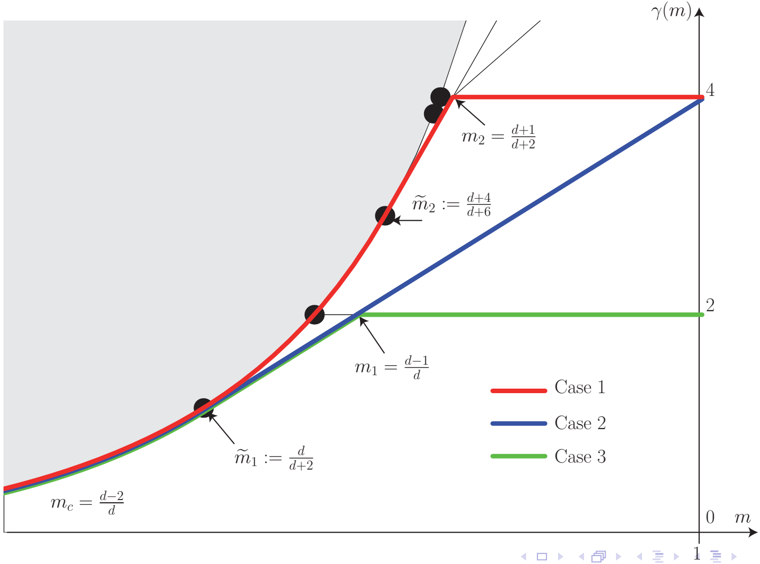
Let $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{F}[v(t,\cdot)] \leq C\, e^{-2\,\gamma(m)\,t} \quad \forall\, t \geq 0$$

where

$$\gamma(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

Spectral gaps and best constants



Comments

1. A recent result by [Denzler, Koch, McCann] *Higher order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach*
2. The constant C in

$$\mathcal{F}[v(t,\cdot)] \le C\, e^{-2\,\gamma(m)\,t} \quad \forall\ t \ge 0$$

can be made explicit, under additional restrictions on the initial data [Bonforte, J.D., Grillo, Vázquez]

An explicit constant C ?

$$\frac{d}{dt}\mathcal{F}[w(t,\cdot)] = -\mathcal{I}[w(t,\cdot)] \quad \forall\, t>0$$

$$h^{m-2}\int_{\mathbb{R}^d}|f|^2\,V_D^{2-m}\,dx \le 2\,\mathcal{F}[w] \le h^{2-m}\int_{\mathbb{R}^d}|f|^2\,V_D^{2-m}\,dx$$

where $f := (w-1)\,V_D^{m-1}$, $h := \max\{\sup_{\mathbb{R}^d} w(t,\cdot), 1/\inf_{\mathbb{R}^d} w(t,\cdot)\}$

$$\int_{\mathbb{R}^d}|\nabla f|^2\,V_D\,dx \le h^{5-2m}\,\mathcal{I}[w] + d\,(1-m)\,\big[h^{4(2-m)}-1\big]\int_{\mathbb{R}^d}|f|^2\,V_D^{2-m}\,dx$$

$$0 \le h-1 \le \mathsf{C}\,\mathcal{F}^{\frac{1-m}{d+2-(d+1)m}}$$

Corollary

$\mathcal{F}[w(t,\cdot)] \le G\big(t, h(0), \mathcal{F}[w(0,\cdot)]\big)$ for any $t \ge 0$, where

$$\frac{dG}{dt} = -2\frac{\Lambda_{\alpha,d}-Y(h)}{[1+X(h)]\,h^{2-m}}\,G\,,\ h = 1+\mathsf{C}\,G^{\frac{1-m}{d+2-(d+1)m}}\,,\ G(0)=\mathcal{F}[w(0,\cdot)]$$

Gagliardo-Nirenberg and Sobolev inequalities : improvements

[J.D., G. Toscani]

Best matching Barenblatt profiles

(Repeating) Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[u \left(\sigma^{\frac{d}{2}(m-m_c)} \nabla u^{m-1} - 2\,x \right) \right] = 0 \quad t>0\,, \quad x \in \mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2\, u(x,t)\, dx\,, \quad K_M := \int_{\mathbb{R}^d} |x|^2\, B_1(x)\, dx$$

where

$$B_\lambda(x) := \lambda^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\lambda}\,|x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\lambda^m - m\, B_\lambda^{m-1}\, (u - B_\lambda) \right]\, dx$$

Three ingredients for *global improvements*

1. $\inf_{\lambda>0} \mathcal{F}_\lambda[u(x,t)] = \mathcal{F}_{\sigma(t)}[u(x,t)]$ so that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(x,t)] = -\mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

where the relative Fisher information is

$$\mathcal{J}_\lambda[u] := \lambda^{\frac{d}{2}(m-m_c)} \frac{m}{1-m} \int_{\mathbb{R}^d} u \left|\nabla u^{m-1} - \nabla B_\lambda^{m-1}\right|^2 dx$$

2. In the *Bakry-Emery method*, there is *an additional (good) term*

$$4\left[1 + 2\,C_{m,d}\,\frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{M^\gamma\,\sigma_0^{\frac{d}{2}\,(1-m)}}\right]\frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right) \ge \frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

3. The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u] \ge \frac{m}{8\int_{\mathbb{R}^d} B_1^m\,dx}\,C_M^2\,\|u - B_\sigma\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$$

improved decay for the relative entropy

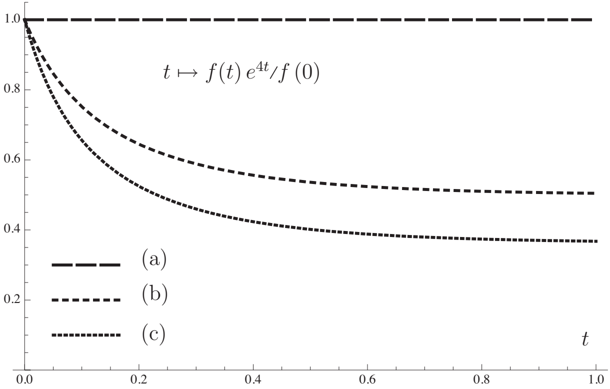


Figure: Upper bounds on the decay of the relative entropy: $t \mapsto f(t)\, e^{4t}/f(0)$
(a): estimate given by the entropy-entropy production method
(b): exact solution of a simplified equation
(c): numerical solution (found by a shooting method)

A Csiszár-Kullback(-Pinsker) inequality

Let $m \in (\widetilde{m}_1, 1)$ with $\widetilde{m}_1 = \frac{d}{d+2}$ and consider the relative entropy

$$\mathcal{F}_\sigma[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\sigma^m - m\, B_\sigma^{m-1} \,(u - B_\sigma) \right] \,dx$$

Theorem

Let $d \geq 1$, $m \in (\widetilde{m}_1, 1)$ and assume that u is a nonnegative function in $\mathrm{L}^1(\mathbb{R}^d)$ such that u^m and $x \mapsto |x|^2\, u$ are both integrable on $\mathbb{R}^d$. If $\|u\|_{\mathrm{L}^1(\mathbb{R}^d)} = M$ and $\int_{\mathbb{R}^d} |x|^2\, u \,dx = \int_{\mathbb{R}^d} |x|^2\, B_\sigma \,dx$, then

$$\frac{\mathcal{F}_\sigma[u]}{\sigma^{\frac{d}{2}(1-m)}} \geq \frac{m}{8 \int_{\mathbb{R}^d} B_1^m \,dx} \left(C_M \|u - B_\sigma\|_{\mathrm{L}^1(\mathbb{R}^d)} + \frac{1}{\sigma} \int_{\mathbb{R}^d} |x|^2\, |u - B_\sigma| \,dx \right)^2$$

Csiszár-Kullback(-Pinsker): proof (1/2)

Let $v := u/B_\sigma$ and $d\mu_\sigma := B_\sigma^m\,dx$

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma = \int_{\mathbb{R}^d} B_\sigma^{m-1}\,(u-B_\sigma)\,dx$$

$$= \sigma^{\frac{d}{2}\,(1-m)}\,C_M\int_{\mathbb{R}^d}(u-B_\sigma)\,dx + \sigma^{\frac{d}{2}(m_c-m)}\int_{\mathbb{R}^d}|x|^2\,(u-B_\sigma)\,dx = 0$$

$$\int_{\mathbb{R}^d}(v-1)\,d\mu_\sigma = \int_{v>1}(v-1)\,d\mu_\sigma - \int_{v<1}(1-v)\,d\mu_\sigma = 0$$

$$\int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma = \int_{v>1}(v-1)\,d\mu_\sigma + \int_{v<1}(1-v)\,d\mu_\sigma$$

$$\int_{\mathbb{R}^d}|u-B_\sigma|\,B_\sigma^{m-1}\,dx = \int_{\mathbb{R}^d}|v-1|\,d\mu_\sigma = 2\int_{v<1}|v-1|\,d\mu_\sigma$$

Csiszár-Kullback(-Pinsker): proof (2/2)

A Taylor expansion shows that

$$\mathcal{F}_\sigma[u] = \frac{1}{m-1}\int_{\mathbb{R}^d}\left[v^m - 1 - m\,(v-1)\right]d\mu_\sigma = \frac{m}{2}\int_{\mathbb{R}^d}\xi^{m-2}\,|v-1|^2\,d\mu_\sigma \ge \frac{m}{2}\int_{v<1}|v-1|^2\,d\mu_\sigma$$

Using the Cauchy-Schwarz inequality, we get

$$\left(\int_{v<1}|v-1|\,d\mu_\sigma\right)^2 = \left(\int_{v<1}|v-1|\,B_\sigma^{\frac{m}{2}}\,B_\sigma^{\frac{m}{2}}\,dx\right)^2 \le \int_{v<1}|v-1|^2\,d\mu_\sigma\int_{\mathbb{R}^d}B_\sigma^m\,dx$$

and finally obtain that

$$\mathcal{F}_\sigma[u] \ge \frac{m}{2}\,\frac{\left(\int_{v<1}|v-1|\,d\mu_\sigma\right)^2}{\int_{\mathbb{R}^d}B_\sigma^m\,dx} = \frac{m}{8}\,\frac{\left(\int_{\mathbb{R}^d}|u-B_\sigma|\,B_\sigma^{m-1}\,dx\right)^2}{\int_{\mathbb{R}^d}B_\sigma^m\,dx}$$

An improved Gagliardo-Nirenberg inequality: the setting

The inequality

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathcal{C}_{p,d}^{\mathrm{GN}} \, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta} \, \|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

with $\theta = \theta(p) := \frac{p-1}{p} \, \frac{d}{d+2-p\,(d-2)}$, $1 < p \le \frac{d}{d-2}$ if $d \ge 3$ and $1 < p < \infty$ if $d = 2$, can be rewritten, in a non-scale invariant form, as

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} |f|^{p+1} \, dx \ge \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2p} \, dx \right)^{\gamma}$$

with $\gamma = \gamma(p,d) := \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$. Optimal function are given by

$$f_{M,y,\sigma}(x) = \frac{1}{\sigma^{\frac{d}{2}}} \left(C_M + \frac{|x-y|^2}{\sigma} \right)^{-\frac{1}{p-1}} \quad \forall \, x \in \mathbb{R}^d$$

where C_M is determined by $\int_{\mathbb{R}^d} f_{M,y,\sigma}^{2p} \, dx = M$

$$\mathfrak{M}_d := \left\{ f_{M,y,\sigma} \; : \; (M,y,\sigma) \in \mathcal{M}_d := (0,\infty) \times \mathbb{R}^d \times (0,\infty) \right\}$$

An improved Gagliardo-Nirenberg inequality (1/2)

Relative entropy functional

$$\mathcal{R}^{(p)}[f] := \inf_{g\in\mathfrak{M}_d^{(p)}} \int_{\mathbb{R}^d} \left[g^{1-p}\left(|f|^{2p} - g^{2p}\right) - \tfrac{2p}{p+1}\left(|f|^{p+1} - g^{p+1}\right)\right] dx$$

Theorem

Let $d \ge 2$, $p > 1$ and assume that $p < d/(d-2)$ if $d \ge 3$. If

$$\frac{\int_{\mathbb{R}^d} |x|^2\,|f|^{2p}\,dx}{\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^\gamma} = \frac{d\,(p-1)\,\sigma_*\,M_*^{\gamma-1}}{d+2-p\,(d-2)}\,,\;\sigma_*(p) := \left(4\,\frac{d+2-p\,(d-2)}{(p-1)^2\,(p+1)}\right)^{\frac{4p}{d-p\,(d-4)}}$$

for any $f \in \mathrm{L}^{p+1}\cap\mathcal{D}^{1,2}(\mathbb{R}^d)$, then we have

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^\gamma \ge \mathsf{C}_{p,d}\,\frac{\left(\mathcal{R}^{(p)}[f]\right)^2}{\left(\int_{\mathbb{R}^d}|f|^{2p}\,dx\right)^\gamma}$$

An improved Gagliardo-Nirenberg inequality (2/2)

A Csiszár-Kullback inequality

$$\mathcal{R}^{(p)}[f] \ge \mathsf{C}_{\rm CK}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-2)} \inf_{g\in\mathfrak{M}_d^{(p)}} \||f|^{2p} - g^{2p}\|_{\mathrm{L}^1(\mathbb{R}^d)}^2$$

with $\mathsf{C}_{\rm CK} = \frac{p-1}{p+1}\,\frac{d+2-p\,(d-2)}{32\,p}\,\sigma_*^{d\,\frac{p-1}{4p}}\,M_*^{1-\gamma}$. Let

$$\mathfrak{C}_{p,d} := \mathsf{C}_{d,p}\,{\mathsf{C}_{\rm CK}}^2$$

Corollary

Under previous assumptions, we have

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx + \int_{\mathbb{R}^d} |f|^{p+1}\,dx - \mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d} |f|^{2p}\,dx\right)^{\gamma}$$
$$\ge \mathfrak{C}_{p,d}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-4)} \inf_{g\in\mathfrak{M}_d(p)} \||f|^{2p} - g^{2p}\|_{\mathrm{L}^1(\mathbb{R}^d)}^4$$

Conclusion 1: improved inequalities

- We have found an improvement of an optimal Gagliardo-Nirenberg inequality, which provides an explicit measure of the distance to the manifold of optimal functions.
- The method is based on the nonlinear flow
- The explicit improvement gives (is equivalent to) an improved entropy – entropy production inequality

Conclusion 2: improved rates

If $m \in (m_1, 1)$, with

$$f(t) := \mathcal{F}_{\sigma(t)}[u(\cdot,t)]$$
$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2\, u(x,t)\, dx$$
$$j(t) := \mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

$$\mathcal{J}_\sigma[u] := \frac{m\,\sigma^{\frac{d}{2}(m-m_c)}}{1-m} \int_{\mathbb{R}^d} u\, \left|\nabla u^{m-1} - \nabla \mathfrak{B}_\sigma^{m-1}\right|^2 dx$$

we can write a system of coupled ODEs

$$\begin{cases} f' = -j \le 0 \\ \sigma' = -2\,d\,\frac{(1-m)^2}{m\,K_M}\,\sigma^{\frac{d}{2}(m-m_c)}\,f \le 0 \\ j' + 4\,j = \frac{d}{2}\,(m-m_c)\left[\frac{j}{\sigma} + 4\,d\,(1-m)\,\frac{f}{\sigma}\right]\sigma' - \mathrm{r} \end{cases} \tag{3}$$

In the rescaled variables, we have found an *improved decay* (algebraic rate) of the relative entropy. This is a new nonlinear effect, which matters for the initial time layer

Conclusion 3: Best matching Barenblatt profiles are delayed

Let u be such that

$$v(\tau,x)=\frac{\mu^d}{R(D\,\tau)^d}\,u\left(\frac{1}{2}\log R(D\,\tau),\frac{\mu\,x}{R(D\,\tau)}\right)$$

with $\tau\mapsto R(\tau)$ given as the solution to

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left(\frac{\mu^2}{K_M}\int_{\mathbb{R}^d}|x|^2\,v(\tau,x)\,dx\right)^{-\frac{d}{2}(m-m_c)}\,,\quad R(0)=1$$

Then

$$\frac{1}{R}\,\frac{dR}{d\tau}=\left[R^2(\tau)\,\sigma\left(\frac{1}{2}\log R(D\,\tau)\right)\right]^{-\frac{d}{2}(m-m_c)}$$

that is $R(\tau)=R_0(\tau)\le R_0(\tau)$ where $\frac{1}{R}\,\frac{dR_0}{d\tau}=\left(R_0^2(\tau)\,\sigma\,(0)\right)^{-\frac{d}{2}(m-m_c)}$
and asymptotically as $\tau\to\infty$, $R(\tau)=R_0(\tau-\delta)$ for some delay $\delta>0$

Thank you for your attention !