Entropy methods in kinetic equations, parabolic equations and quantum models

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in memory of Naoufel Ben Abdallah

Outline

- Entropy in non-dissipative kinetic theory
  - Vlasov-Poisson system with injection boundary conditions the electrostatic case (coll. Naoufel)
  - Dynamical stability
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Entropy in non-dissipative kinetic theory

- Vlasov-Poisson system with injection boundary conditions the electrostatic case (coll. Naoufel)
- Dynamical stability
- Gravitational Vlasov-Poisson (Newton) system: Variational methods and functional inequalities
- Relative equilibria

Vlasov-Poisson system with injection boundary conditions: the electrostatic case

Vlasov-Poisson system with injection boundary conditions

ω is a bounded domain in $\mathbb{R}^d$, $\partial\omega$ is of class C^1
$\Omega = \omega \times \mathbb{R}^d$ and $\Gamma = \partial\Omega = \partial\Omega \times \mathbb{R}^d$ are the phase space and its boundary (incoming and outgoing boundary)

$$\Sigma^\pm(x) = \{v \in \mathbb{R}^d \ : \ \pm v \cdot \nu(x) > 0\}$$

$$\Gamma^\pm = \{(x,v) \in \Gamma \ : \ v \in \Sigma^\pm(x)\}$$

Vlasov-Poisson system with injection boundary conditions

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - (\nabla_x \varphi + \nabla_x \varphi_0) \cdot \nabla_v f = 0 \\ f_{|t=0} = f_0 \,, \ f_{|\Gamma^- \times \mathbb{R}^+}(x,v,t) = \gamma(\tfrac{1}{2}|v|^2 + \varphi_0(x)) \\ -\Delta\varphi = \rho = \displaystyle\int_{\mathbb{R}^d} f \ dv \,, (x,t) \in \omega \times \mathbb{R}^+ \\ \qquad \text{and} \quad \varphi(x,t) = 0 \,, (x,t) \in \partial\Omega \times \mathbb{R}^+ \end{cases}$$

Assumptions

The initial condition f_0 is a nonnegative bounded function

The external electrostatic potential is nonnegative and smooth

The function γ is defined on $(\min_{x\in\omega}\varphi_0(x),+\infty)$, bounded, smooth, strictly decreasing with values in $(0,\infty)$, and

$$\sup_{x\in\omega}\int_0^{+\infty} s^{d/2}\gamma(s+\varphi_0(x))\,ds<+\infty$$

γ is strictly decreasing: $\beta=-\int_0^g\gamma^{-1}(z)\,dz$ is strictly convex

Stationary solution: the nonlinear Poisson equation

Let $U : L^1(\Omega) \to W_0^{1,d/(d-1)}(\omega)$: $U[g] = u$ the unique solution in $W_0^{1,d/(d-1)}(\omega)$ of

$$-\Delta u = \int_{\mathbb{R}^d} g(x,v)\,dv$$

$M(x,v) = \gamma\left(\frac{1}{2}|v|^2 + U[M](x) + \varphi_0(x)\right)$ is a stationary solution:

$$v \cdot \nabla_x M - (\nabla_x \varphi_0 + \nabla_x U[M]) \cdot \nabla_v M = 0$$

It is the unique critical point in $H_0^1(\omega)$ of the strictly convex coercive functional

$$U \mapsto \frac{1}{2}\int_\omega |\nabla U|^2\,dx - \int_\omega G(U+\varphi_0)\,dx$$

where $G'(u) = g(u) = \int_{\mathbb{R}^d} \gamma\left(\frac{1}{2}|v|^2 + u\right)\,dv$

$$= 2^{d/2-1}|S^{d-1}| \cdot \int_0^{+\infty} s^{d/2-1}\,\gamma(s+u)\,ds$$

Relative entropy

$$\mathcal{F}[g|h] := \int_\Omega \left(\beta(g) - \beta(h) - (g-h)\,\beta'(h)\right)\,dx\,dv + \tfrac{1}{2}\int_\omega |\nabla U[g-h]|^2\,dx$$

where β is the real function defined by $\beta(g) = -\int_0^g \gamma^{-1}(z)\,dz$

$$\int_\Omega f\left(\tfrac{1}{2}|v|^2 + \tfrac{1}{2}U[f] + \varphi_0\right)dx\,dv$$
$$= \int_\Omega \left[\tfrac{1}{2}(f-M)U[f-M] - \tfrac{1}{2}M\,U[M] - f\beta'(M)\right]dx\,dv$$

$$\mathcal{F}[f] = \int_\Omega \left(\beta(f) + \left(\tfrac{1}{2}|v|^2 + \tfrac{1}{2}U[f] + \varphi_0\right)f\right)dx\,dv$$
$$- \int_\Omega \left(\beta(M) + \left(\tfrac{1}{2}|v|^2 + \tfrac{1}{2}U[M] + \varphi_0\right)M\right)dx\,dv$$

Relative entropy and irreversibility

Theorem

Assume that $f_0 \in L^1 \cap L^\infty_+$ is such that $\mathcal{F}[f_0|M] < +\infty$

$$\frac{d}{dt}\mathcal{F}[f(t)|M] = -\mathcal{F}^+[f(t)|M]$$

$\mathcal{F}^+$ is the boundary relative entropy flux

$$\mathcal{F}^+[g|h] = \int_{\Gamma^+} \left(\beta(g) - \beta(h) - (g-h)\,\beta'(h)\right)\,d\sigma$$

Proof. Integration by parts, like in [Darrozès, Guiraud]

$$\frac{d}{dt}\int_\Omega \beta(f)\,dx\,dv = \sum_\pm \mp \int_{\Gamma^\mp} \beta(f)\,d\sigma$$

$\frac{d}{dt}\int_\Omega f\left(\frac{1}{2}|v|^2 + \frac{1}{2}U[f] + \varphi_0\right)\,dx\,dv = \sum_\pm \pm \int_{\Gamma^\mp} f\left(\frac{1}{2}|v|^2 + \varphi_0\right)\,d\sigma$
$\int_{\Gamma^\pm} f\left(\frac{1}{2}|v|^2 + \varphi_0(x)\right)\,d\sigma = -\int_{\Gamma^\pm} f\beta'(M)\,d\sigma$ $\square$

The large time limit

$\mathcal{F}[f(t)|M] + \int_0^t \mathcal{F}^+[f(s)|M]ds \leq \mathcal{F}[f_0|M]$

$$(f^n(x,v,t), \varphi^n(x,t)) = (f(x,v,t+t_n), \varphi(x,t+t_n))$$

$$\lim_{n\to+\infty} \int_{\mathbb{R}^+} \mathcal{F}^+[f^n(s)|M]\, ds = 0$$

$$\sup_{t>0} \mathcal{F}[f^n(t)|M] \leq C\,.$$

+ Dunford-Pettis criterion: $(f^n, \varphi^n) \rightharpoonup (f^\infty, \varphi^\infty)$ weakly in $L^1_{\rm loc}(dt, L^1(\Omega)) \times L^1_{\rm loc}(dt, H^1_0(\omega))$

$$\boxed{f^\infty \equiv M \text{ on } \Gamma}$$

Is f^∞ stationary ?

$f^\infty \equiv M$ on Γ: $f^\infty \equiv M$ on Ω ? A partial answer for $d = 1$.

Theorem

Consider a solution $(f^\infty, \varphi^\infty)$ such that $f^\infty \equiv M$ on Γ on the interval $\omega = (0,1)$. If φ is analytic in x with C^∞ (in time) coefficients and if φ_0 is analytic with $-\frac{d^2\varphi_0}{dx^2} \geq 0$ on ω, then (f, φ) is the unique stationary solution, given by: $f = M$, $\varphi = U[M]$

Proof. Let $\varphi_0(0) = 0$, $\varphi_0(1) \geq 0$: $\varphi_0'(0) \geq 0$. Characteristics

$$\frac{\partial X}{\partial t} = V \,, \qquad \frac{\partial V}{\partial t} = -\frac{\partial \varphi}{\partial x}(X,t) - \frac{d\varphi_0}{dx}(X) \,,$$

$$X(s; x, v, s) = x \,, \qquad V(s; x, v, s) = v$$

are defined on $(\mathcal{T}_{in}(s;x,v), \mathcal{T}_e(s;x,v))$: either $\mathcal{T}_{in}(s;x,v) = -\infty$ or $(X_{in}, V_{in})(s;x,v) \in \Gamma^-$; either $\mathcal{T}_e(s;x,v) = +\infty$ or $(X_e, V_e)(s;x,v) \in \Gamma^+$.

Step 1: the electric field is repulsive at $x = 0$.

Step 2 : Analysis of the characteristics in a neighborhood of $(0,0,t)$.

$$f(X_{in}, V_{in}, \mathcal{T}_{in}) = \gamma\left(\tfrac{1}{2}\,|V_{in}|^2\right)$$
$$f(X_e, V_e, \mathcal{T}_e) = \gamma\left(\tfrac{1}{2}\,|V_e|^2\right)$$

The function γ is strictly decreasing:

$$|V_{in}(t_0, x_0, 0)| = |V_e(t_0, x_0, 0)|$$

Characteristics are parametrized by

$$\frac{dt^\pm}{dX} = \frac{1}{V}\,, \quad \frac{dV}{dX} = -\frac{1}{V}\frac{\partial \Phi}{\partial x}(X, t^\pm)$$

Let $e_\pm(X) = \frac{1}{2}V^2(X)$

$$t^\pm(X) = t_0 \mp \int_{x_0}^{X} \frac{dY}{\sqrt{2e_\pm(Y)}} \quad \forall\, X \in [0, x_0]$$

$$\frac{de_\pm}{dX} = -\frac{\partial \Phi}{\partial x}(X, t^\pm(X))\,, \quad e_\pm(x_0) = 0$$

Rescaling: $x_0 = \varepsilon^2\,,\quad X = \varepsilon^2(1-x)$ and $e_\pm(X) := \frac{\varepsilon^2}{2}\,e_\pm^\varepsilon(x)$

$$\boxed{\frac{de_\pm^\varepsilon}{dx} = 2\,\frac{\partial \Phi}{\partial x}\left(\varepsilon^2(1-x),\, t_0 \pm \varepsilon \int_0^x \frac{dy}{\sqrt{e_\pm^\varepsilon(y)}}\right)}$$

with the condition $e_+^\varepsilon(1) = e_-^\varepsilon(1)$ for any $\varepsilon > 0$ small enough

$$e_\pm^\varepsilon = \sum_{n=0}^{+\infty} \varepsilon^n e_n^\pm$$

Lemma

With the above notations, for all $n \in \mathbb{N}$, we have the following identities:

(i) $\displaystyle \frac{de_{2n}^\pm}{dx}(x) = \frac{2(1-x)^n}{n!}\,\partial_x^{n+1}\Phi(0,t_0)$

(ii) $\displaystyle \frac{de_{2n+1}^\pm}{dx}(x) = \frac{\pm 2(1-x)^n}{(n+1)!}\left(\int_0^x \frac{dy}{\sqrt{e_0(y)}}\right)\partial_t\,\partial_x^{n+1}\Phi(0,t_0),$

(iii) $\partial_t\,\partial_x^{n+1}\Phi(0,t_0) = 0$

Stability

[M. J. Cáceres, J. A. Carrillo, and J. Dolbeault. Nonlinear stability in Lp for a confined system of charged particles. SIAM J. Math. Anal., 34 : 478-494, 2002]

Theorem

Let φ_0 *be s.t.* $(x,s)\mapsto s^{3/2-1}\gamma(s+\varphi_0(x))\in L^1\cap L^\infty(\mathbb{R}^3,L^1(\mathbb{R}))$. *If* f *is a weak solution of Vlasov-Poisson with* f_0 *in* $L^1\cap L^{p_0}$, $p_0=(12+3\sqrt{5})/11$, *s.t.* $\beta(f_0)$, $(|\varphi_0|+|v|^2)f_0\in L^1(\mathbb{R}^3)$ *and if for some* $p\in[1,2]$, $\inf_{s\in(0,+\infty)}\beta''(s)/s^{p-2}>0$, *then*

$$\|f-f_\infty\|^2_{L^p}\le\frac{C}{2}\int_{\mathbb{R}^3}|\nabla(\varphi_0-\varphi_\infty)|^2\,dx$$
$$+\,C\int_{\mathbb{R}^6}[\beta(f_0)-\beta(f_\infty)-\beta'(f_\infty)(f_0-f_\infty)]dx\,dv$$

where $\left(f_\infty(x,v)=\gamma(\frac{1}{2}|v|^2+\varphi_0(x)+\varphi_\infty(x)),\varphi_\infty\right)$. *Here* $\gamma^{-1}=-\beta'$

Stability

Value of p_0 [Hörst and Hunze] (weak solutions). Renormalization [DiPerna and Lions, Mischler]. [Braasch, Rein and Vukadinović, 1998].
Csiszár-Kullback inequality: [AMTU, 2000]

Theorem

$p_0 = 2$ and $\beta(s) = s \log s - s$. There exists a convex functional $\mathcal{F}$ reaching its minimum at $f_\infty(x,v) = \frac{e^{-|v|^2/2}}{(2\pi)^{3/2}}\,\rho_\infty(x)$ s.t.

$$\|f(t,\cdot) - f_\infty\|^2_{L^2} \leq \mathcal{F}[f_0]$$

To be compared with the standard Csiszár-Kullback results !
de la Vallée - Poussin type of approach
Here $p = 1$, $\gamma(s) = e^{-s}$

$$-\Delta\varphi_\infty = \rho_\infty = \|f_0\|_{L^1}\,\frac{e^{-(\varphi_\infty+\varphi_0)}}{\int e^{-(\varphi_\infty+\varphi_0)}\,dx}$$

Gravitational Vlasov-Poisson (Newton) system: Variational methods and functional inequalities

Gravitational Vlasov-Poisson (Newton) system:

Consider solutions of the gravitational Vlasov-Poisson system

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \varphi \cdot \nabla_v f = 0 \\ \varphi = -\frac{1}{4\pi\,|\cdot|} * \rho\,, \quad \rho = \int_{\mathbb{R}^3} f\,dv \end{cases}$$

By minimizing the *free energy* functional

$$\mathcal{F}[f] = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\,dx\,dv + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f\,dx\,dv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\varphi|^2\,dx$$

stationary solutions can be found, s.t.

$$f(x,v) = \gamma\left(\lambda + \tfrac{1}{2}\,|v|^2 + \varphi(x)\right)$$

where $\gamma = (\beta')^{-1}$. With $g(s) := \int_{\mathbb{R}^3} \gamma\left(s + \frac{1}{2}\,|v|^2\right)\,dv$, the potential is given by the nonlinear Poisson equation:

$$\Delta\varphi = g\left(\lambda + \varphi(x)\right)$$

Variational methods and functional inequalities and dynamical stability

- Use concentration-compactness methods (for instance) + a priori estimates and minimize $\mathcal{F}[f]$ under a mass constraint $\int_{\mathbb{R}^3} f\, dv = M$
- Power law (polytropic cases) case: $\beta(f) = \frac{1}{q}\, \kappa_q^{q-1}\, f^q$, $q \in (9/7, \infty)$
- Minimization of $\mathcal{F}$ is exactly equivalent to build explicit *interpolation inequalities* of Hardy-Littlewood-Sobolev type, with optimal constants explicitly related to $\inf \mathcal{F}$
- Stability is a consequence of the conservation of the free energy (resp. free energy bound) for classical solutions (resp. weak solutions) of the Vlasov-Poisson system

Relative equilibria

Relative equilibria

[J. Campos, M. del Pino, and J. Dolbeault. Relative equilibria in continuous stellar dynamics. *Communications in Mathematical Physics*, 300:765–788, 2010]

Consider solutions of the gravitational Vlasov-Poisson system

$$\left\{\begin{array}{l}\partial_t f + v\cdot\nabla_x f - \nabla_x\varphi\cdot\nabla_v f = 0\\ \varphi = -\frac{1}{4\pi\,|\cdot|} * \rho\;, \quad \rho = \int_{\mathbb{R}^3} f\,dv\end{array}\right.$$

which are rotating at constant angular velocity ω:
$x = (x', x^3) \mapsto (e^{i\,\omega\,t}\,x', x^3)$ and $v = (v', v^3) \mapsto (i\,\omega\,x' + e^{i\,\omega\,t}\,v', v^3)$

$$\left\{\begin{array}{l}\partial_t f + v\cdot\nabla_x f - \nabla_x\varphi\cdot\nabla_v f - \omega^2\,x'\cdot\nabla_{v'} f + 2\,\omega\,i\,v'\cdot\nabla_{v'} f = 0\;,\\ \varphi = -\frac{1}{4\pi\,|\cdot|} * \rho\;, \quad \rho = \int_{\mathbb{R}^3} f\,dv\;,\end{array}\right.$$

A *relative equilibrium* of (2) is a stationary solution of (2) and can be obtained as a critical point of the *free energy* functional

$$\mathcal{F}[f] = \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\,dx\,dv + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left(|v|^2 - \omega^2\,|x'|^2\right) f\,dx\,dv - \frac{1}{2}\int_{\mathbb{R}^3}|\nabla\varphi|^2\,dx$$

Critical points

If $\omega \neq 0$, $\mathcal{F}$ is not bounded from below

The *polytropic gas model:* $\beta(f) = \frac{1}{q}\,\kappa_q^{q-1}\,f^q$ for some $q \in (9/7, \infty)$

$$f(x,v) = \gamma\big(\lambda + \tfrac{1}{2}\,|v|^2 + \varphi(x) - \tfrac{1}{2}\,\omega^2\,|x'|^2\big)$$

where $\gamma(s) = \kappa_q^{-1}\,(-s)_+^{1/(q-1)}$. Nonlinear Poisson equation:

$$\Delta\varphi = g\left(\lambda + \varphi(x) - \tfrac{1}{2}\,\omega^2\,|x'|^2\right) \quad \text{if } x \in \operatorname{supp}(\rho)$$

with $g(\mu) = (-\mu)_+^p$ and $p = \frac{1}{q-1} + \frac{3}{2}$. With $u = -\varphi$,

$$-\Delta u = \sum_{i=1}^{N} \rho_i^\omega \quad \text{in } \mathbb{R}^3\,, \quad \rho_i^\omega = \left(u - \lambda_i + \tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^p \chi_i$$

under the asymptotic boundary condition $\lim_{|x|\to\infty} u(x) = 0$

$$m_i = \int_{\mathbb{R}^3} \rho_i^\omega\;dx \text{ and } \xi_i^\omega = \frac{1}{m_i}\int_{\mathbb{R}^3} x\,\rho_i^\omega\;dx$$

A result

Theorem

Let $N \geq 2$ and $p \in (3/2, 3) \cup (3, 5)$. For almost any m_i, $i = 1, \dots N$, and for any sufficiently small $\omega > 0$, there exist at least $[2^{N-1}(N-2)+1]\,(N-2)\,!$ *distinct stationary solutions f_ω of (2) s.t.*

$$\int_{\mathbb{R}^3} f_\omega \, dv = \sum_{i=1}^N \rho_i^\omega + o(1) \quad \text{as}\, \omega \to 0_+$$

where $o(1)$ means that the remainder term uniformly converges to 0

$$\forall\, i = 1, \dots N, \quad \rho_i^\omega(x - \xi_i^\omega) = \lambda_i^p \, \rho_*\big(\lambda_i^{(p-1)/2} \, x\big) + o(1)$$

where ρ_* is non-negative, radially symmetric, non-increasing, compactly supported function. $m_i = \lambda_i^{(3-p)/2} \int_{\mathbb{R}^3} \rho_* \, dx + o(1)$ and $\xi_i^\omega = \omega^{-2/3} (\zeta_i^\omega, 0)$ are s.t. $\zeta_i^\omega \in \mathbb{R}^2$ converges to a critical point of

$$\mathcal{V}(\zeta_1, \dots \zeta_N) = \frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i \, m_j}{|\zeta_i - \zeta_j|} + \frac{1}{2} \sum_{i=1}^{N} m_i \, |\zeta_i|^2$$

[J.I. Palmore]: classification of relative equilibria for the N-body problems + find critical points of

$$\mathsf{J}[u]=\frac{1}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx-\frac{1}{p+1}\sum_{i=1}^{N}\int_{\mathbb{R}^3}\left(u-\lambda_i+\tfrac{1}{2}\,\omega^2\,|x'|^2\right)_+^{p+1}\chi_i\,dx$$

by using the solution of

$$-\Delta w_*=(w_*-1)_+^p=:\rho_*\quad\text{in }\mathbb{R}^3$$

as "building brick" on each of the connected components. With $W_\xi:=\sum_{i=1}^N w_i$, $w_i(x)=\lambda_i\,w_*\big(\lambda_i^{(p-1)/2}\,(x-\xi_i)\big)$ and $\xi=(\xi_1,\ldots\xi_N)$, we want to solve the problem

$$\Delta\varphi+\sum_{i=1}^{N}p\big(W_\xi-\lambda_i+\tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^{p-1}\chi_i\,\varphi=-\mathsf{E}-\mathsf{N}[\varphi]$$

with $\lim_{|x|\to\infty}\varphi(x)=0$, where $\mathsf{E}=\Delta W_\xi+\sum_{i=1}^N\big(W_\xi-\lambda_i+\tfrac{1}{2}\,\omega^2\,|x'|^2\big)_+^p\,\chi_i$

$$\mathsf{J}\,[W_\xi] = \sum_{i=1}^N \lambda_i^{(5-p)/2}\,\mathsf{e}_* - \omega^{2/3}\,\mathcal{V}(\zeta_1,\ldots\zeta_N) + O(\omega^{4/3})$$

where $\mathsf{e}_* = \frac{1}{2}\int_{\mathbb{R}^3}|\nabla w|^2\,dx - \frac{1}{p+1}\int_{\mathbb{R}^3}(w-1)_+^{p+1}\,dx$ and $\zeta_i = \omega^{2/3}\,\xi_i'$ if the points ξ_i are such that, for a large, fixed $\mu > 0$, and all small $\omega > 0$, we have $|\xi_i| < \mu\,\omega^{-2/3}$ and $|\xi_i - \xi_j| > \mu^{-1}\,\omega^{-2/3}$. To localize each K_i in a neighborhood of ξ_i, we impose the orthogonality conditions

$$\int_{\mathbb{R}^3}\varphi\,\partial_{x_j}w_i\,\chi_i\,dx = 0 \quad \forall\; i = 1,\,2\ldots N,\; j = 1,\,2,\,3$$

to the price of Lagrange multipliers and solve the problem by fixed point methods

Since $\xi \mapsto \mathsf{J}\,[W_\xi]$ is a finite dimensional function, if $\xi_i = (\zeta_i, 0)$ is such that $(\zeta_1,\ldots\zeta_N)$ is in a neighborhood of a non-degenerate critical point of $\mathcal{V}$, we can find a critical point φ for which the Lagrange multipliers are all equal to zero

Diffusive limits, parabolic equations

Some references

- Diffusive limits

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- Fast diffusion equations

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- Hypocoercivity

  - [J. Dolbeault, C. Mouhot, and C. Schmeiser. *Hypocoercivity for linear kinetic equations conserving mass.* Preprint]
  - [J. Dolbeault, C. Mouhot, and C. Schmeiser. *Hypocoercivity for kinetic equations with linear relaxation terms.* Comptes Rendus Mathématique, 347 : 511-516, 2009]

Entropy for quantum models

[J. Dolbeault, P. Felmer, M. Loss, and E. Paturel. *Lieb-Thirring type inequalities and Gagliardo-Nirenberg inequalities for systems.* J. Funct. Anal., 238: 193-220, 2006]

[J. Dolbeault, P. Felmer, and J. Mayorga-Zambrano. *Compactness properties for trace-class operators and applications to quantum mechanics.* Monatshefte für Mathematik, 155: 43-66, 2008]

[J. Dolbeault, P. Felmer, and M. Lewin. *Orbitally stable states in generalized Hartree-Fock theory.* Mathematical Models and Methods in Applied Sciences, 19: 347-367, 2009]

[G. L. Aki, J. Dolbeault, and C. Sparber. *Thermal effects in gravitational Hartree systems.* To appear in Annales Henri Poincaré]

Lieb-Thirring type inequalities and Gagliardo-Nirenberg inequalities for systems

A little bit of functional analysis

- [J. Dolbeault, P. Felmer, M. Loss, and E. Paturel. *Lieb-Thirring type inequalities and Gagliardo-Nirenberg inequalities for systems.* J. Funct. Anal., 238 (1): 193–220, 2006]

Let $H = -\Delta + V$ in $\mathbb{R}^d$, $d > 1$ with lowest $\lambda_1(V)$ and consider

$$C_1 = \sup_{V \in \mathcal{D}(\mathbb{R}^d)} \frac{|\lambda_1(V)|^\gamma}{\int_{\mathbb{R}^d} |V_-|^{\gamma+\frac{d}{2}}\,dx}$$

$$C_{\rm GN}(\gamma) = \inf_{u \in H^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2\gamma+d}} \|u\|_{L^2(\mathbb{R}^d)}^{\frac{2\gamma}{2\gamma+d}}}{\|u\|_{L^{2q}(\mathbb{R}^d)}}$$

Theorem

Let $d \in \mathbb{N}^*$. For any $\gamma > \max(0, 1-\frac{d}{2})$

$$C_1 = \kappa_1(\gamma)\left[C_{\rm GN}(\gamma)\right]^{-\kappa_2(\gamma)}, \quad \kappa_1(\gamma) = \frac{2\gamma}{d}\left(\frac{d}{2\gamma+d}\right)^{1+\frac{d}{2\gamma}}, \quad \kappa_2(\gamma) = 2 + \frac{d}{\gamma}$$

How to relate the two inequalities ?

The estimate on the first eigenvalue amounts to the inequality

$$|\lambda_1(V)|^\gamma \leq C_1 \int_{\mathbb{R}^d} |V_-|^{\gamma+\frac{d}{2}}\,dx$$

while the second inequality is a Gagliardo-Nirenberg inequality

$$\|u\|_{L^{2q}(\mathbb{R}^d)} \leq \frac{1}{C_{\rm GN}(\gamma)}\,\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2\gamma+d}}\|u\|_{L^2(\mathbb{R}^d)}^{\frac{2\gamma}{2\gamma+d}}$$

To relate the two inequalities, consider the free energy

$$u \mapsto \int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} V\,|u|^2\,dx - C_1 \int_{\mathbb{R}^d} |V_-|^{\gamma+\frac{d}{2}}\,dx$$

and optimize either on u such that $\|u\|^2_{L^2(\mathbb{R}^d)} = 1$ or on V

Lieb-Thirring inequalities

Given a smooth bounded nonpositive potential V on $\mathbb{R}^d$, if

$$\lambda_1(V) < \lambda_2(V) \leq \lambda_3(V) \leq \ldots \lambda_N(V) < 0$$

is the finite sequence of all negative eigenvalues of

$$H = -\Delta + V$$

then we have the Lieb-Thirring inequality

$$\sum_{i=1}^N |\lambda_i(V)|^\gamma \leq C_{\mathrm{LT}}(\gamma) \int_{\mathbb{R}^3} |V|^{\gamma+\frac{d}{2}}\, dx \tag{3.1}$$

For $\gamma = 1$, $\sum_{i=1}^N |\lambda_i(V)|$ is the *complete ionization energy* [...], [Laptev-Weidl] for $\gamma \geq 3/2$ the sharp constant is semiclassical

Lieb-Thirring conjecture: $d = 1$, $1/2 < \gamma < 3/2$, $C_{\mathrm{LT}}(\gamma) = C_1$

Interpolation inequalities for systems

Let $m>1$ and consider $\beta(\nu):=c_m\,\nu^m$, $c_m:=\frac{(m-1)^{m-1}}{m^m}$, $m=\frac{\gamma}{\gamma-1}$

$$q:=\frac{2\gamma+d}{2\gamma+d-2}\quad\text{and}\quad\mathcal{K}^{-1}:=q\,[C_{\mathrm{LT}}(\gamma)\,(\gamma+d/2)]^{q-1}$$

Corollary

For any $m\in(1,+\infty)$,

$$K[\boldsymbol{\nu},\boldsymbol{\psi}]+c_m\sum_{i\in\mathbb{N}^*}\nu_i^m\geq\mathcal{K}\int_{\mathbb{R}^3}\rho^q\,dx$$

$$\Big(K[\boldsymbol{\nu},\boldsymbol{\psi}]\Big)^{\theta}\Big(\sum_{i\in\mathbb{N}^*}\nu_i^m\Big)^{(1-\theta)}\geq\mathcal{L}\int_{\mathbb{R}^3}\rho^q\,dx\,,\quad\theta=\frac{d}{2(\gamma-1)+d}$$

A second type of Lieb-Thirring inequality

Let V be a nonnegative unbounded smooth potential on $\mathbb{R}^d$: the eigenvalues of H_V are

$$0 < \lambda_1(V) < \lambda_2(V) \le \lambda_3(V) \le \dots \lambda_N(V) \dots$$

Theorem

For any $\gamma > d/2$, for any nonnegative $V \in C^\infty(\mathbb{R}^d)$ such that $V^{d/2-\gamma} \in L^1(\mathbb{R}^d)$,

$$\sum_{i=1}^{N} \lambda_i(V)^{-\gamma} \le \mathcal{C}(\gamma) \int_{\mathbb{R}^3} V^{\frac{d}{2}-\gamma}\, dx$$

$$\mathcal{C}(\gamma) = (2\pi)^{-d/2}\, \frac{\Gamma(\gamma - d/2)}{\Gamma(\gamma)}$$

Let f be a nonnegative function on $\mathbb{R}_+$ such that

$$\int_0^\infty f(t)\,\left(1+t^{-d/2}\right)\,\frac{dt}{t}<\infty$$

$$F(s):=\int_0^\infty e^{-t\,s}\,f(t)\,\frac{dt}{t}\quad\text{and}\quad G(s):=\int_0^\infty e^{-t\,s}\,(4\pi\,t)^{-d/2}\,f(t)\,\frac{dt}{t}$$

Theorem

Let V be in $L^1_{\rm loc}(\mathbb{R}^d)$ and bounded from below. If $G(V)\in L^1(\mathbb{R}^d)$, then

$$\sum_{i\in\mathbb{N}^*}F(\lambda_i(V))=\mathrm{tr}\,[F\,(-\Delta+V)]\le\int_{\mathbb{R}^d}G(V(x))\,dx$$

If $F(s)=s^{-\gamma}$, then $G(s)=\frac{\Gamma(\gamma-\frac d2)}{(4\pi)^{\frac d2}\,\Gamma(\gamma)}\,s^{\frac d2-\gamma}$

The exponential case

If $F(s)=e^{-s}$, then $f(s)=\delta(s-1)$ and $G(s)=(4\pi)^{-d/2}\,e^{-s}$

Corollary

$$\sum_{i\in\mathbb{N}^*} e^{-\lambda_i(V)} \le \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-V(x)}\,dx$$

Interpolation: Gagliardo-Nirenberg inequalities for systems

$$\sum_{i\in\mathbb{N}^*}\beta(\nu_i)+\sum_{i\in\mathbb{N}^*}\nu_i\int_{\mathbb{R}^3}\left(|\nabla\psi_i|^2+V\,|\psi_i|^2\right)\,dx+\int_{\mathbb{R}^3}G(V(x))\,dx\geq 0$$

for any sequence of nonnegative occupation numbers $(\nu_i)_{i\in\mathbb{N}^*}$ and any sequence $(\psi_i)_{i\in\mathbb{N}^*}$ of orthonormal $L^2(\mathbb{R}^d)$ functions

Method: For fixed $\boldsymbol{\nu}=(\nu_i)_{i\in\mathbb{N}^*}$, $\boldsymbol{\psi}=(\psi_i)_{i\in\mathbb{N}^*}$

$$K[\boldsymbol{\nu},\boldsymbol{\psi}]:=\int_{\mathbb{R}^d}\sum_{i\in\mathbb{N}^*}\nu_i\,|\nabla\psi_i|^2\,dx\quad\text{and}\quad\rho:=\sum_{i\in\mathbb{N}^*}\nu_i\,|\psi_i|^2$$

$$H(s):=-\left[G\circ(G')^{-1}(-s)+s\,(G')^{-1}(-s)\right]$$

Assume that G' is invertible and optimize on V: the optimal potential V has to satisfy

$$G'(V)+\rho=0$$

$$\int_{\mathbb{R}^3}V\,\rho\,dx+\int_{\mathbb{R}^3}G(V(x))\,dx=-\int_{\mathbb{R}^3}H(\rho(x))\,dx$$

Theorem

$$K[\boldsymbol{\nu},\boldsymbol{\psi}]+\sum_{i\in\mathbb{N}^*}\beta(\nu_i)\geq\int_{\mathbb{R}^3}H(\rho)\,dx$$

with $\rho=\sum_{i\in\mathbb{N}^*}\nu_i\,|\psi_i|^2$

Here $(\nu_i)_{i\in\mathbb{N}^*}$ *is any nonnegative sequence of occupation numbers and* $(\psi_i)_{i\in\mathbb{N}^*}$ *is any sequence of orthonormal* $L^2(\mathbb{R}^d)$ *functions*

If $\beta(\nu):=\nu\log\nu-\nu$, then $\beta'(\nu)=\log\nu=-\lambda$, $F(s)=e^{-s}$, $G(s)=(4\pi)^{-d/2}\,e^{-s}$ and $H(\rho)=\rho\,\log\rho$, and we get a logarithmic Sobolev inequality for systems

Corollary

$$K[\boldsymbol{\nu},\boldsymbol{\psi}]+\sum_{i\in\mathbb{N}^*}\nu_i\log\nu_i\geq\int_{\mathbb{R}^3}\rho\log\rho\,dx+\frac{d}{2}\,\log(4\pi)\int_{\mathbb{R}^3}\rho\,dx$$

Stability for the linear Schrödinger equation

$E[\psi] := \int_{\mathbb{R}^d} (|\nabla\psi|^2 + V\,|\psi|^2)\,dx$, eigenvalues of $H := -\Delta + V$

$$\lambda_i(V) := \inf_{\substack{F \subset L^2(\mathbb{R}^d)\\ \dim(F) = i}} \sup_{\psi \in F} E[\psi]$$

The eigenfunction $\bar\psi_i$ form an orthonormal sequence:

$$(\bar\psi_i, \bar\psi_j)_{L^2(\mathbb{R}^d)} = \delta_{ij} \quad \forall\, i, j \in \mathbb{N}^*$$

Free energy of the *mixed state* $(\boldsymbol{\nu}, \boldsymbol{\psi}) = ((\nu_i)_{i\in\mathbb{N}^*}, (\psi_i)_{i\in\mathbb{N}^*})$:

$$\mathcal{F}[\boldsymbol{\nu}, \boldsymbol{\psi}] := \sum_{i\in\mathbb{N}^*} \beta(\nu_i) + \sum_{i\in\mathbb{N}^*} \nu_i\, E[\psi_i]$$

Assumption (H) holds if β is a strictly convex function, $\beta(0) = 0$,

$$\Big|\sum_{i\in\mathbb{N}^*} \beta(\bar\nu_i)\Big| < \infty \quad \text{and} \quad \Big|\sum_{i\in\mathbb{N}^*} \bar\nu_i\,\lambda_i(V)\Big| < \infty$$

where $\bar\nu_i := (\beta')^{-1}(-\lambda_i(V))$ for any $i \in \mathbb{N}^*$

A discrete Csiszár-Kullback inequality

Lemma

Under Assumption (H), *if* $\boldsymbol{\psi} = (\psi_i)_{i\in\mathbb{N}^*}$ *is an orthonormal sequence,*

$$\mathcal{F}_n[\boldsymbol{\nu},\boldsymbol{\psi}] - \mathcal{F}_n[\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\psi}}]$$

$$= \sum_{i=1}^n \Big(\beta(\nu_i) - \beta(\bar\nu_i) - \beta'(\bar\nu_i)(\nu_i - \bar\nu_i)\Big) + \sum_{i=1}^n \nu_i \left(E[\psi_i] - E[\bar\psi_i]\right)$$

Corollary

Assume that $\inf_{s>0} \beta''(s)\, s^{2-p} =: \alpha > 0$, $p \in [1,2]$. *If* $\sum_{i\in\mathbb{N}^*} \beta(\nu_i)$ *and* $\sum_{i\in\mathbb{N}^*} \nu_i\, \beta'(\bar\nu_i)$ *are absolutely convergent, then* $(\nu_i - \bar\nu_i)_{i\in\mathbb{N}^*} \in \ell^p$ *and*

$$\sum_{i\in\mathbb{N}^*} \Big(\beta(\nu_i) - \beta(\bar\nu_i) - \beta'(\bar\nu_i)(\nu_i - \bar\nu_i)\Big) \geq \frac{\alpha\, \|\boldsymbol{\nu} - \bar{\boldsymbol{\nu}}\|_{\ell^p}^2}{2^{2/p}} \cdot \min\left\{\|\boldsymbol{\nu}\|_{\ell^p}^{p-2}, \|\bar{\boldsymbol{\nu}}\|_{\ell^p}^{p-2}\right\}$$

Stability for the Hartree-Fock system with temperature

Orbitally stable states in generalized Hartree-Fock theory: the repulsive case

- J. Dolbeault, P. Felmer, and M. Lewin.
Orbitally stable states in generalized Hartree-Fock theory.
Mathematical Models and Methods in Applied Sciences, 19 (3): 347–367, 2009.

- P.A. Markowich, G. Rein, G. Wolansky.
Existence and Nonlinear Stability of Stationary States of the Schrödinger-Poisson System. Journal of Statistical Physics, Vol. 106, Nos. 5/6, March 2002

- We consider *free energy* functionals of the form

$$\gamma \mapsto \mathcal{E}^{\mathrm{HF}}(\gamma) - T\,S(\gamma)$$

where $\mathcal{E}^{\mathrm{HF}}$ is the Hartree-Fock energy and the entropy takes the form

$$S(\gamma) := -\ \mathrm{tr}\left(\beta(\gamma)\right)$$

for some convex function β on $[0,1]$. Has the free energy a minimizer ? The Hartree-Fock energy $\mathcal{E}^{\mathrm{HF}}(\gamma)$ is

$$\mathrm{tr}\left(\left(-\Delta\right)\gamma\right) - Z\int_{\mathbb{R}^3}\frac{\rho_\gamma(x)}{|x|}\,dx + \frac{1}{2}\,D(\rho_\gamma,\rho_\gamma) - \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{|\gamma(x,y)|^2}{|x-y|}\,dx\,dy$$

with $D(f,g) := \iint_{\mathbb{R}^3\times\mathbb{R}^3}\frac{f(x)\,g(y)}{|x-y|}\,dx\,dy$ (*direct term* of the interaction). The last term is the *exchange term*

- Evolution is described by the *von Neumann equation*

$$i\,\frac{d\gamma}{dt} = [H_\gamma,\gamma]$$

Here H_γ is a self-adjoint operator depending on γ but not on β. Orbital stability of the solution obtained by minimization ?

Assumptions on the entropy term

(A1) β is a strictly convex C^1 function on $(0,1)$

(A2) $\beta(0)=0$ and $\beta\geq 0$ on $[0,1]$

Fermions !... we introduce a modified Legendre transform of β

$$g(\lambda) := \operatorname*{argmin}_{0\leq\nu\leq 1}\left(\lambda\,\nu+\beta(\nu)\right)$$

$$g(\lambda)=\sup\left\{\inf\left\{(\beta')^{-1}(-\lambda),\,1\right\},\;0\right\}$$

Notice that g is a nonincreasing function with $0\leq g\leq 1$. Also define

$$\beta^*(\lambda) := \lambda\,g(\lambda)+(\beta\circ g)(\lambda)$$

(A3) β is a nonnegative C^1 function on $[0,1)$ and $\beta'(0)=0$ (no loss of generality)

(A4) $\displaystyle\sum_{j\geq 1} j^2\left|\beta^*\left(-Z^2/(4\,T\,j^2)\right)\right|<\infty$

The ground state free energy is finite (the eigenvalues of $-\Delta-Z/|x|$ are $-Z^2/(4\,j^2)$, with multiplicity j^2)

Minimization of the free energy

Theorem (Minimization for the generalized HF model)

Assume that β *satisfies* **(A1)–(A4)** *for some* $T > 0$.

1. *For every* $q \geq 0$ *, the following statements are equivalent:*
   (i) *all minimizing sequences* $(\gamma_n)_{n\in\mathbb{N}}$ *for* $I_Z^\beta(q)$ *are precompact in* $\mathcal{K}$
   (ii) $I_Z^\beta(q) < I_Z^\beta(q')$ *for all* q *,* q' *such that* $0 \leq q' < q$
2. *Any minimizer* γ *of* $I_Z^\beta(q)$ *satisfies the self-consistent equation*

$$\gamma = g\big((H_\gamma - \mu)/T\big)\,, \quad H_\gamma = -\Delta - \frac{Z}{|x|} + \rho_\gamma * |\cdot|^{-1} - \frac{\gamma(x,y)}{|x-y|}$$

   for some $\mu \leq 0$
3. *The minimization problem* $I_Z^\beta(q)$ *has no minimizer if* $q \geq 2\,Z + 1$
4. *Problem* I_Z^β *always has a minimizer* $\bar\gamma$ *. It satisfies the self-consistent equation*

$$\bar\gamma = g(H_{\bar\gamma}/T)$$

The Hartree model of boson stars with temperature

Non-zero temperature solutions of the gravitational Hartree system

[Gonca L. Aki, Jean Dolbeault and Christof Sparber
Thermal effects in gravitational Hartree systems
To appear in Annales Henri Poincaré]

The free energy is well-defined and bounded from below

- Potential energy term: By the Hardy-Littlewood-Sobolev inequality and Sobolev's embedding

$$\mathcal{E}_{\rm pot}[\rho] = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{n_\rho(x)n_\rho(y)}{|x-y|}\,\mathrm{d}x\,\mathrm{d}y \le C\,\|n_\rho\|_{L^1}^{3/2}\,\mathrm{tr}(-\Delta\rho)^{1/2}$$

This yields

$$\mathcal{E}_H[\rho] \ge \mathrm{tr}(-\Delta\rho) - CM^{3/2}\;\mathrm{tr}(-\Delta\rho)^{1/2} \ge -\frac{1}{4}\,C^2M^3$$

- Entropy term: $\mathcal{S}[\rho] = -\,\mathrm{tr}\,\beta(\rho)$

β is convex and $\beta(0)=0 \Longrightarrow 0 \le \beta(\rho) \le \beta(M)\rho$ for all $\rho\in\mathfrak{H}$
Then $\beta(\rho)\in\mathfrak{S}_1$, provided $\rho\in\mathfrak{S}_1$

- The free energy $\mathcal{F}_T[\rho] := \mathcal{F}_T[\rho] = \mathrm{tr}(-\Delta\rho) - \frac{1}{2}\,\mathrm{tr}(V_\rho\rho) + T\,\mathrm{tr}\,\beta(\rho)$ is well defined

Sub-additivity of $i_{M,T}$ w.r.t. M

(*i*) *As a function of* M, $i_{M,T}$ *is sub-additive. In addition, for any* $M>0$, $m\in(0,M)$ *and* $T>0$, *we have*

$$i_{M,T}\le i_{M-m,T}+i_{m,T}$$

Consider two states as "almost minimizers" $\rho\in\mathfrak{H}_{M-m}$ and $\sigma\in\mathfrak{H}_m$

$$\rho=\sum_{j=1}^{J}\lambda_j|\varphi_j\rangle\langle\varphi_j|,\quad \sigma=\sum_{j=1}^{J}\bar\lambda_j|\varphi_j\rangle\langle\varphi_j|$$

with smooth eigenfunctions $(\varphi_j)_{j=1}^J$ having compact support in a ball $B(0,R)$. Next translate one of the states so that they have disjoint supports, observe that $\rho+\sigma\in\mathfrak{H}_M$

(*ii*) *The function* $i_{M,T}$ *is a decreasing function of* M *and an increasing function of* T

Maximal temperature T^*

Let $T^*(M) := \sup\{T > 0 \;:\; i_{M,T} < 0\}$.

(*iii*) *For any* $M > 0$, $T^*(M) > 0$ *is positive, maybe even infinite. As a function of* M *it is increasing and satisfies*

$$T^*(M) \geq \max_{0 \leq m \leq M} \frac{m^3}{\beta(m)} \; |i_{1,0}|$$

If $T < T^*$, *we have* $i_{M,T} < 0$

- $i_{M,0} = M^3 \; i_{1,0}$ by the homogeneity of zero-temp. minimal energy
- By sub-additivity $i_{M,T} \leq n \; i_{M/n,T} \leq n\beta\left(\frac{M}{n}\right) T - \frac{M^3}{n^2} \; |i_{1,0}|$

(*iv*) *As a consequence,* $T^*(M) = +\infty$ *for any* $M > 0$, *if*

$$\lim_{s \to 0_+} \frac{\beta(s)}{s^3} = 0$$

Euler-Lagrange equations and Lagrange multiplier

Theorem (Euler-Lagrange equations)

Let $M>0$, $T\in(0,T^*(M)]$ and assume that ($\beta1$)–($\beta2$) hold. Consider a density matrix operator $\rho\in\mathfrak{H}_M$ which minimizes $\mathcal{F}_T$. Then ρ satisfies the self-consistent equation

$$\rho=(\beta')^{-1}\left(\tfrac{1}{T}\,(\mu-H_\rho)\right)$$

where $\mu\le 0$ denotes the Lagrange multiplier associated to the mass constraint. Explicitly, it is given by

$$\mu=\frac{1}{M}\left(\operatorname{tr}\left(H_\rho+T\,\beta'(\rho)\right)\rho\right)$$

Existence of minimizers below T^*

Theorem (Existence of minimizers)

Assume that (β1)–(β3) hold. Let $M > 0$ and consider $T^* = T^*(M)$ as the maximal temperature. For all $T < T^*$, there exists an operator

$$\rho \quad \text{in} \quad \mathfrak{H}_M \qquad \text{such that} \quad \mathcal{F}_T[\rho] = i_{M,T}$$

Moreover, every minimizing sequence $(\rho_n)_{n\in\mathbb{N}}$ for $i_{M,T}$ is relatively compact in $\mathfrak{H}$ up to translations

The proof relies on the concentration-compactness method once it is known that $i_{M,T} < 0$:

- Vanishing: can be ruled out by the fact that $n_\rho \in L^{7/5}$
- Dichotomy: splitting behaviour: $i_{M,T} = i_{M^{(1)},T} + i_{M-M^{(1)},T}$ contradicts the binding inequality $i_{M^{(1)}+M^{(2)},T} < i_{M^{(1)},T} + i_{M^{(2)},T}$
- Compactness

Orbital stability

A direct consequence of this variational approach is orbital stability: Consider the set of minimizers $\mathfrak{M}_M \subset \mathfrak{H}_M$ and denote

$$\mathrm{dist}_{\mathfrak{M}_M}(\rho(t), \rho) = \inf_{\rho \in \mathfrak{M}_M} \|\rho(t) - \rho\|$$

Here $\rho(t)$ solves the corresponding time-dependent system

$$i \frac{\mathrm{d}}{\mathrm{d}t} \rho(t) = [H_{\rho(t)}, \rho(t)] \,, \quad \rho(0) = \rho_{\mathrm{in}}$$

where $H_\rho := -\Delta - n_\rho * \frac{1}{|\cdot|}$

Corollary (Orbital stability)

For given $M > 0$ let $T < T^*(M)$. Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $\rho_{\mathrm{in}} \in \mathfrak{H}_M$ and $\rho \in \mathfrak{M}_M$ with $\mathrm{dist}_{\mathfrak{M}_M}(\rho_{\mathrm{in}}, \rho) \leq \delta$ it holds:

$$\sup_{t \in \mathbb{R}_+} \mathrm{dist}_{\mathfrak{M}_M}(\rho(t), \rho) \leq \varepsilon$$

Pure states, mixed states and critical temperature

Let $\rho_0 = M|\psi_0\rangle\langle\psi_0|$ be the (appropriately scaled) minimizer for $T = 0$. Then the corresponding Hamiltonian operator

$$H_0 := -\Delta - |\psi_0|^2 * \frac{1}{|\cdot|}$$

admits countably many (negative) eigenvalues [1]

$$(\mu_j^0)_{j\in\mathbb{N}} \qquad \text{with} \quad \mu_j^0 \nearrow 0$$

Claim:

A critical temperature $T_c \in (0, T^*)$ exists, and depends on the entropy function β such that, for $T < T_c$ minimizers $\rho \in \mathfrak{M}_M$ are only pure states

Positivity of the critical temperature for all $M > 0$

$$T_c(M) := \max\{T > 0 : \; i_{M,T} = i_{M,0} + \tau\beta(M) \;\; \forall\tau \in (0,T]\}$$

Assume that (β1)–(β3) hold. Then $T_c(M)$ is positive for any $M > 0$

To see this, take $T_n \to 0$ and consider a sequence of minimizers $\rho^{(n)}$
Since $\rho^{(n)}$ is also a minimizing sequence for $\mathcal{F}_{T=0}$, we know

$$\mu_j^{(n)} \xrightarrow{n\to\infty} \mu_j^0 \le 0$$

We assume by contradiction that $\liminf_{n\to\infty} \lambda_1^{(n)} = \epsilon > 0$
Then the Euler-Lagrange equation implies $\mu^{(n)} > \mu_1^{(n)}$, yields a contradiction:

$$M = \lambda_0^{(0)} \ge \lim_{n\to\infty} \lambda_0^{(n)} \ge \lim_{n\to\infty} (\beta')^{-1}\left((\mu_1^0 - \mu_0^{(n)})/T_n\right) = +\infty$$

Hence $\exists\,[0, T_c]$ with $T_c > 0$ s.t. $\mu^{(n)} < \mu_1^{(n)}$ for any $T_n \in [0, T_c]$. Thus $\rho^{(n)}$ is of rank one

Characterization of the critical temperature T_c

Corollary

Assume that (β1)–(β3) hold. There is a pure state minimizer of mass M if and only if $T \in [0, T_c]$

For any $M > 0$ the critical temperature satisfies

$$T_c = \frac{\mu_1^0 - \mu_0^0}{\beta'(M)}$$

where $\mu_0^0 < \mu_1^0$ are the two lowest eigenvalues of H_0

Step 1: Prove $T_c \leq (\mu_1^0 - \mu_0^0)/\beta'(M)$ by using $\mu(T) = \mu_0^0 + T\beta'(M)$ for pure states ($T \leq T_c$)
Step 2: ($T > T_c$) Prove the equality (approaching to T_c from above) by using

$$M\mu^{(n)} = \sum_{j\in\mathbb{N}} \lambda_j^{(n)} \left(\mu_j^{(n)} + T^{(n)}\beta'(\lambda_j^{(n)})\right)$$

Remarks on the maximal temperature

- A case in which $T^* = \infty$:

$T^*(M) = +\infty$ for any $M > 0$, if

$$\lim_{s \to 0_+} \frac{\beta(s)}{s^3} = 0$$

- A case in which T^* is finite:

If $p \in (1, 7/5)$ given in the entropy generating function $\beta(s) = s^p$, then the maximal temperature, $T^*(M)$ is finite

- Limit case:

Assume $T^* < +\infty$. Then, $\lim_{T \to T^*_-} i_{M,T} = 0$ and $\lim_{T \to T^*_-} \mu(T) = 0$

- Open: If $p \in (7/5, 3)$ then T^* is finite ?

Thank you for your attention !