Inégalités fonctionnelles optimales, diffusions non linéaires et brisure de symétrie

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Nancy

Outline

- An introduction to symmetry and symmetry breaking results in weighted elliptic PDEs
- Caffarelli-Kohn-Nirenberg inequalities
  - ▷ The symmetry issue
  - ▷ The result
- The proof
  - ▷ a change of variables and a Sobolev type inequality
  - ▷ the fast diffusion flow and the nonlinear Fisher information
  - ▷ regularity, decay and integrations by parts
- Concavity of the Rényi entropy powers: role of the nonlinear flow
- The Bakry-Emery method: curvature, linear and nonlinear flows
- Conclusion

In collaboration with M.J. Esteban and M. Loss

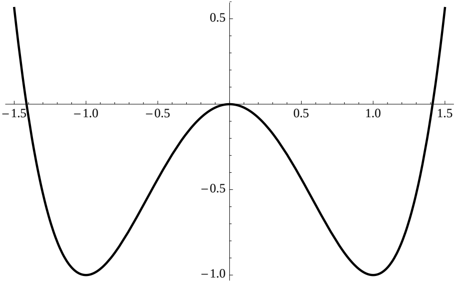
An introduction to symmetry and symmetry breaking results in weighted elliptic PDEs

▷ *The typical issue is the competition between a potential or a weight and a nonlinearity*

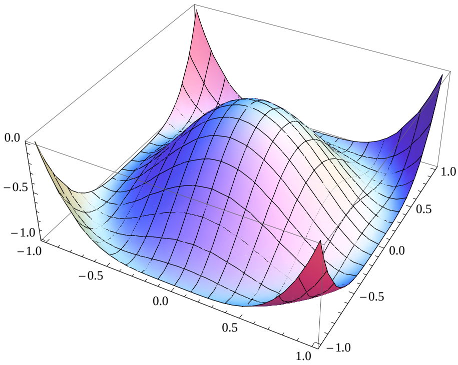
The *mexican hat* potential

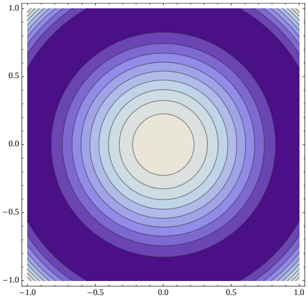
Let us consider a nonlinear Schrödinger equation in presence of a radial external potential with a minimum which is not at the origin

$$-\Delta u + V(x)\,u - f(u) = 0$$



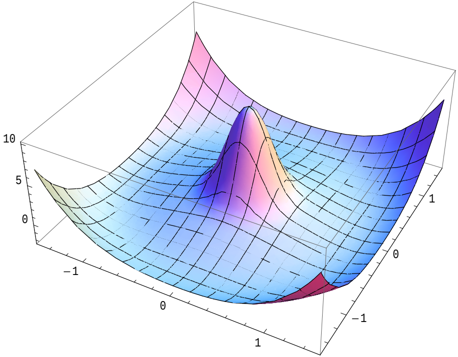
A one-dimensional potential $V(x)$

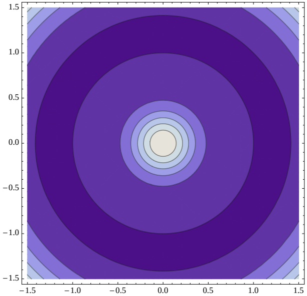




A two-dimensional potential $V(x)$ *with mexican hat shape*

Radial solutions to $-\Delta u + V(x)\, u - F'(u) = 0$

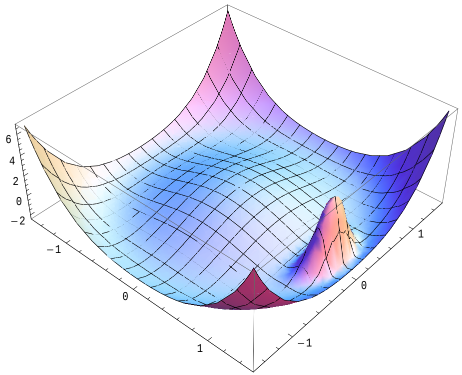


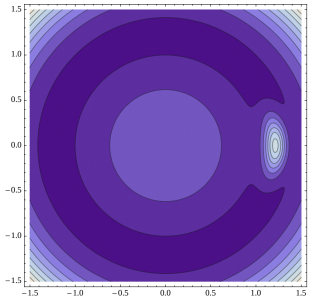


... *give rise to a radial density of energy* $x \mapsto V\,|u|^2 + F(u)$

symmetry breaking

... but in some cases minimal energy solutions





... *give rise to a non-radial density of energy* $x \mapsto V\,|u|^2 + F(u)$

Symmetry and symmetry breaking

Proving symmetry breaking

The most classical method is by perturbation of a radial solution and energy descent
... but there are other methods, like direct energy estimates

Methods for proving symmetry

Classical methods (a non exhaustive list)

- Alexandrov moving planes and the result of [B. Gidas, W. Ni, L. Nirenberg (1979, 1980)]

$$-\Delta u = f(|x|, u) \quad \text{in} \quad \mathbb{R}^d \; d \geq 3$$

If f *is of class* C^1, $\frac{\partial f}{\partial r} < 0$, $u \geq 0$ *is of class* C^2 *and sufficiently decaying at infinity, then* u *is a radial function and* $\frac{\partial u}{\partial r} < 0$.

- Reflexion with respect planes and unique continuation [O. Lopes]
- Symmetrization methods: Schwarz, Steiner, *etc.*
- A priori estimates, direct energy estimates
- Uniqueness or rigidity: [B. Gidas, J. Spruck], [M.-F. Bidault-Véron, L. Véron, 1991]
- ... probabilistic methods and *carré du champ* methods [D. Bakry, M. Emery, 1984]

▷ *A new method based on entropy functionals and evolution under the action of a nonlinear flow: flow interpretation, non-compact case*

Caffarelli-Kohn-Nirenberg inequalities

▷ *Nonlinear flows (fast diffusion equation) can be used as a tool for the investigation of sharp functional inequalities*

Caffarelli-Kohn-Nirenberg inequalities and the symmetry breaking issue

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

▷ *With*

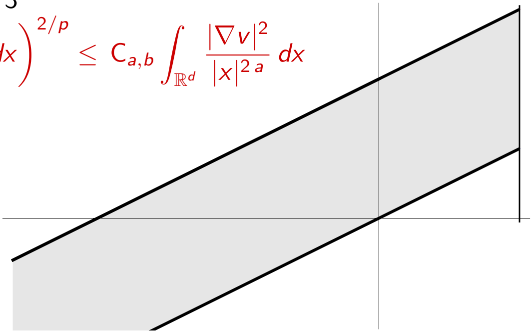
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

do we have $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry)*
or $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le C_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a\le b\le a+1$ if $d\ge 3$

$a<b\le a+1$ if $d=2$, $a+1/2<b\le a+1$ if $d=1$

and $a<a_c:=(d-2)/2$

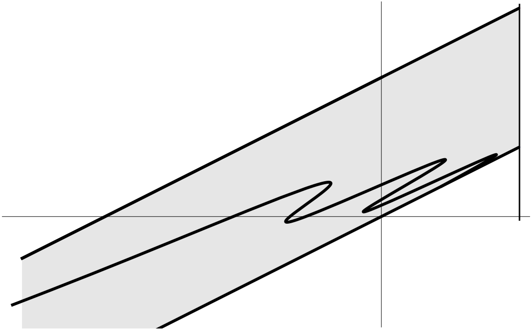
$$p=\frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]
[F. Catrina, Z.-Q. Wang (2001)]

Symmetry and symmetry breaking

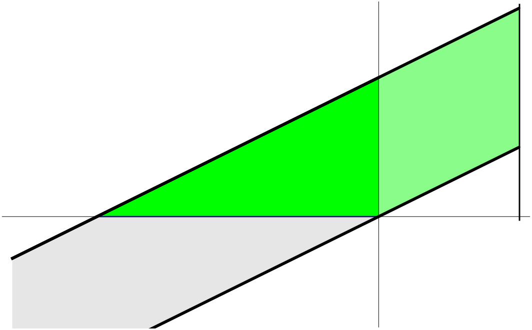
Proving symmetry breaking

[F. Catrina, Z.-Q. Wang], [V. Felli, M. Schneider (2003)]



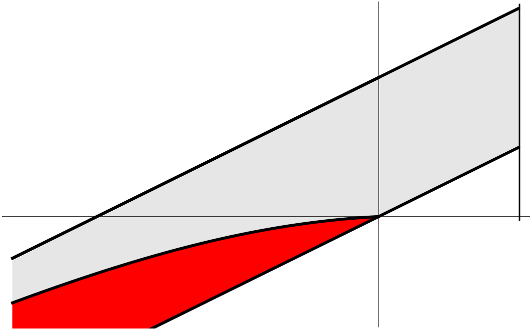
[J.D., Esteban, Loss, Tarantello, 2009] There is a curve which separates the symmetry region from the symmetry breaking region, which is parametrized by a function $p \mapsto a + b$

Moving planes and symmetrization techniques



[Chou, Chu], [Horiuchi]
[Betta, Brock, Mercaldo, Posteraro]
+ Perturbation results: [CS Lin, ZQ Wang], [Smets, Willem], [JD, Esteban, Tarantello 2007], [J.D., Esteban, Loss, Tarantello, 2009]

Linear instability of radial minimizers: the Felli-Schneider curve

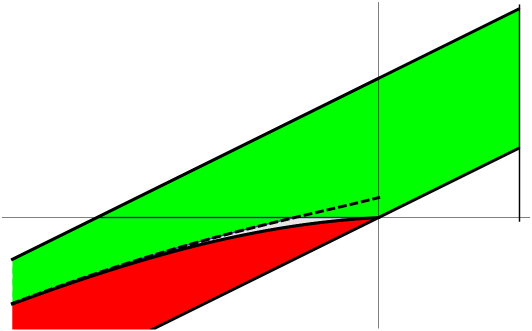


[Catrina, Wang], [Felli, Schneider] The functional

$$C^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

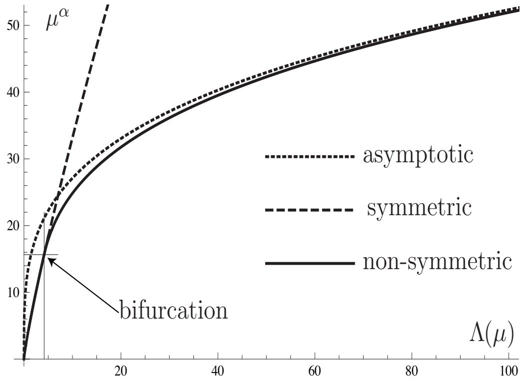
is linearly instable at $v = v_\star$

Direct spectral estimates



[J.D., Esteban, Loss, 2011]: sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

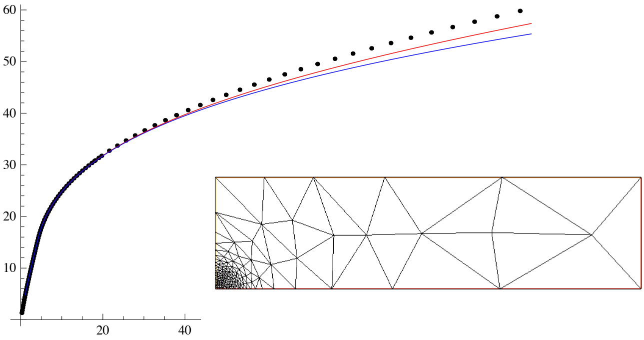
Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

Other evidences

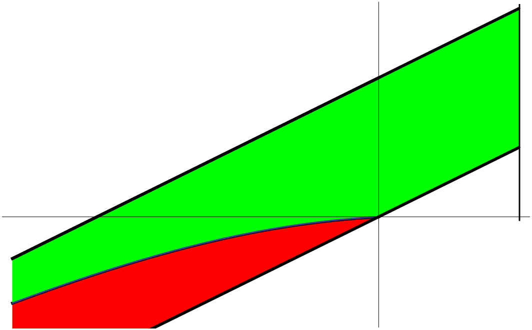
- Further numerical results [J.D., Esteban, 2012] (coarse / refined / self-adaptive grids)



- Formal commutation of the non-symmetric branch near the bifurcation point [J.D., Esteban, 2013]
- Asymptotic energy estimates [J.D., Esteban, 2013]

Symmetry *versus* symmetry breaking: the sharp result

A result based on entropies and nonlinear flows



[J.D., Esteban, Loss, 2015]: http://arxiv.org/abs/1506.03664

The symmetry result

The Felli & Schneider curve is defined by

$$b_{\rm FS}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\rm FS}(a)$, then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, Caffarelli-Kohn-Nirenberg inequalities on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r=|x|\,, \quad s=-\log r \quad \text{and} \quad \omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}\quad\forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda:=(a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a=a_c\pm\sqrt{\Lambda} \quad \text{and} \quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

Generalizations and comments

Generalized Caffarelli-Kohn-Nirenberg inequalities (CKN)

Let $2^* = \infty$ if $d = 1$ or $d = 2$, $2^* = 2d/(d-2)$ if $d \geq 3$ and define

$$\vartheta(p,d) := \frac{d\,(p-2)}{2\,p}$$

[Caffarelli-Kohn-Nirenberg-84] Let $d \geq 1$. For any $\theta \in [\vartheta(p,d), 1]$, with $p = \frac{2\,d}{d-2+2\,(b-a)}$, there exists a positive constant $\mathsf{C}_{\rm CKN}(\theta,p,a)$ such that

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}_{\rm CKN}(\theta,p,a) \left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a - a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and

$$\mathsf{C}_{\rm CKN}(\theta,p,a) \geq \mathsf{C}^*_{\rm CKN}(\theta,p,a) = \mathsf{C}^*_{\rm CKN}(\theta,p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\rm CKN}(\theta,p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}} \left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}} \left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta} \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]$$

Implementing the method of Catrina-Wang / Felli-Schneider

Among functions $w \in H^1(\mathcal{C})$ which depend only on s, the minimum of

$$\mathcal{J}[w] := \int_{\mathcal{C}} \left(|\nabla w|^2 + \tfrac{1}{4}\,(d-2-2\,a)^2\,|w|^2\right)\,dx - [\mathsf{C}^*(\theta,p,a)]^{-\frac{1}{\theta}}\,\frac{\left(\int_{\mathcal{C}} |w|^p\,dx\right)^{\frac{2}{p\,\theta}}}{\left(\int_{\mathcal{C}} |w|^2\,dx\right)^{\frac{1-\theta}{\theta}}}$$

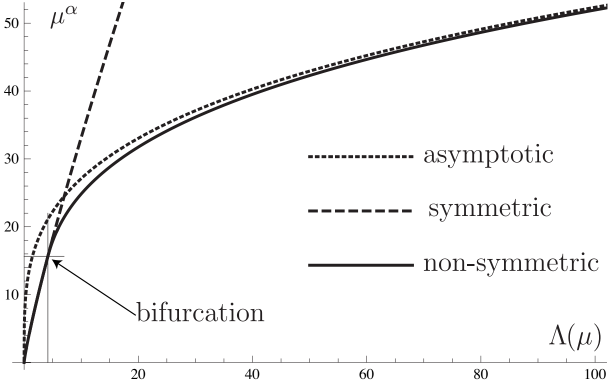
is achieved by $\overline{w}(y) := \left[\cosh(\lambda\,s)\right]^{-\frac{2}{p-2}}$, $y=(s,\omega)\in\mathbb{R}\times\mathbb{S}=\mathcal{C}$ with $\lambda := \frac{1}{4}\,(d-2-2\,a)\,(p-2)\,\sqrt{\frac{p+2}{2\,p\,\theta-(p-2)}}$ as a solution of

$$\lambda^2\,(p-2)^2\,w'' - 4\,w + 2\,p\,|w|^{p-2}\,w = 0$$

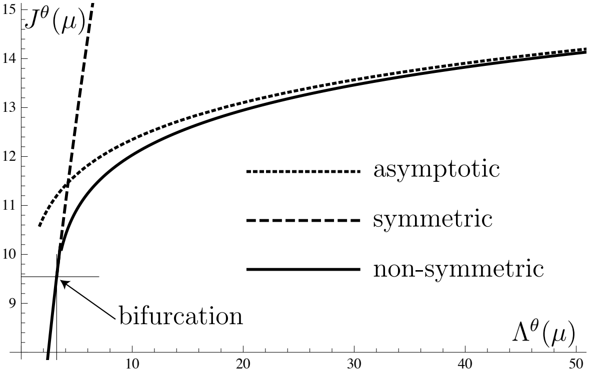
Spectrum of $\mathcal{L} := -\Delta + \kappa\,\overline{w}^{p-2} + \mu$ is given for $\sqrt{1+4\,\kappa/\lambda^2} \ge 2\,j+1$ by $\lambda_{i,j} = \mu + i\,(d+i-2) - \frac{\lambda^2}{4}\left(\sqrt{1+4\,\kappa/\lambda^2} - (1+2\,j)\right)^2 \quad \forall\, i\,,\; j \in \mathbb{N}$

- The eigenspace of $\mathcal{L}$ corresponding to $\lambda_{0,0}$ is generated by $\overline{w}$
- The eigenfunction $\phi_{(1,0)}$ associated to $\lambda_{1,0}$ is not radially symmetric and such that $\int_{\mathcal{C}} \overline{w}\,\phi_{(1,0)}\,dx = 0$ and $\int_{\mathcal{C}} \overline{w}^{p-1}\,\phi_{(1,0)}\,dx = 0$
- If $\lambda_{1,0} < 0$, *optimal functions for (CKN) cannot be radially symmetric* and $\mathsf{C}(\theta,p,a) > \mathsf{C}^*(\theta,p,a)$

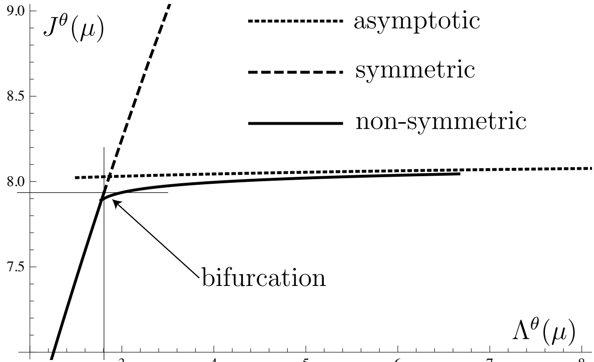
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 1$



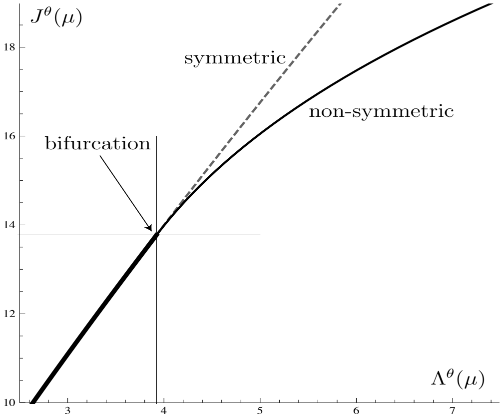
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.8$



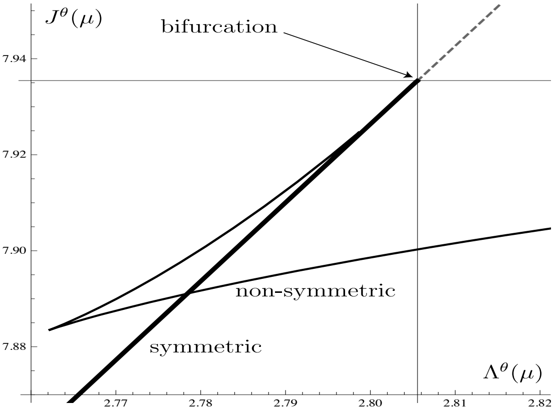
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 2.8$, $d = 5$, $\theta = 0.72$



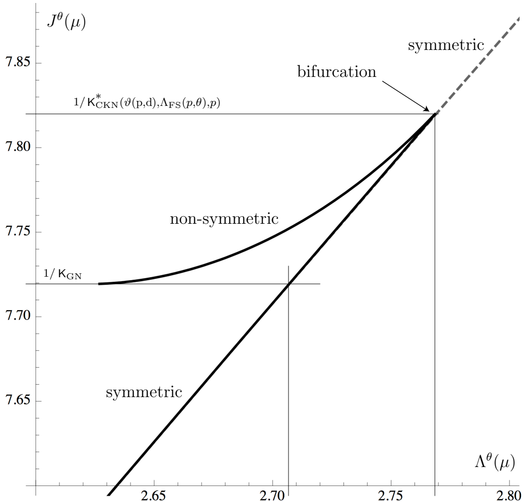
Enlargement for $p = 2.8$, $d = 5$, $\theta = 0.95$



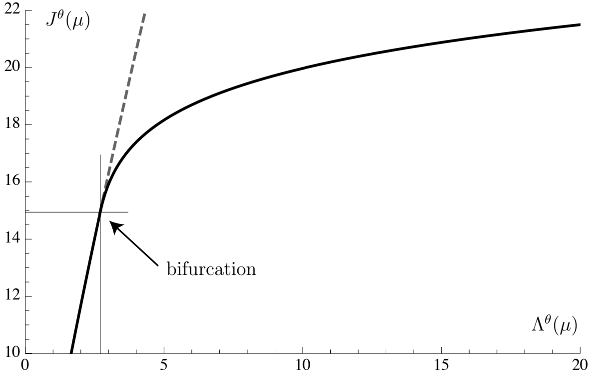
Enlargement for $p = 2.8$, $d = 5$, $\theta = 0.72$



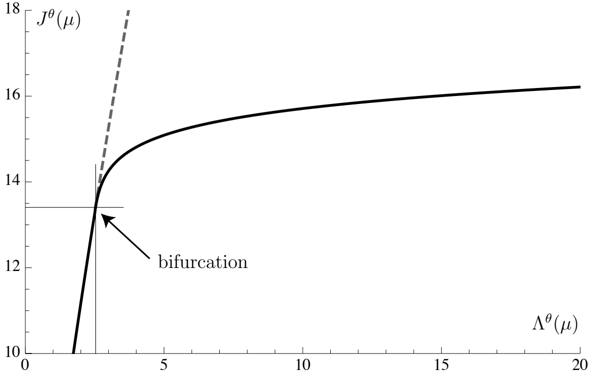
Critical case $\theta = \vartheta(p, d)$



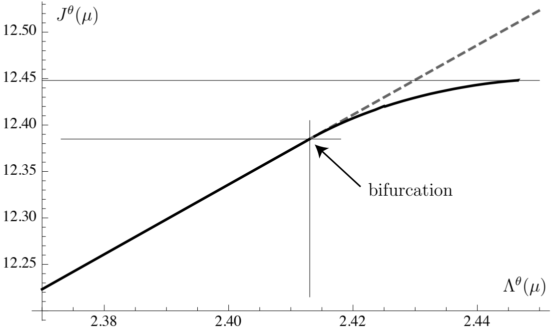
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 1$



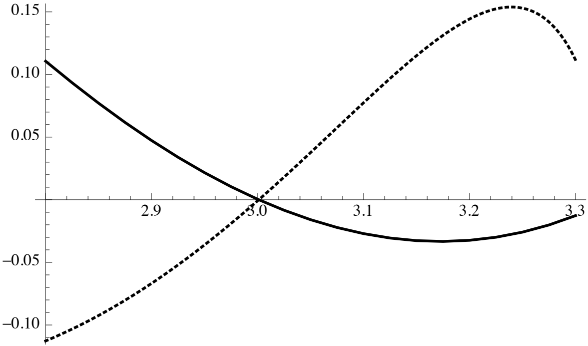
Parametric plot of $\mu \mapsto (\Lambda^\theta(\mu), J^\theta(\mu))$ for $p = 3.15$, $d = 5$, $\theta = 0.95$



Case $p = 3.15$, $d = 5$, $\theta = \vartheta(3.15, 5) \approx 0.9127$



Local and asymptotic criteria for $\theta = \vartheta(p, d)$



- *Local criterion: based on an expansion of the solutions near the bifurcation point, it decides whether the branch goes to the right to to the left.*
- *Asymptotic criterion: based on the energy of the branch as* $\Lambda \to +\infty$ *and an analysis in a semi-classical regime*

The main steps of the proof

- A change of variables: an equivalent inequality of Sobolev type
- The fast diffusion flow and the nonlinear Fisher information
- Proving the decay along the flow
- The justification of the integration by parts: decay estimates on the cylinder

A change of variables

With $(r=|x|,\,\omega=x/r)\in\mathbb{R}^+\times\mathbb{S}^{d-1}$, the Caffarelli-Kohn-Nirenberg inequality is

$$\left(\int_0^\infty\int_{\mathbb{S}^{d-1}}|v|^p\,r^{d-b\,p}\,\frac{dr}{r}\,d\omega\right)^{\frac{2}{p}}\le\mathsf{C}_{a,b}\int_0^\infty\int_{\mathbb{S}^{d-1}}|\nabla v|^2\,\,r^{d-2\,a}\,\frac{dr}{r}\,d\omega$$

Change of variables $r\mapsto r^\alpha$, $v(r,\omega)=w(r^\alpha,\omega)$

$$\alpha^{1-\frac{2}{p}}\left(\int_0^\infty\int_{\mathbb{S}^{d-1}}|w|^p\,r^{\frac{d-b\,p}{\alpha}}\,\frac{dr}{r}\,d\omega\right)^{\frac{2}{p}}$$
$$\le\mathsf{C}_{a,b}\int_0^\infty\int_{\mathbb{S}^{d-1}}\left(\alpha^2\left|\frac{\partial w}{\partial r}\right|^2+\frac{1}{r^2}\,|\nabla_\omega w|^2\right)\,r^{\frac{d-2\,a-2}{\alpha}+2}\,\frac{dr}{r}\,d\omega$$

Choice of α

$$n=\frac{d-b\,p}{\alpha}=\frac{d-2\,a-2}{\alpha}+2$$

Then $p=\frac{2\,n}{n-2}$ is the critical Sobolev exponent associated with n

A Sobolev type inequality

The parameters α and n vary in the ranges $0 < \alpha < \infty$ and $d < n < \infty$ and the *Felli-Schneider curve* in the (α, n) variables is given by

$$\alpha = \sqrt{\frac{d-1}{n-1}} =: \alpha_{\rm FS}$$

With

$$\mathsf{D}w = \left(\alpha\,\frac{\partial w}{\partial r}, \frac{1}{r}\,\nabla_\omega w\right)\,, \quad d\mu := r^{n-1}\,dr\,d\omega$$

the inequality becomes

$$\alpha^{1-\frac{2}{p}}\left(\int_{\mathbb{R}^d}|w|^p\,d\mu\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b}\int_{\mathbb{R}^d}|\mathsf{D}w|^2\,d\mu$$

Proposition

Let $d \ge 2$. Optimality is achieved by radial functions and $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ if $\alpha \le \alpha_{\rm FS}$

Gagliardo-Nirenberg inequalities on general cylinders: similar

Notations

When there is no ambiguity, we will omit the index $_\omega$ and from now on write that $\nabla = \nabla_\omega$ denotes the gradient with respect to the angular variable $\omega \in \mathbb{S}^{d-1}$ and that Δ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. We define the self-adjoint operator $\mathcal{L}$ by

$$\mathcal{L}\, w := -\,\mathsf{D}^*\,\mathsf{D}\, w = \alpha^2\, w'' + \alpha^2\, \frac{n-1}{r}\, w' + \frac{\Delta\, w}{r^2}$$

The fundamental property of $\mathcal{L}$ is the fact that

$$\int_{\mathbb{R}^d} w_1\, \mathcal{L}\, w_2\, d\mu = -\int_{\mathbb{R}^d} \mathsf{D} w_1 \cdot \mathsf{D} w_2\, d\mu \quad \forall\, w_1,\, w_2 \in \mathcal{D}(\mathbb{R}^d)$$

▷ Heuristics: we look for a monotonicity formula along a well chosen nonlinear flow, based on the analogy with the decay of the Fisher information along the fast diffusion flow in $\mathbb{R}^d$

Fisher information

Let $u^{\frac{1}{2}-\frac{1}{n}} = |w| \quad \Longleftrightarrow \quad u = |w|^p,\ p = \frac{2\,n}{n-2}$

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|\mathrm{Dp}|^2\,d\mu\,, \quad \mathsf{p} = \frac{m}{1-m}\,u^{m-1} \quad \text{and} \quad m = 1 - \frac{1}{n}$$

Here $\mathcal{I}$ is the *Fisher information* and p is the *pressure function*

Proposition

With $\Lambda = 4\,\alpha^2/(p-2)^2$ and for some explicit numerical constant κ, we have

$$\kappa\,\mu(\Lambda) = \inf\left\{\mathcal{I}[u]\ :\ \|u\|_{\mathrm{L}^1(\mathbb{R}^d,d\mu)} = 1\right\}$$

▷ Optimal solutions solutions of the elliptic PDE) are (constrained) critical point of $\mathcal{I}$

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}\, u^m\,, \quad m = 1 - \frac{1}{n}$$

Barenblatt self-similar solutions

$$u_\star(t,r,\omega) = t^{-n}\left(\mathsf{c}_\star + \frac{r^2}{2\,(n-1)\,\alpha^2\,t^2}\right)^{-n}$$

Lemma

Barenblatt solutions realize the minimum of $\mathcal{I}$ among radial functions:

$$\kappa\,\mu_\star(\Lambda) = \mathcal{I}[u_\star(t,\cdot)] \quad \forall\, t > 0$$

▷ Strategy:
1) prove that $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] \le 0$,
2) prove that $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = 0$ means that $u = u_\star$ up to a time shift

Decay of the Fisher information along the flow ?

The *pressure function* $\mathsf{p} = \frac{m}{1-m}\, u^{m-1}$ satisfies

$$\frac{\partial \mathsf{p}}{\partial t} = \frac{1}{n}\, \mathsf{p}\, \mathcal{L}\, \mathsf{p} - |\mathsf{D}\mathsf{p}|^2$$

$$\mathcal{Q}[\mathsf{p}] := \frac{1}{2}\, \mathcal{L}\, |\mathsf{D}\mathsf{p}|^2 - \mathsf{D}\mathsf{p} \cdot \mathsf{D}\mathcal{L}\, \mathsf{p}$$

$$\mathcal{K}[\mathsf{p}] := \int_{\mathbb{R}^d} \left(\mathcal{Q}[\mathsf{p}] - \frac{1}{n}\, (\mathcal{L}\, \mathsf{p})^2 \right) \mathsf{p}^{1-n}\, d\mu$$

Lemma

If u solves the weighted fast diffusion equation, then

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = -\, 2\, (n-1)^{n-1}\, \mathcal{K}[\mathsf{p}]$$

If u is a critical point, then $\mathcal{K}[\mathsf{p}] = 0$

▷ Boundary terms ! Regularity !

Proving decay (1/2)

$$\mathsf{k}[\mathsf{p}] := \mathcal{Q}(\mathsf{p}) - \frac{1}{n}\,(\mathcal{L}\,\mathsf{p})^2 = \frac{1}{2}\,\mathcal{L}\,|\mathsf{D}\mathsf{p}|^2 - \mathsf{D}\mathsf{p}\cdot\mathsf{D}\,\mathcal{L}\,\mathsf{p} - \frac{1}{n}\,(\mathcal{L}\,\mathsf{p})^2$$

$$\mathsf{k}_{\mathfrak{M}}[\mathsf{p}] := \frac{1}{2}\,\Delta\,|\nabla\mathsf{p}|^2 - \nabla\mathsf{p}\cdot\nabla\Delta\,\mathsf{p} - \tfrac{1}{n-1}\,(\Delta\,\mathsf{p})^2 - (n-2)\,\alpha^2\,|\nabla\mathsf{p}|^2$$

Lemma

Let $n \neq 1$ be any real number, $d \in \mathbb{N}$, $d \geq 2$, and consider a function $\mathsf{p} \in C^3((0,\infty)\times\mathfrak{M})$, where $(\mathfrak{M}, g)$ is a smooth, compact Riemannian manifold. Then we have

$$\mathsf{k}[\mathsf{p}] = \alpha^4\left(1-\frac{1}{n}\right)\left[\mathsf{p}'' - \frac{\mathsf{p}'}{r} - \frac{\Delta\,\mathsf{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla\mathsf{p}' - \frac{\nabla\mathsf{p}}{r}\right|^2 + \frac{1}{r^4}\,\mathsf{k}_{\mathfrak{M}}[\mathsf{p}]$$

Proving decay (2/2)

Lemma

Assume that $d \geq 3$, $n > d$ and $\mathfrak{M} = \mathbb{S}^{d-1}$. For some $\zeta_\star > 0$ we have

$$\int_{\mathbb{S}^{d-1}} \mathrm{k}_{\mathfrak{M}}[\mathrm{p}]\,\mathrm{p}^{1-n}\,d\omega \geq \left(\lambda_\star - (n-2)\,\alpha^2\right) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{p}|^2\,\mathrm{p}^{1-n}\,d\omega + \zeta_\star\,(n-d) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{p}|^4\,\mathrm{p}^{1-n}\,d\omega$$

Proof based on the Bochner-Lichnerowicz-Weitzenböck formula

Corollary

Let $d \geq 2$ and assume that $\alpha \leq \alpha_{\mathrm{FS}}$. Then for any nonnegative function $u \in \mathrm{L}^1(\mathbb{R}^d)$ with $\mathcal{I}[u] < +\infty$ and $\int_{\mathbb{R}^d} u\,d\mu = 1$, we have

$$\mathcal{I}[u] \geq \mathcal{I}_\star$$

When $\mathfrak{M} = \mathbb{S}^{d-1}$, $\lambda_\star = (n-2)\,\frac{d-1}{n-1}$

A perturbation argument

- If u is a critical point of $\mathcal{I}$ under the mass constraint $\int_{\mathbb{R}^d} u\,d\mu = 1$, then

$$o(\varepsilon) = \mathcal{I}[u + \varepsilon\,\mathcal{L}\,u^m] - \mathcal{I}[u] = -\,2\,(n-1)^{n-1}\,\varepsilon\,\mathcal{K}[\mathsf{p}] + o(\varepsilon)$$

because $\varepsilon\,\mathcal{L}\,u^m$ is an admissible perturbation (formal). Indeed, we know that

$$\int_{\mathbb{R}^d} (u + \varepsilon\,\mathcal{L}\,u^m)\,d\mu = \int_{\mathbb{R}^d} u\,d\mu = 1$$

but positivity of $u + \varepsilon\,\mathcal{L}\,u^m$ is an issue: compute

$$0 = D\mathcal{I}[u]\cdot\mathcal{L}\,u^m = -\,\mathcal{K}[\mathsf{p}]$$

- Regularity issues (uniform decay of various derivatives up to order 3) and boundary terms

- If $\alpha \le \alpha_{\rm FS}$, then $\mathcal{K}[\mathsf{p}] = 0$ implies that $u = u_\star$

The justification of the integration by parts: decay estimates on the cylinder

After then Emden-Fowler transformation, a critical point satisfies the Euler-Lagrange equation

$$-\,\partial_s^2\,\varphi - \,\Delta_\omega\,\varphi + \Lambda\,\varphi = \varphi^{p-1} \quad \text{in} \quad \mathcal{C} = \mathbb{R}\times\mathcal{M}$$

(up to a multiplication by a constant; $\mathcal{M} = \mathbb{S}^{d-1}$ e.g.)

Proposition

For all $(s,\omega)\in\mathcal{C}$, we have $C_1\,e^{-\sqrt{\Lambda}\,|s|} \le \varphi(s,\omega) \le C_2\,e^{-\sqrt{\Lambda}\,|s|}$

$$|\varphi'(s,\omega)|\,,\;|\varphi''(s,\omega)|\,,\;|\nabla\varphi(s,\omega)|\,,\;|\Delta\,\varphi(s,\omega)| \le C_2\,e^{-\sqrt{\Lambda}\,|s|}$$

$\int_{\mathfrak{M}}|\mathsf{p}'(r,\omega)|^2\,d\,v_g \le O(1)$, $\int_{\mathfrak{M}}|\nabla\mathsf{p}(r,\omega)|^2\,d\,v_g \le O(r^2)$,
$\int_{\mathfrak{M}}|\mathsf{p}''(r,\omega)|^2\,d\,v_g \le O(1/r^2)$
$\int_{\mathfrak{M}}\left|\nabla\mathsf{p}'(r,\omega) - \frac{1}{r}\,\nabla\mathsf{p}(r,\omega)\right|^2 d\,v_g \le O(1)$,
$\int_{\mathfrak{M}}|\Delta\mathsf{p}(r,\omega)|^2\,d\,v_g \le O(1/r^2)$

Some comments on the method

The flow interpretation is very useful to organize the computations. It unifies *rigidity* methods and *carré du champ* (or Γ_2) techniques in a common framework which has a clear variational interpretation and opens a door for improvements

However, the proof is done at a purely *variational* level: we consider a critical point and test it against a special test function built on top of the solution, in order to obtain another identity involving sum of squares: each of these square has to be equal to zero, which provides *additional equations* and allow to identify the critical point with a known one (uniqueness result)

In a sense, this method relates with similar estimates:

▷ *Nehari manifolds:* one tests the equation with the function u itself. At variational level, this amounts to test the homogeneity of the energy (if any).

▷ *Pohozaev's method:* one tests the equation with $x \cdot \nabla u$, which corresponds to a local dilation. At variational level, this amounts to test the energy under scalings (eventually localized) and remarkably, this is a method to produce uniqueness results [Schmitt], [Schaeffer]

▷ Here we test the equation with Δu^m, which has to do with *convexity* (displacement convexity) and produces uniqueness (rigidity) results.

Two ingredients for the proof

▷ Rényi entropy powers and fast diffusion

▷ Flows on the sphere

Rényi entropy powers and fast diffusion

▷ Rényi entropy powers, the entropy approach without rescaling: [Savaré, Toscani]: scalings, nonlinearity and a concavity property inspired by information theory

▷ faster rates of convergence: [Carrillo, Toscani], [JD, Toscani]

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(x, t = 0) = u_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} u_0\, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2\, u_0\, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t, x) := \frac{1}{\left(\kappa\, t^{1/\mu}\right)^d}\, \mathcal{B}_\star\left(\frac{x}{\kappa\, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\, m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, u^{m-1}$$

If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The concavity property

Theorem

[Toscani-Savaré] *Assume that* $m \ge 1 - \frac{1}{d}$ *if* $d > 1$ *and* $m > 0$ *if* $d = 1$. *Then* $F(t)$ *is increasing,* $(1-m)\,\mathsf{F}''(t) \le 0$ *and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \ge \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \ge \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \le m < 1$. Hint: $u^{m-1/2} = \frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}}$, $q = \frac{1}{2\,m-1}$

The proof

Lemma

If u solves $\frac{\partial u}{\partial t} = \Delta u^m$ with $\frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} u\,|\nabla \mathsf{p}|^2\,dx = -\,2 \int_{\mathbb{R}^d} u^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2 \right) dx$$

$$\|\mathrm{D}^2 \mathsf{p}\|^2 = \frac{1}{d}\,(\Delta \mathsf{p})^2 + \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2$$

$$\begin{aligned} \frac{1}{\sigma\,(1-m)}\,\mathsf{E}^{2-\sigma}\,(\mathsf{E}^\sigma)'' &= (1-m)\,(\sigma-1) \left(\int_{\mathbb{R}^d} u\,|\nabla \mathsf{p}|^2\,dx \right)^2 \\ &\quad - 2 \left(\frac{1}{d} + m - 1 \right) \int_{\mathbb{R}^d} u^m\,dx \int_{\mathbb{R}^d} u^m\,(\Delta \mathsf{p})^2\,dx \\ &\quad - 2 \int_{\mathbb{R}^d} u^m\,dx \int_{\mathbb{R}^d} u^m \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2 dx \end{aligned}$$

Flows on the sphere

▷ The heat flow introduced by D. Bakry and M. Emery (*carré du champ* method) does not cover all exponents up to the critical one

[Bakry, Emery, 1984]
[Bidault-Véron, Véron, 1991], [Bakry, Ledoux, 1996]
[Demange, 2008][JD, Esteban, Loss, 2014 & 2015]

The interpolation inequalities

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \neq 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2}\int_{\mathbb{S}^d} |u|^2\,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\,v_g - \left(\int_{\mathbb{S}^d} \rho\,d\,v_g\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\,v_g$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\,v_g$$

Bakry-Emery (carré du champ): use the heat flow $\frac{\partial \rho}{\partial t} = \Delta \rho$ where Δ denotes the Laplace-Beltrami operator on $\mathbb{S}^d$, and compute

$$\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho] \quad \text{and} \quad \frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$$

$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \Longrightarrow \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$ with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

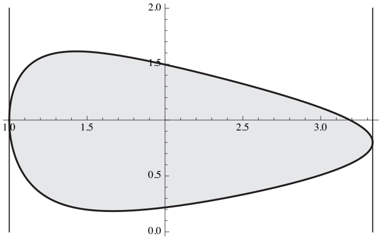
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{1}$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0 \,,$$



(p, m) *admissible region*, $d = 5$

Sobolev's inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho\,\phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \geq 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\, v(x)$$

- $p = 2^*$, $\mathsf{S}_d = \frac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \geq \mathsf{S}_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right]^{\frac{d-2}{d}} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

Schwarz symmetrization and the ultraspherical setting

$(\xi_0, \xi_1, \dots \xi_d) \in \mathbb{S}^d$, $\xi_d = z$, $\sum_{i=0}^d |\xi_i|^2 = 1$ [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on $\xi_d = z$

- Let $d\sigma(\theta) := \frac{(\sin\theta)^{d-1}}{Z_d}\,d\theta$, $Z_d := \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}$: $\forall\, v \in \mathrm{H}^1([0,\pi], d\sigma)$

$$\frac{p-2}{d}\int_0^\pi |v'(\theta)|^2\,d\sigma + \int_0^\pi |v(\theta)|^2\,d\sigma \ge \left(\int_0^\pi |v(\theta)|^p\,d\sigma\right)^{\frac{2}{p}}$$

- Change of variables $z = \cos\theta$, $v(\theta) = f(z)$

$$\frac{p-2}{d}\int_{-1}^1 |f'|^2\,\nu\,d\nu_d + \int_{-1}^1 |f|^2\,d\nu_d \ge \left(\int_{-1}^1 |f|^p\,d\nu_d\right)^{\frac{2}{p}}$$

where $\nu_d(z)\,dz = d\nu_d(z) := Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$, consider the space $\mathrm{L}^2((-1,1), d\nu_d)$ with scalar product

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1\, f_2\, d\nu_d\,, \quad \|f\|_{\mathrm{L}^p(\mathbb{S}^d)} = \left(\int_{-1}^{1} f^p\, d\nu_d\right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\, f := (1 - z^2)\, f'' - d\, z\, f' = \nu\, f'' + \frac{d}{2}\,\nu'\, f'$$

which satisfies $\langle f_1, \mathcal{L}\, f_2 \rangle = -\int_{-1}^{1} f_1'\, f_2'\, \nu\, d\nu_d$

Proposition

Let $p \in [1,2) \cup (2, 2^*]$, $d \ge 1$. For any $f \in \mathrm{H}^1([-1,1], d\nu_d)$,

$$-\langle f, \mathcal{L}\, f \rangle = \int_{-1}^{1} |f'|^2\, \nu\, d\nu_d \ge d\, \frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

Heat flow and the Bakry-Emery method

With $g = f^p$, $i.e.$ $f = g^\alpha$ with $\alpha = 1/p$

(Ineq.) $\quad -\langle f, \mathcal{L} f\rangle = -\langle g^\alpha, \mathcal{L} g^\alpha\rangle =: \mathcal{I}[g] \geq d \, \frac{\|g\|^{2\alpha}_{\mathrm{L}^1(\mathbb{S}^d)} - \|g^{2\alpha}\|_{\mathrm{L}^1(\mathbb{S}^d)}}{p-2} =: \mathcal{F}[g]$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

$$\frac{d}{dt}\|g\|_{\mathrm{L}^1(\mathbb{S}^d)} = 0\,, \quad \frac{d}{dt}\|g^{2\alpha}\|_{\mathrm{L}^1(\mathbb{S}^d)} = -\,2\,(p-2)\,\langle f, \mathcal{L} f\rangle = 2\,(p-2)\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

which finally gives

$$\frac{d}{dt}\mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\,\frac{d}{dt}\,\|g^{2\alpha}\|_{\mathrm{L}^1(\mathbb{S}^d)} = -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

Ineq. $\iff \frac{d}{dt}\mathcal{F}[g(t,\cdot)] \leq -\,2\,d\,\mathcal{F}[g(t,\cdot)] \Longleftarrow \frac{d}{dt}\mathcal{I}[g(t,\cdot)] \leq -\,2\,d\,\mathcal{I}[g(t,\cdot)]$

The equation for $g=f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t}=\mathcal{L}\,f+(p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d=\frac{1}{2}\,\frac{d}{dt}\,\langle f,\mathcal{L}\,f\rangle=\langle\mathcal{L}\,f,\mathcal{L}\,f\rangle+(p-1)\,\langle\frac{|f'|^2}{f}\,\nu,\mathcal{L}\,f\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)]+2\,d\,\mathcal{I}[g(t,\cdot)]=\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d+2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$=-2\int_{-1}^{1}\left(|f''|^2+(p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2}-2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2+(p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2}-2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2\le(p-1)\,\frac{d}{d+2}\iff p\le\frac{2\,d^2+1}{(d-1)^2}=2^\#<\frac{2\,d}{d-2}=2^*$$

... up to the critical exponent: a proof in two slides

$$\left[\frac{d}{dz},\mathcal{L}\right]u=(\mathcal{L}\,u)'-\mathcal{L}\,u'=-2\,z\,u''-d\,u'$$

$$\int_{-1}^1(\mathcal{L}\,u)^2\,d\nu_d=\int_{-1}^1|u''|^2\,\nu^2\,d\nu_d+d\int_{-1}^1|u'|^2\,\nu\,d\nu_d$$

$$\int_{-1}^1(\mathcal{L}\,u)\,\frac{|u'|^2}{u}\,\nu\,d\nu_d=\frac{d}{d+2}\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d-2\,\frac{d-1}{d+2}\int_{-1}^1\frac{|u'|^2\,u''}{u}\,\nu^2\,d\nu_d$$

On $(-1,1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t=u^{2-2\beta}\left(\mathcal{L}\,u+\kappa\,\frac{|u'|^2}{u}\,\nu\right)$$

If $\kappa=\beta\,(p-2)+1$, the L^p norm is conserved

$$\frac{d}{dt}\int_{-1}^1u^{\beta p}\,d\nu_d=\beta\,p\,(\kappa-\beta\,(p-2)-1)\int_{-1}^1u^{\beta(p-2)}\,|u'|^2\,\nu\,d\nu_d=0$$

$$f=u^\beta,\ \|f'\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\tfrac{d}{p-2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right)\ge 0\ ?$$

$$\mathcal{A}:=\int_{-1}^1|u''|^2\,\nu^2\,d\nu_d-2\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\int_{-1}^1 u''\,\frac{|u'|^2}{u}\,\nu^2\,d\nu_d$$
$$+\left[\kappa\,(\beta-1)+\frac{d}{d+2}\,(\kappa+\beta-1)\right]\int_{-1}^1\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d$$

$\mathcal{A}$ is nonnegative for some β if

$$\frac{8\,d^2}{(d+2)^2}\,(p-1)\,(2^*-p)\ge 0$$

$\mathcal{A}$ is a sum of squares if $p\in(2,2^*)$ for an arbitrary choice of β in a certain interval (depending on p and d)

$$\mathcal{A}=\int_{-1}^1\left|u''-\frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\,d\nu_d\ge 0\quad\text{if } p=2^* \text{ and } \beta=\frac{4}{6-p}$$

The rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$

$$-\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with the two-homogenous terms

Constraints and improvements

Integral constraints

Proposition

For any $p \in (2, 2^\#)$, the inequality

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{\lambda}{p-2} \, \|f\|_2^2 \ge \frac{\lambda}{p-2} \, \|f\|_p^2$$

$$\forall \, f \in \mathrm{H}^1((-1,1), d\nu_d) \text{ s.t. } \int_{-1}^{1} z \, |f|^p \, d\nu_d = 0$$

holds with

$$\lambda \ge d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

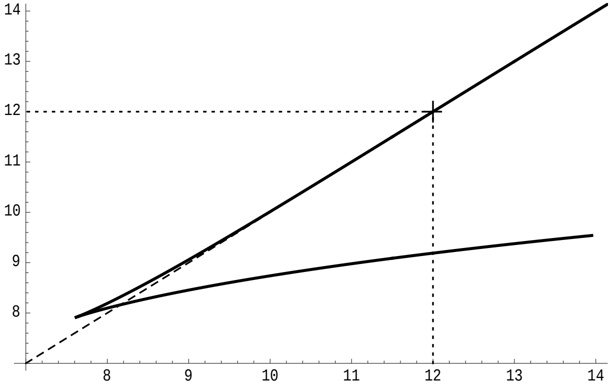
If $p \in (1,2) \cup (2,2^*)$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \ge \frac{d}{p-2}\left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1}\right]\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

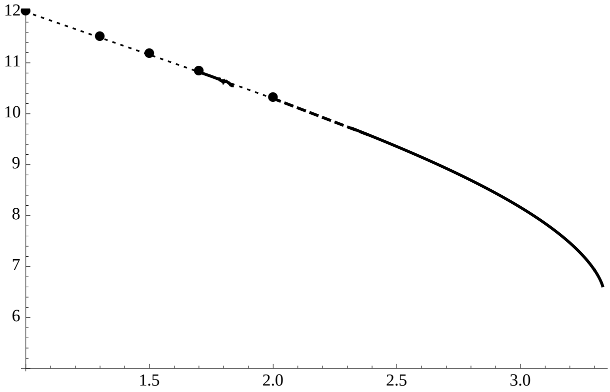
$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \ge \frac{d}{2}\frac{(d+3)^2}{(d+1)^2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\,v_g$$

The larger picture: branches of antipodal solutions



Case $d = 5$, $p = 3$: values of the shooting parameter a as a function of λ

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d = 5$ *and* $1 \le p \le 10/3 \approx 3.33$*. The limiting value of the constant is numerically found to be equal to* $\lambda_\star = 2^{1-2/p}\,d \approx 6.59754$ *with* $d = 5$ *and* $p = 10/3$

Conclusion

▷ The flow interpretation is a powerful method to organize computations which are otherwise nasty

▷ Outcomes of the method: various improvements. We are able to deal with non-compact cases and weights but the price to pay for controlling the boundary terms is high

▷ A better understanding of the variational structure of the problem opens a new direction for research on constrained inequalities, but generic difficulties (non-monotone branches) have to be understood

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

Thank you for your attention !