Symétrisation basée sur des méthodes d’entropie et des flots non-linéaires

Jean Dolbeault

http://www.ceremade.dauphine.fr/~dolbeaul

Ceremade, Université Paris-Dauphine

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Outline

- **Symmetry breaking and linearization**
  - The critical Caffarelli-Kohn-Nirenberg inequality
  - A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
  - Linearization and spectrum

- **Without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows**
  - The Bakry-Emery method on the sphere
  - Rényi entropy powers
  - Self-similar variables and relative entropies
  - The role of the spectral gap

- **With weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**
  - Towards a parabolic proof
  - Large time asymptotics and spectral gaps
  - A discussion of optimality cases
Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large
time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and
linearization for the evolution equations)

...and also

S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk ...
Symmetry breaking and linearization
Entropy methods without weights
Weighted nonlinear flows and CKN inequalities

Background references (partial)


- Entropy methods in PDEs
  - Rényi entropy powers (information theory) (Savaré, Toscani, 2014), (Dolbeault, Toscani)

J. Dolbeault
Symétrisation, entropie, flots non-linéaires
Symmetry and symmetry breaking results

* The critical Caffarelli-Kohn-Nirenberg inequality
* A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities
* Linearization and spectrum
Critical Caffarelli-Kohn-Nirenberg inequality

Let $D_{a,b} := \left\{ v \in L^p(\mathbb{R}^d, |x|^{-b} \, dx) : |x|^{-a} |\nabla v| \in L^2(\mathbb{R}^d, dx) \right\}$

$$\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^b} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx \quad \forall v \in D_{a,b}$$

holds under conditions on $a$ and $b$

$$p = \frac{2 \, d}{d - 2 + 2(b - a)} \quad \text{(critical case)}$$

▷ An optimal function among radial functions:

$$v_\star(x) = \left( 1 + |x|^{(p-2)(a_c - a)} \right)^{-\frac{2}{p-2}}$$

and

$$C_{a,b}^\star = \frac{\| |x|^{-b} v_\star \|^2_p}{\| |x|^{-a} \nabla v_\star \|^2_2}$$

Question: $C_{a,b} = C_{a,b}^\star$ (symmetry) or $C_{a,b} > C_{a,b}^\star$ (symmetry breaking)?
Critical CKN: range of the parameters

Figure: $d = 3$

\[
\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|
abla v|^2}{|x|^{2a}} \, dx
\]

\[
a \leq b \leq a + 1 \text{ if } d \geq 3
\]
\[
a < b \leq a + 1 \text{ if } d = 2, \quad a + 1/2 < b \leq a + 1 \text{ if } d = 1
\]
and $a < a_c := (d - 2)/2$

\[
p = \frac{2d}{d - 2 + 2(b - a)}
\]

(Glaser, Martin, Grosse, Thirring (1976))
(Caffarelli, Kohn, Nirenberg (1984))
[F. Catrina, Z.-Q. Wang (2001)]
Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

\[ b_{FS}(a) := \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c \]

[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

The functional

\[ C^*_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2a}} \, dx - \left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{bp}} \, dx \right)^{2/p} \]

is linearly instable at \( v = v_\star \)
Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric.
The Emden-Fowler transformation and the cylinder

With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder

\[ v(r, \omega) = r^{a-c} \varphi(s, \omega) \quad \text{with} \quad r = |x|, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r} \]

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the subcritical interpolation inequality

\[ \| \partial_s \varphi \|_{L^2(C)}^2 + \| \nabla \omega \varphi \|_{L^2(C)}^2 + \Lambda \| \varphi \|_{L^2(C)}^2 \geq \mu(\Lambda) \| \varphi \|_{L^p(C)}^2 \quad \forall \varphi \in H^1(C) \]

where \( \Lambda := (a_c - a)^2 \), \( C = \mathbb{R} \times S^{d-1} \) and the optimal constant \( \mu(\Lambda) \) is

\[ \mu(\Lambda) = \frac{1}{C_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda} \]
Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_*(s, \omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$H^1(C) \ni \varphi \mapsto \|\partial_s \varphi\|_{L^2(C)}^2 + \|\nabla \omega \varphi\|_{L^2(C)}^2 + \Lambda \|\varphi\|_{L^2(C)}^2$$

under a constraint on $$\|\varphi\|_{L^p(C)}^2$$

$$\varphi_*$$ is not optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta \omega + \Lambda - \varphi_*^{p-2} = -\partial_s^2 - \Delta \omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has a negative eigenvalue
Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: $\|w\|_{L^{q,\gamma}(\mathbb{R}^d)} := \left( \int_{\mathbb{R}^d} |w|^q |x|^{-\gamma} \, dx \right)^{1/q}$, $\|w\|_{L^q(\mathbb{R}^d)} := \|w\|_{L^{q,0}(\mathbb{R}^d)}$


$$\|w\|_{L^{2p,\gamma}(\mathbb{R}^d)} \leq C_{\beta,\gamma,p} \|\nabla w\|_{L^{2,\beta}(\mathbb{R}^d)}^{\vartheta} \|w\|_{L^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta}$$

(CKN)

Here $C_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \geq 2, \quad \gamma-2 < \beta < \frac{d-2}{d} \gamma, \quad \gamma \in (-\infty, d), \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2}$$

and the exponent $\vartheta$ is determined by the scaling invariance, i.e.,

$$\vartheta = \frac{(d-\gamma)(p-1)}{p \left( d+\beta+2-2 \gamma-p(d-\beta-2) \right)}$$

Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = (1 + |x|^{2+\beta-\gamma})^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d$$

Do we know (symmetry) that the equality case is achieved among radial functions?
Here $p$ is given

\[ \beta = \frac{d-2}{d} \gamma \]

\[ \beta = d - 2 + \frac{\gamma - d}{p} \]

Range of the parameters
Let us define
\[ \beta_{FS}(\gamma) := d - 2 - \sqrt{(d - \gamma)^2 - 4(d - 1)} \]

**Theorem**

*Symmetry breaking holds in (CKN) if*

\[ \gamma < 0 \quad \text{and} \quad \beta_{FS}(\gamma) < \beta < \frac{d - 2}{d} \gamma \]

In the range \( \beta_{FS}(\gamma) < \beta < \frac{d-2}{d} \gamma \), \( w_\star(x) = (1 + |x|^{2+\beta-\gamma})^{-1/(p-1)} \) is not optimal

(JD, Esteban, Loss, Muratori, 2016)

**Theorem**

*Symmetry holds in (CKN) if*

\[ \gamma \geq 0, \quad \text{or} \quad \gamma \leq 0 \quad \text{and} \quad \gamma - 2 \leq \beta \leq \beta_{FS}(\gamma) \]
The green area is the region of symmetry, while the red area is the region of symmetry breaking. The threshold is determined by the hyperbola

\[(d - \gamma)^2 - (\beta - d + 2)^2 - 4(d - 1) = 0\]
A useful change of variables

With

\[ \alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2 \frac{d - \gamma}{\beta + 2 - \gamma}, \]

(CKN) can be rewritten for a function \( v(|x|^{\alpha-1} x) = w(x) \) as

\[
\|v\|_{L^{2p,d-n}(\mathbb{R}^d)} \leq K_{\alpha,n,p} \|D_{\alpha}v\|_{L^{2,d-n}(\mathbb{R}^d)}^{\frac{d}{2}} \|v\|_{L^{p+1,d-n}(\mathbb{R}^d)}^{1-\frac{d}{2}}
\]

with the notations \( s = |x|, D_{\alpha}v = (\alpha \frac{\partial v}{\partial s}, \frac{1}{s} \nabla_\omega v). \) Parameters are in the range

\[ d \geq 2, \quad \alpha > 0, \quad n > d \quad \text{and} \quad p \in (1, p_*], \quad p_* := \frac{n}{n-2} \]

By our change of variables, \( w_* \) is changed into

\[ v_*(x) := (1 + |x|^2)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}^d \]

The symmetry breaking condition (Felli-Schneider) now reads

\[ \alpha < \alpha_{FS} \quad \text{with} \quad \alpha_{FS} := \sqrt{\frac{d-1}{n-1}} \]
The second variation

\[ \mathcal{J}[v] := \vartheta \log (\| D_\alpha v \|_{L^2,d-n(\mathbb{R}^d)}) + (1 - \vartheta) \log (\| v \|_{L^{p+1},d-n(\mathbb{R}^d)}) + \log K_{\alpha,n,p} - \log (\| v \|_{L^{2p},d-n(\mathbb{R}^d)}) \]

Let us define \( d\mu_\delta := \mu_\delta(x) \, dx \), where \( \mu_\delta(x) := (1 + |x|^2)^{-\delta} \). Since \( v_* \) is a critical point of \( \mathcal{J} \), a Taylor expansion at order \( \varepsilon^2 \) shows that

\[ \| D_\alpha v_* \|^2_{L^2,d-n(\mathbb{R}^d)} \mathcal{J}[v_* + \varepsilon \mu_\delta/2 \, f] = \frac{1}{2} \varepsilon^2 \vartheta \, \mathcal{Q}[f] + o(\varepsilon^2) \]

with \( \delta = \frac{2p}{p-1} \) and

\[ \mathcal{Q}[f] = \int_{\mathbb{R}^d} |D_\alpha f|^2 |x|^{n-d} \, d\mu_\delta - \frac{4p \alpha^2}{p-1} \int_{\mathbb{R}^d} |f|^2 |x|^{n-d} \, d\mu_{\delta+1} \]

We assume that \( \int_{\mathbb{R}^d} f |x|^{n-d} \, d\mu_{\delta+1} = 0 \) (mass conservation)
Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let \(d \geq 2, \alpha \in (0, +\infty), n > d\) and \(\delta \geq n\). If \(f\) has 0 average, then

\[
\int_{\mathbb{R}^d} |D_\alpha f|^2 |x|^{n-d} \, d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2 |x|^{n-d} \, d\mu_{\delta+1}
\]

with optimal constant \(\Lambda = \min\{2 \alpha^2 (2 \delta - n), 2 \alpha^2 \delta \eta\}\) where \(\eta\) is the unique positive solution to \(\eta(\eta + n - 2) = (d - 1)/\alpha^2\). The corresponding eigenfunction is not radially symmetric if \(\alpha^2 > \frac{(d - 1) \delta^2}{n(2 \delta - n)(\delta - 1)}\)

\(Q \geq 0\) iff \(\frac{4p \alpha^2}{p-1} \leq \Lambda\) and symmetry breaking occurs in (CKN) if

\[
2 \alpha^2 \delta \eta < \frac{4p \alpha^2}{p-1} \iff \eta < 1
\]

\[
\iff \frac{d - 1}{\alpha^2} = \eta(\eta + n - 2) < n - 1 \iff \alpha > \alpha_{FS}
\]
Inequalities without weights and fast diffusion equations

▷ The Bakry-Emery method for a Fokker-Planck equation on a domain of the Euclidean space, on the sphere and its extension by non-linear diffusion flows

▷ [Rényi entropy powers]

▷ Self-similar variables and relative entropies

▷ The role of the spectral gap
The Fokker-Planck equation: $\varphi$-entropies and Bakry-Emery method

The linear Fokker-Planck (FP) equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \varphi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \varphi) \cdot \nu = 0 \quad \text{on} \quad \partial \Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \varphi \cdot \nabla v =: \mathcal{L} v$$

(Bakry, Emery, 1985), (Arnold, Markowich, Toscani, Unterreiter, 2001)

The unique stationary solution of (FP) (with mass normalized to 1) is

$$e^{-\varphi} \int_{\Omega} e^{-\varphi} \, dx$$
The Bakry-Emery method

With $\nu$ such that $\int_{\Omega} \nu \, d\gamma = 1$, $q \in (1, 2]$, the $q$-entropy is defined by

$$\mathcal{E}_q[\nu] := \frac{1}{q-1} \int_{\Omega} (\nu^q - 1 - q (\nu - 1)) \, d\gamma$$

Under the action of (OU), with $w = \nu^{q/2}$, $\mathcal{I}_q[\nu] := \frac{4}{q} \int_{\Omega} |\nabla w|^2 \, d\gamma$,

$$\frac{d}{dt} \mathcal{E}_q[\nu(t, \cdot)] = -\mathcal{I}_q[\nu(t, \cdot)] \quad \text{and} \quad \frac{d}{dt} (\mathcal{I}_q[\nu] - 2\lambda \mathcal{E}_q[\nu]) \leq 0$$

with $\lambda := \inf_{w \in H^1(\Omega, d\gamma) \setminus \{0\}} \frac{\int_{\Omega} \left(2 \frac{q-1}{q} \|\text{Hess } w\|^2 + \text{Hess } \phi : \nabla w \otimes \nabla w\right) \, d\gamma}{\int_{\Omega} |w|^2 \, d\gamma}$

Proposition

(Bakry, Emery, 1984) (JD, Nazaret, Savaré, 2008) Let $\Omega$ be convex. If $\lambda > 0$ and $\nu$ is a solution of (OU), then $\mathcal{I}_q[\nu(t, \cdot)] \leq \mathcal{I}_q[\nu(0, \cdot)] e^{-2\lambda t}$ and $\mathcal{E}_q[\nu(t, \cdot)] \leq \mathcal{E}_q[\nu(0, \cdot)] e^{-2\lambda t}$ for any $t \geq 0$ and, as a consequence,

$$\mathcal{I}_q[\nu] \geq 2\lambda \mathcal{E}_q[\nu] \quad \forall \nu \in H^1(\Omega, d\gamma)$$
The interpolation inequalities

On the $d$-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|_{L^2(S^d)}^2 + \frac{d}{p-2} \|u\|_{L^2(S^d)}^2 \geq \frac{d}{p-2} \|u\|_{L^p(S^d)}^2 \quad \forall u \in H^1(S^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $S^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \geq 1$, $p \neq 2$, is such that

$$p \leq 2^* := \frac{2d}{d-2}$$

if $d \geq 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|_{L^2(S^d)}^2 \geq \frac{d}{2} \int_{S^d} |u|^2 \log \left( \frac{|u|^2}{\|u\|_{L^2(S^d)}^2} \right) d\mu \quad \forall u \in H^1(S^d, d\mu) \setminus \{0\}$$
**The Bakry-Emery method**

Entropy functional

\[ \mathcal{E}_p[\rho] := \frac{1}{p-2} \left[ \int_{\mathbb{S}^d} \rho^{\frac{2}{p}} \, d\mu - \left( \int_{\mathbb{S}^d} \rho \, d\mu \right)^{\frac{2}{p}} \right] \quad \text{if} \quad p \neq 2 \]

\[ \mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log \left( \frac{\rho}{\|\rho\|_{L^1(\mathbb{S}^d)}} \right) \, d\mu \]

Fisher information functional

\[ \mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\mu \]

Bakry-Emery (carré du champ) method: use the heat flow

\[ \frac{\partial \rho}{\partial t} = \Delta \rho \]

and compute \( \frac{d}{dt} \mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho] \) and \( \frac{d}{dt} \mathcal{I}_p[\rho] \leq -d \mathcal{I}_p[\rho] \) to get

\[ \frac{d}{dt} \left( \mathcal{I}_p[\rho] - d \mathcal{E}_p[\rho] \right) \leq 0 \quad \Rightarrow \quad \mathcal{I}_p[\rho] \geq d \mathcal{E}_p[\rho] \]

with \( \rho = |u|^p \), if \( p \leq 2\# := \frac{2d^2+1}{(d-1)^2} \)
The evolution under the fast diffusion flow

To overcome the limitation \( p \leq 2^# \), one can consider a nonlinear diffusion of fast diffusion / porous medium type

\[
\frac{\partial \rho}{\partial t} = \Delta \rho^m. \tag{1}
\]

(Demange), (JD, Esteban, Kowalczyk, Loss): for any \( p \in [1, 2^*] \)

\[
\mathcal{K}_p[\rho] := \frac{d}{dt} \left( \mathcal{I}_p[\rho] - d \mathcal{E}_p[\rho] \right) \leq 0
\]

\((p, m)\) admissible region, \( d = 5 \)
The rigidity point of view (elliptic PDEs)

In cylindrical coordinates with \( z \in [-1, 1] \), let
\[
\mathcal{L} f := (1 - z^2) f''' - d z f' = \nu f'' + \frac{d}{2} \nu' f'
\]
be the ultraspherical operator and consider
\[
-\mathcal{L} u - (\beta - 1) \frac{|u'|^2}{u} \nu + \frac{\lambda}{p - 2} u = \frac{\lambda}{p - 2} u^\kappa
\]
Multiply by \( \mathcal{L} u \) and integrate
\[
\ldots \int_{-1}^{1} \mathcal{L} u u^\kappa \, d\nu_d = -\kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \, d\nu_d
\]
Multiply by \( \kappa \frac{|u'|^2}{u} \) and integrate
\[
\ldots = +\kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \, d\nu_d
\]
\[
\int_{-1}^{1} \left| u'' - p + 2 \frac{|u'|^2}{u} \right|^2 \nu^2 \, d\nu_d = 0 \quad \text{if} \quad p = 2^* \quad \text{and} \quad \beta = \frac{4}{6 - p}
\]
Consequences, improvements, related problems

- Improved inequalities in the subcritical range
- Improved constants under orthogonality constraints for $p \leq 2$
- Improved constants under antipodal symmetry for $p \leq 2^*$
- The extension to Riemannian manifolds
- Onofri inequality (JD, Esteban, Jankowiak, 2015)
- Lin-Ni problems (JD, Kowalczyk)
- Keller-Lieb-Thirring inequalities on manifolds: estimates for Schrödinger operator that (really) differ from the semi-classical estimates (JD, Esteban, Laptev, Loss)
Rényi entropy powers and fast diffusion

- Rényi entropy powers, the entropy approach without rescaling: (Savaré, Toscani): scalings, nonlinearity and a concavity property inspired by information theory

- Faster rates of convergence: (Carrillo, Toscani), (JD, Toscani)
The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in \( \mathbb{R}^d, d \geq 1 \)

\[
\frac{\partial v}{\partial t} = \Delta v^m
\]

with initial datum \( v(x, t = 0) = v_0(x) \geq 0 \) such that \( \int_{\mathbb{R}^d} v_0 \, dx = 1 \) and \( \int_{\mathbb{R}^d} |x|^2 v_0 \, dx < +\infty \). The large time behavior of the solutions is governed by the source-type Barenblatt solutions

\[
U_\star(t, x) := \frac{1}{(\kappa t^{1/\mu})^d} B_\star\left(\frac{x}{\kappa t^{1/\mu}}\right)
\]

where

\[
\mu := 2 + d(m - 1), \quad \kappa := \left|\frac{2 \mu m}{m - 1}\right|^{1/\mu}
\]

and \( B_\star \) is the Barenblatt profile

\[
B_\star(x) := \begin{cases} 
(C_\star - |x|^2)^{1/(m-1)} & \text{if } m > 1 \\
(C_\star + |x|^2)^{1/(m-1)} & \text{if } m < 1 
\end{cases}
\]
The Rényi entropy power $F$

The entropy is defined by

$$E := \int_{\mathbb{R}^d} v^m \, dx$$

and the Fisher information by

$$I := \int_{\mathbb{R}^d} v |\nabla p|^2 \, dx \quad \text{with} \quad p = \frac{m}{m - 1} v^{m-1}$$

If $v$ solves the fast diffusion equation, then

$$E' = (1 - m) I$$

To compute $I'$, we will use the fact that

$$\frac{\partial p}{\partial t} = (m - 1) p \Delta p + |\nabla p|^2$$

$$F := E^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d (1 - m)} = 1 + \frac{2}{1 - m} \left( \frac{1}{d} + m - 1 \right) = \frac{2}{d} \frac{1}{1 - m} - 1$$

has a linear growth asymptotically as $t \to +\infty$.
The concavity property

**Theorem**

[Toscani-Savaré] Assume that $m \geq 1 - \frac{1}{d}$ if $d > 1$ and $m > 0$ if $d = 1$. Then $F(t)$ is increasing, $(1 - m) F''(t) \leq 0$ and

$$
\lim_{t \to +\infty} \frac{1}{t} F(t) = (1 - m) \sigma \lim_{t \to +\infty} E^{\sigma-1} I = (1 - m) \sigma E_*^{\sigma-1} I_*
$$

[Dolbeault-Toscani] The inequality

$$
E^{\sigma-1} I \geq E_*^{\sigma-1} I_*
$$

is equivalent to the Gagliardo-Nirenberg inequality

$$
\| \nabla w \|_{L^2(\mathbb{R}^d)}^\theta \| w \|_{L^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq C_{GN} \| w \|_{L^q(\mathbb{R}^d)}
$$

if $1 - \frac{1}{d} \leq m < 1$. Hint: $v^{m-1/2} = \frac{w}{\| w \|_{L^2(\mathbb{R}^d)}}$, $q = \frac{1}{2m-1}$
The proof

Lemma

If \( v \) solves \( \frac{\partial v}{\partial t} = \Delta v^m \) with \( \frac{1}{d} \leq m < 1 \), then

\[
I' = \frac{d}{dt} \int_{\mathbb{R}^d} v |\nabla p|^2 \, dx = -2 \int_{\mathbb{R}^d} v^m \left( \|D^2 p\|^2 + (m - 1)(\Delta p)^2 \right) \, dx
\]

Explicit arithmetic geometric inequality

\[
\|D^2 p\|^2 - \frac{1}{d} (\Delta p)^2 = \left\| D^2 p - \frac{1}{d} \Delta p \text{Id} \right\|^2
\]

There are no boundary terms in the integrations by parts
**Remainder terms**

\[ F'' = -\sigma (1 - m) R[v]. \] The *pressure variable* is \[ P = \frac{m}{1-m} v^{m-1}. \]

\[ R[v] := (\sigma - 1) (1 - m) E^{\sigma-1} \int_{\mathbb{R}^d} v^m \left| \Delta P - \frac{\int_{\mathbb{R}^d} v |\nabla P|^2 \, dx}{\int_{\mathbb{R}^d} v^m \, dx} \right|^2 \, dx \]

\[ + 2 E^{\sigma-1} \int_{\mathbb{R}^d} v^m \| D^2 P - \frac{1}{d} \Delta P \text{Id} \|^2 \, dx \]

Let

\[ G[v] := \frac{F[v]}{\sigma (1 - m)} = \left( \int_{\mathbb{R}^d} v^m \, dx \right)^{\sigma-1} \int_{\mathbb{R}^d} v |\nabla P|^2 \, dx \]

The Gagliardo-Nirenberg inequality is equivalent to \( G[v_0] \geq G[v_*] \)

**Proposition**

\[ G[v_0] \geq G[v_*] + \int_0^\infty R[v(t, \cdot)] \, dt \]
Self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type Barenblatt solutions

$$v_*(t, x) := \frac{1}{\kappa^d (\mu t)^{d/\mu}} B_*(\frac{x}{\kappa (\mu t)^{1/\mu}}) \text{ where } \mu := 2 + d (m - 1)$$

where $B_*$ is the Barenblatt profile (with appropriate mass)

$$B_*(x) := (1 + |x|^2)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t, x) = \frac{1}{\kappa^d R^d} u(\tau, \frac{x}{\kappa R}) \text{ where } \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \frac{1}{2} \log \left( \frac{R(t)}{R_0} \right)$$

Then the function $u$ solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0$$
Free energy and Fisher information

The function $u$ solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0$$

(Ralston, Newman, 1984) Lyapunov functional:

Generalized entropy or Free energy

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left( -\frac{u^m}{m} + |x|^2 u \right) \, dx - \mathcal{E}_0$$

Entropy production is measured by the Generalized Fisher information

$$\frac{d}{dt} \mathcal{E}[u] = -\mathcal{I}[u] , \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2x \right|^2 \, dx$$
Without weights: relative entropy, entropy production

- **Stationary solution:** choose $C$ such that $\|u_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$u_\infty(x) := (C + |x|^2)^{-1/(1-m)}_+$$

- **Relative entropy:** Fix $E_0$ so that $E[u_\infty] = 0$

- **Entropy – entropy production inequality** (del Pino, J.D.)

**Theorem**

$$d \geq 3, \ m \in \left[\frac{d-1}{d}, +\infty\right), \ m > \frac{1}{2}, \ m \neq 1$$

$$\mathcal{I}[u] \geq 4 \mathcal{E}[u]$$

**Corollary**

(del Pino, J.D.) **A solution** $u$ **with initial data** $u_0 \in L^1_+(\mathbb{R}^d)$ **such that**

$$|x|^2 u_0 \in L^1(\mathbb{R}^d), \ u_0^m \in L^1(\mathbb{R}^d)$$

**satisfies**

$$\mathcal{E}[u(t, \cdot)] \leq \mathcal{E}[u_0] e^{-4t}$$
A computation on a large ball, with boundary terms

\[
\frac{\partial u}{\partial \tau} + \nabla \cdot \left[ u \left( \nabla u^{m-1} - 2x \right) \right] = 0 \quad \tau > 0, \quad x \in B_R
\]

where \( B_R \) is a centered ball in \( \mathbb{R}^d \) with radius \( R > 0 \), and assume that \( u \) satisfies zero-flux boundary conditions

\[
\left( \nabla u^{m-1} - 2x \right) \cdot \frac{x}{|x|} = 0 \quad \tau > 0, \quad x \in \partial B_R.
\]

With \( z(\tau, x) := \nabla Q(\tau, x) := \nabla u^{m-1} - 2x \), the relative Fisher information is such that

\[
\frac{d}{d\tau} \int_{B_R} u |z|^2 \, dx + 4 \int_{B_R} u |z|^2 \, dx
\]

\[
+ 2 \frac{1-m}{m} \int_{B_R} u^m \left( \|D^2 Q\|^2 - (1-m)(\Delta Q)^2 \right) \, dx
\]

\[
= \int_{\partial B_R} u^m (\omega \cdot \nabla |z|^2) \, d\sigma \leq 0 \quad \text{(by Grisvard’s lemma)}
\]
Another improvement of the GN inequalities

Let us define the relative entropy

\[ \mathcal{E}[u] := -\frac{1}{m} \int_{\mathbb{R}^d} (u^m - B^*_m - m B^{m-1}_* (u - B_*)) \, dx \]

the relative Fisher information

\[ \mathcal{I}[u] := \int_{\mathbb{R}^d} u |z|^2 \, dx = \int_{\mathbb{R}^d} u |\nabla u^{m-1} - 2x|^2 \, dx \]

and

\[ \mathcal{R}[u] := 2 \frac{1 - m}{m} \int_{\mathbb{R}^d} u^m \left( \|D^2 Q\|^2 - (1 - m)(\Delta Q)^2 \right) \, dx \]

**Proposition**

If \( 1 - 1/d \leq m < 1 \) and \( d \geq 2 \), then

\[ \mathcal{I}[u_0] - 4 \mathcal{E}[u_0] \geq \int_0^\infty \mathcal{R}[u(\tau, \cdot)] \, d\tau \]
Entropy – entropy production, Gagliardo-Nirenberg ineq.

$$4 \mathcal{E}[u] \leq \mathcal{I}[u]$$

Rewrite it with $p = \frac{1}{2m-1}$, $u = w^{2p}$, $u^m = w^{p+1}$ as

$$\frac{1}{2} \left( \frac{2m}{2m-1} \right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 \, dx + \left( \frac{1}{1-m} - d \right) \int_{\mathbb{R}^d} |w|^{1+p} \, dx - K \geq 0$$

- for some $\gamma$, $K = K_0 \left( \int_{\mathbb{R}^d} u \, dx = \int_{\mathbb{R}^d} w^{2p} \, dx \right)^\gamma$
- $w = w_\infty = \nu^{1/2p}_\infty$ is optimal

**Theorem**

[Del Pino, J.D.] *With $1 < p \leq \frac{d}{d-2}$ (fast diffusion case) and $d \geq 3$*

$$\|w\|_{L^{2p} (\mathbb{R}^d)} \leq C_{p, d}^{GN} \|\nabla w\|_{L^2 (\mathbb{R}^d)}^{\theta} \|w\|_{L^{p+1} (\mathbb{R}^d)}^{1-\theta}$$

$$C_{p, d}^{GN} = \left( \frac{\Gamma(p-1)^2}{2\pi d} \right)^{\frac{\theta}{2}} \left( \frac{2y-d}{2y} \right)^{\frac{1}{2p}} \left( \frac{\Gamma(y)}{\Gamma(y-d/2)} \right)^{\frac{\theta}{d}}$$

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)}$$

$$y = \frac{p+1}{p-1}$$

J. Dolbeault

*Symérisation, entropie, flots non-linéaires*
Sharp asymptotic rates of convergence

Assumptions on the initial datum \( v_0 \)

**\( (H1) \)** \( V_{D_0} \leq v_0 \leq V_{D_1} \) for some \( D_0 > D_1 > 0 \)

**\( (H2) \)** if \( d \geq 3 \) and \( m \leq m^* \), \( (v_0 - V_D) \) is integrable for a suitable \( D \in [D_1, D_0] \)

**Theorem**

(Blanchet, Bonforte, J.D., Grillo, Vázquez) **Under Assumptions** (H1)-(H2), if \( m < 1 \) and \( m \neq m^* := \frac{d-4}{d-2} \), the entropy decays according to

\[
\mathcal{E}[v(t, \cdot)] \leq C e^{-2(1-m)\Lambda_{\alpha, d} t} \quad \forall \ t \geq 0
\]

where \( \Lambda_{\alpha, d} > 0 \) is the best constant in the Hardy–Poincaré inequality

\[
\Lambda_{\alpha, d} \int_{\mathbb{R}^d} |f|^2 \, d\mu_{\alpha-1} \leq \int_{\mathbb{R}^d} |\nabla f|^2 \, d\mu_\alpha \quad \forall \ f \in H^1(d\mu_\alpha)
\]

with \( \alpha := 1/(m - 1) < 0 \), \( d\mu_\alpha := h_\alpha \, dx \), \( h_\alpha(x) := (1 + |x|^2)^\alpha \)
Spectral gaps and best constants

\[ m_c = \frac{d-2}{d} \]

\[ m_1 = \frac{d-1}{d} \]

\[ \tilde{m}_1 := \frac{d}{d+2} \]

\[ \tilde{m}_2 := \frac{d+4}{d+6} \]

\[ m_2 = \frac{d+1}{d+2} \]

Case 1

Case 2

Case 3

\( \gamma(m) \)
The spectral gap corresponding to the red curves relies on a refined notion of relative entropy with respect to best matching Barenblatt profiles (J.D., Toscani)

A result by (Denzler, Koch, McCann) Higher order time asymptotics of fast diffusion in Euclidean space: a dynamical systems approach

The constant $C$ in

$$\mathcal{E}[\nu(t, \cdot)] \leq C e^{-2\gamma^{(m)} t} \quad \forall \ t \geq 0$$

can be made explicit, under additional restrictions on the initial data (Bonforte, J.D., Grillo, Vázquez)
Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

▷ A parabolic proof?
▷ Entropy and Caffarelli-Kohn-Nirenberg inequalities
▷ Large time asymptotics and spectral gaps
▷ Optimality cases
When symmetry holds, (CKN) can be written as an entropy – entropy production inequality

\[ \frac{1 - m}{m} (2 + \beta - \gamma)^2 \mathcal{E}[v] \leq \mathcal{I}[v] \]

and equality is achieved by \( \mathcal{B}_{\beta, \gamma}(x) := (1 + |x|^{2+\beta-\gamma}) \frac{1}{m-1} \)

Here the free energy and the relative Fisher information are defined by

\[ \mathcal{E}[v] := \frac{1}{m - 1} \int_{\mathbb{R}^d} \left( v^m - \mathcal{B}_{\beta, \gamma}^m - m \mathcal{B}_{\beta, \gamma}^{m-1} (v - \mathcal{B}_{\beta, \gamma}) \right) \frac{dx}{|x|^{\gamma}} \]

\[ \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathcal{B}_{\beta, \gamma}^{m-1} \right|^2 \frac{dx}{|x|^\beta} \]

If \( v \) solves the Fokker-Planck type equation

\[ v_t + |x|^{\gamma} \nabla \cdot \left[ |x|^{-\beta} v \nabla (v^{m-1} - |x|^{2+\beta-\gamma}) \right] = 0 \]  \( \text{(WFDE-FP)} \)

then \( \frac{d}{dt} \mathcal{E}[v(t, \cdot)] = - \frac{m}{1 - m} \mathcal{I}[v(t, \cdot)] \)
Proposition

Let \( m = \frac{p+1}{2p} \) and consider a solution to \((WFDE-FP)\) with nonnegative initial datum \( u_0 \in L^{1,\gamma}(\mathbb{R}^d) \) such that \( \| u_0^m \|_{L^{1,\gamma}(\mathbb{R}^d)} \) and \( \int_{\mathbb{R}^d} u_0 |x|^{2+\beta-2\gamma} \, dx \) are finite. Then

\[
\mathcal{E}[v(t, \cdot)] \leq \mathcal{E}[u_0] \, e^{-(2+\beta-\gamma)^2 t} \quad \forall \, t \geq 0
\]

if one of the following two conditions is satisfied:

(i) either \( u_0 \) is a.e. radially symmetric
(ii) or symmetry holds in \((CKN)\)
Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, artificial dimension $n > d$ (here $n$ is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}x) = w(x), \quad \alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2 \frac{d - \gamma}{\beta + 2 - \gamma},$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}x) = w(x)$ as

$$\|v\|_{L^{2p,d-n}(\mathbb{R}^d)} \leq K_{\alpha,n,p} \|D_\alpha v\|_{L^{2,d-n}(\mathbb{R}^d)} \|v\|_{1-d}^{L^{p+1,d-n}(\mathbb{R}^d)} \quad \forall v \in H^{p,d-n,d-n}_d(\mathbb{R}^d)$$

with the notations $s = |x|$, $D_\alpha v = (\alpha \frac{\partial v}{\partial s}, \frac{1}{s} \nabla_\omega v)$ and

$$d \geq 2, \quad \alpha > 0, \quad n > d \quad \text{and} \quad p \in (1, p_\star].$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := (1 + |x|^2)^{-1/(p-1)} \quad \forall x \in \mathbb{R}^d$$
The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

\[
G[u] := \left( \int_{\mathbb{R}^d} u^m \, d\mu \right)^{\sigma^{-1}} \int_{\mathbb{R}^d} u |D_\alpha P|^2 \, d\mu
\]

where \( \sigma = \frac{2}{d} \frac{1}{1-m} - 1 \). Here \( d\mu = |x|^{n-d} \, dx \) and the pressure is

\[
P := \frac{m}{1-m} u^{m-1}
\]

Looking for an optimal function in (CKN) is equivalent to minimize \( G \) under a mass constraint.
With $L_\alpha = -D_\alpha^* D_\alpha = \alpha^2 (u'' + \frac{n-1}{s} u') + \frac{1}{s^2} \Delta_\omega u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = L_\alpha u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$- \frac{d}{dt} G[u(t, \cdot)] (\int_{\mathbb{R}^d} u^m \, d\mu)^{1-\sigma} \geq (1 - m)(\sigma - 1) \int_{\mathbb{R}^d} u^m \left| L_\alpha P - \frac{\int_{\mathbb{R}^d} u |D_\alpha P|^2 \, d\mu}{\int_{\mathbb{R}^d} u^m \, d\mu} \right|^2 \, d\mu$$

$$+ 2 \int_{\mathbb{R}^d} \left( \alpha^4 \left( 1 - \frac{1}{n} \right) \left| P'' - \frac{P'}{s} - \frac{\Delta_\omega P}{\alpha^2 (n-1) s^2} \right|^2 + \frac{2\alpha^2}{s^2} \left| \nabla_\omega P' - \frac{\nabla_\omega P}{s} \right|^2 \right) u^m \, d\mu$$

$$+ 2 \int_{\mathbb{R}^d} \left( (n - 2) \left( \alpha_{FS}^2 - \alpha^2 \right) |\nabla_\omega P|^2 + c(n, m, d) \frac{|\nabla_\omega P|^4}{P^2} \right) u^m \, d\mu =: \mathcal{H}[u]$$

for some numerical constant $c(n, m, d) > 0$. Hence if $\alpha \leq \alpha_{FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if $P$ is an affine function of $|x|^2$, which proves the symmetry result. A quantifier elimination problem (Tarski, 1951)?
This method has a hidden difficulty: integrations by parts! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if \( u \) solves the Euler-Lagrange equation, we test by \( \mathbb{L}_\alpha u^m \)

\[
0 = \int_{\mathbb{R}^d} dG[u] \cdot \mathbb{L}_\alpha u^m \, d\mu \geq \mathcal{H}[u] \geq 0
\]

\( \mathcal{H}[u] \) is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by \( |x|^\gamma \text{div} \left( |x|^{-\beta} \nabla w^{1+p} \right) \) the equation

\[
\frac{(p - 1)^2}{p(p + 1)} w^{1-3p} \text{div} \left( |x|^{-\beta} w^{2p} \nabla w^{1-p} \right) + |\nabla w^{1-p}|^2 + |x|^{-\gamma} \left( c_1 w^{1-p} - c_2 \right) = 0
\]
Towards a parabolic proof

For any $\alpha \geq 1$, let $D_\alpha W = (\alpha \partial_r W, r^{-1} \nabla_\omega W)$ so that

$$D_\alpha = \nabla + (\alpha - 1) \frac{x}{|x|^2} (x \cdot \nabla) = \nabla + (\alpha - 1) \omega \partial_r$$

and define the diffusion operator $L_\alpha$ by

$$L_\alpha = -D_\alpha^* D_\alpha = \alpha^2 \left( \partial_r^2 + \frac{n-1}{r} \partial_r \right) + \frac{\Delta_\omega}{r^2}$$

where $\Delta_\omega$ denotes the Laplace-Beltrami operator on $S^{d-1}$

$$\frac{\partial g}{\partial t} = L_\alpha g^m$$

is changed into

$$\frac{\partial u}{\partial \tau} = D_\alpha^* (u z), \quad z := D_\alpha q, \quad q := u^{m-1} - B_\alpha^{m-1}, \quad B_\alpha(x) := \left(1 + \frac{|x|^2}{\alpha^2}\right)^{\frac{1}{m-1}}$$

by the change of variables

$$g(t, x) = \frac{1}{\kappa^n R^n} u\left(\tau, \frac{x}{\kappa R}\right)$$

where

$$\begin{cases}
\frac{dR}{dt} = R^{1-\mu}, & R(0) = R_0 \\
\tau(t) = \frac{1}{2} \log \left(\frac{R(t)}{R_0}\right)
\end{cases}$$

$\omega$
If the weight does not introduce any singularity at $x = 0$...

\[
\frac{m}{1 - m} \frac{d}{d\tau} \int_{B_R} u |z|^2 \, d\mu_n
\]

\[
= \int_{\partial B_R} u^m (\omega \cdot D_\alpha |z|^2) |x|^{n - d} \, d\sigma \quad (\leq 0 \text{ by Grisvard’s lemma})
\]

\[-2 \frac{1 - m}{m} (m - 1 + \frac{1}{n}) \int_{B_R} u^m |L_\alpha q|^2 \, d\mu_n
\]

\[-\int_{B_R} u^m \left( \alpha^4 m_1 \left| q'' - \frac{q'}{r} - \frac{\Delta \omega q}{\alpha^2 (n-1) r^2} \right|^2 + \frac{2 \alpha^2}{r^2} \left| \nabla \omega q' - \frac{\nabla \omega q}{r} \right|^2 \right) \, d\mu_n
\]

\[-(n - 2) \left( \alpha_{FS}^2 - \alpha^2 \right) \int_{B_R} \frac{|\nabla \omega q|^2}{r^4} \, d\mu_n
\]

A formal computation that still needs to be justified
(singularity at $x = 0$ ?)

* Other potential application: the computation of Bakry, Gentil and Ledoux (chapter 6) for non-integer dimensions; weights on manifolds.
Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here $\nu$ solves the *Fokker-Planck type equation*

$$\nu_t + |x|^{\gamma} \nabla \cdot \left[ |x|^{-\beta} \nu \nabla (\nu^{m-1} - |x|^{2+\beta-\gamma}) \right] = 0 \quad (WFDE-FP)$$

Joint work with M. Bonforte, M. Muratori and B. Nazare
Relative uniform convergence

\[ \zeta := 1 - \left( 1 - \frac{2-m}{(1-m)q} \right) \left( 1 - \frac{2-m}{1-m} \theta \right) \]

\[ \theta := \frac{(1-m)(2+\beta-\gamma)}{(1-m)(2+\beta)+2+\beta-\gamma} \text{ is in the range } 0 < \theta < \frac{1-m}{2-m} < 1 \]

**Theorem**

For “good” initial data, there exist positive constants \( K \) and \( t_0 \) such that, for all \( q \in \left[ \frac{2-m}{1-m}, \infty \right] \), the function \( w = v/B \) satisfies

\[ \| w(t) - 1 \|_{L^q,\gamma(\mathbb{R}^d)} \leq K e^{-2 \left( \frac{1-m}{2-m} \right)^2 \Lambda \zeta(t-t_0)} \quad \forall t \geq t_0 \]

in the case \( \gamma \in (0, d) \), and

\[ \| w(t) - 1 \|_{L^q,\gamma(\mathbb{R}^d)} \leq K e^{-2 \left( \frac{1-m}{2-m} \right)^2 \Lambda (t-t_0)} \quad \forall t \geq t_0 \]

in the case \( \gamma \leq 0 \)
The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\text{ess}}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions.
Main steps of the proof:

- Existence of weak solutions, $L^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t, x) := \nu(t, x)/\mathcal{B}(x)$ solves

\[
\begin{cases}
|x|^{-\gamma} w_t = -\frac{1}{\mathcal{B}} \nabla \cdot \left(|x|^{-\beta} \mathcal{B} w \nabla \left((w^{m-1} - 1) \mathcal{B}^{m-1}\right)\right) & \text{in } \mathbb{R}^+ \times \mathbb{R}^d \\
w(0, \cdot) = w_0 := \nu_0/\mathcal{B} & \text{in } \mathbb{R}^d
\end{cases}
\]

- **Regularity:** (Chiarenza, Serapioni), Harnack inequalities; relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap
Asymptotic rates of convergence

Corollary

Assume that $m \in (0, 1)$, with $m \neq m_* := \frac{n-4}{n-2}$. Under the relative mass condition, for any "good solution" $v$ there exists a positive constant $C$ such that

$$E[v(t)] \leq C \, e^{-2(1-m) t} \quad \forall \ t \geq 0.$$ 

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $L^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates can be obtained using "best matching techniques"
From asymptotic to global estimates

When symmetry holds (CKN) can be written as an entropy – entropy production inequality

$$(2 + \beta - \gamma)^2 \mathcal{E}[v] \leq \frac{m}{1 - m} \mathcal{I}[v]$$

so that

$$\mathcal{E}[v(t)] \leq \mathcal{E}[v(0)] e^{-2(1-m)\Lambda^* t} \quad \forall \ t \geq 0 \quad \text{with} \quad \Lambda^* := \frac{(2 + \beta - \gamma)^2}{2(1 - m)}$$

Let us consider again the entropy – entropy production inequality

$$\mathcal{K}(M) \mathcal{E}[v] \leq \mathcal{I}[v] \quad \forall \ v \in L^{1,\gamma}(\mathbb{R}^d) \quad \text{such that} \quad \|v\|_{L^{1,\gamma}(\mathbb{R}^d)} = M,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M) := \frac{m}{2} (1 - m)^{-2} \mathcal{K}(M)$

$$\mathcal{E}[v(t)] \leq \mathcal{E}[v(0)] e^{-2(1-m)\Lambda(M) t} \quad \forall \ t \geq 0$$
Symmetry breaking and global entropy – entropy production inequalities

**Proposition**

- **In the symmetry breaking range of (CKN), for any** $M > 0$, we have 
  \[0 < K(M) \leq \frac{2}{m} (1 - m)^2 \Lambda_{0,1}\]

- **If symmetry holds in (CKN) then**
  \[K(M) \geq \frac{1-m}{m} (2 + \beta - \gamma)^2\]

**Corollary**

Assume that $m \in [m_1, 1)$

(i) For any $M > 0$, if $\Lambda(M) = \Lambda_\ast$ then $\beta = \beta_{FS}(\gamma)$

(ii) If $\beta > \beta_{FS}(\gamma)$ then $\Lambda_{0,1} < \Lambda_\ast$ and $\Lambda(M) \in (0, \Lambda_{0,1}]$ for any $M > 0$

(iii) For any $M > 0$, if $\beta < \beta_{FS}(\gamma)$ and if symmetry holds in (CKN), then $\Lambda(M) > \Lambda_\ast$
Linearization and optimality

Joint work with M.J. Esteban and M. Loss
Linearization and scalar products

With \( u_\epsilon \) such that

\[
  u_\epsilon = B_\star (1 + \epsilon f B_\star^{1-m}) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\epsilon \, dx = M_\star
\]

at first order in \( \epsilon \to 0 \) we obtain that \( f \) solves

\[
  \frac{\partial f}{\partial t} = \mathcal{L} f \quad \text{where} \quad \mathcal{L} f := (1 - m) B_\star^{m-2} |x|^\gamma \, D_\alpha^* (|x|^{-\beta} B_\star \, D_\alpha f)
\]

Using the scalar products

\[
  \langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1 \, f_2 \, B_\star^{2-m} |x|^{-\gamma} \, dx \quad \text{and} \quad \langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} D_\alpha f_1 \, \cdot \, D_\alpha f_2 \, B_\star \, |x|^{-\beta} \, dx
\]

we compute

\[
  \frac{1}{2} \frac{d}{dt} \langle f, f \rangle = \langle f, \mathcal{L} f \rangle = \int_{\mathbb{R}^d} f \, (\mathcal{L} f) \, B_\star^{2-m} |x|^{-\gamma} \, dx = - \int_{\mathbb{R}^d} |D_\alpha f|^2 \, B_\star \, |x|^{-\beta} \, dx
\]

for any \( f \) smooth enough:

\[
  \frac{1}{2} \frac{d}{dt} \langle f, f \rangle = \int_{\mathbb{R}^d} D_\alpha f \, \cdot \, D_\alpha (\mathcal{L} f) \, B_\star \, |x|^{-\beta} \, dx = - \langle f, \mathcal{L} f \rangle
\]
Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue $\lambda_1$ of $\mathcal{L}$

$$- \mathcal{L} f_1 = \lambda_1 f_1$$

so that $f_1$ realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle g, g \rangle = - \langle g, \mathcal{L} g \rangle \geq \lambda_1 \| g - \bar{g} \|^2, \quad \bar{g} := \langle g, 1 \rangle / \langle 1, 1 \rangle$$

$$- \langle g, \mathcal{L} g \rangle \geq \lambda_1 \langle g, g \rangle$$

Proof: expansion of the square:

$$- \langle (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle = \langle \mathcal{L} (g - \bar{g}), \mathcal{L} (g - \bar{g}) \rangle = \| \mathcal{L} (g - \bar{g}) \|^2$$

Key observation:

$$\lambda_1 \geq 4 \iff \alpha \leq \alpha_{FS} := \sqrt{\frac{d-1}{n-1}}$$
Symmetry breaking in CKN inequalities

Symmetry holds in (CKN) if $\mathcal{J}[w] \geq \mathcal{J}[w_*]$ with

$$\mathcal{J}[w] := \vartheta \log (\|D_\alpha w\|_{L^2,\delta(\mathbb{R}^d)}) + (1-\vartheta) \log (\|w\|_{L^{p+1,\delta}(\mathbb{R}^d)}) - \log (\|w\|_{L^{2p,\delta}(\mathbb{R}^d)})$$

with $\delta := d - n$ and

$$\mathcal{J}[w_* + \varepsilon g] = \varepsilon^2 Q[g] + o(\varepsilon^2)$$

where

$$\frac{2}{\vartheta} \|D_\alpha w_*\|_{L^2, d-n(\mathbb{R}^d)}^2 Q[g]$$

$$= \|D_\alpha g\|_{L^2, d-n(\mathbb{R}^d)}^2 + \frac{p(2+\beta-\gamma)}{(p-1)^2} \left[ d - \gamma - p(d - 2 - \beta) \right] \int_{\mathbb{R}^d} |g|^2 \frac{|x|^{n-d}}{1+|x|^2} \, dx$$

$$- p(2p-1) \frac{(2+\beta-\gamma)^2}{(p-1)^2} \int_{\mathbb{R}^d} |g|^2 \frac{|x|^{n-d}}{(1+|x|^2)^2} \, dx$$

is a nonnegative quadratic form if and only if $\alpha \leq \alpha_{FS}$

Symmetry breaking holds if $\alpha > \alpha_{FS}$
Information — production of information inequality

Let $\mathcal{K}[u]$ be such that

$$
\frac{d}{d\tau} \mathcal{I}[u(\tau, \cdot)] = - \mathcal{K}[u(\tau, \cdot)] = - \text{(sum of squares)}
$$

If $\alpha \leq \alpha_{FS}$, then $\lambda_1 \geq 4$ and

$$
u \mapsto \frac{\mathcal{K}[\nu]}{\mathcal{I}[\nu]} - 4
$$

is a nonnegative functional

With $u_\varepsilon = B_\star (1 + \varepsilon f B_\star^{1-m})$, we observe that

$$
4 \leq C_2 := \inf_{\nu} \frac{\mathcal{K}[\nu]}{\mathcal{I}[\nu]} \leq \lim_{\varepsilon \to 0} \inf_{f} \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_{f} \frac{\langle f, \mathcal{L} f \rangle}{\langle f, f \rangle} = \frac{\langle f_1, \mathcal{L} f_1 \rangle}{\langle f_1, f_1 \rangle} = \lambda_1
$$

- if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{FS}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to B_\star$ and determined by the spectral gap of $\mathcal{L}$
- if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{FS}$, then $\mathcal{K}/\mathcal{I} > 4$
If \( \alpha \leq \alpha_{FS} \), the fact that \( \mathcal{K}/\mathcal{I} \geq 4 \) has an important consequence. Indeed we know that

\[
\frac{d}{d\tau} \left( \mathcal{I}[u(\tau, \cdot)] - 4 \mathcal{E}[u(\tau, \cdot)] \right) \leq 0
\]

so that

\[
\mathcal{I}[u] - 4 \mathcal{E}[u] \geq \mathcal{I}[\mathcal{B}_*] - 4 \mathcal{E}[\mathcal{B}_*] = 0
\]

This inequality is equivalent to \( \mathcal{J}[w] \geq \mathcal{J}[w_*] \), which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for \( \alpha \leq \alpha_{FS} \), the function

\[
\tau \mapsto \mathcal{I}[u(\tau, \cdot)] - 4 \mathcal{E}[u(\tau, \cdot)]
\]

is monotone decreasing.

This explains why the method based on nonlinear flows provides the optimal range for symmetry.
Entropy – production of entropy inequality

Using \( \frac{d}{d\tau} (\mathcal{I}[u(\tau, \cdot)] - C_2 \mathcal{E}[u(\tau, \cdot)]) \leq 0 \), we know that

\[
\mathcal{I}[u] - C_2 \mathcal{E}[u] \geq \mathcal{I}[B_\star] - C_2 \mathcal{E}[B_\star] = 0
\]

As a consequence, we have that

\[
C_1 := \inf_u \frac{\mathcal{I}[u]}{\mathcal{E}[u]} \geq C_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]}
\]

With \( u_\varepsilon = B_\star (1 + \varepsilon f B_\star^{-m}) \), we observe that

\[
C_1 \leq \lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{I}[u_\varepsilon]}{\mathcal{E}[u_\varepsilon]} = \inf_f \frac{\langle f, \mathcal{L} f \rangle}{\langle f, f \rangle} = \frac{\langle f_1, \mathcal{L} f_1 \rangle}{\langle f_1, f_1 \rangle} = \lambda_1 = \lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]}
\]

If \( \lim_{\varepsilon \to 0} \inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = C_2 \), then \( C_1 = C_2 = \lambda_1 \)

This happens if \( \alpha = \alpha_{\text{FS}} \) and in particular in the case without weights (Gagliardo-Nirenberg inequalities)
These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
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Thank you for your attention!