Nonlinear flows, functional inequalities and applications

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GDR Dynqua Annual meeting

Scope (1/4): rigidity results

Rigidity results for semilinear elliptic PDEs on manifolds...

Let $(\mathfrak{M}, g)$ be a smooth compact Riemannian manifold
of dimension $d \geq 2$, no boundary, Δ_g is the Laplace-Beltrami operator
the Ricci tensor $\mathfrak{R}$ has good properties (which ones ?)

Let $p \in (2, 2^*)$, with $2^* = \frac{2\,d}{d-2}$ if $d \geq 3$, $2^* = \infty$ if $d = 2$

For which values of $\lambda > 0$ the equation

$$-\Delta_g v + \lambda\, v = v^{p-1}$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv \lambda^{\frac{1}{p-2}}$?

A typical *rigidity result* is: there exists $\lambda_0 > 0$ such that
$v \equiv \lambda^{\frac{2}{p-2}}$ if $\lambda \in (0, \lambda_0]$

Assumptions ?
Optimal λ_0 ?

Scope (2/4): interpolation inequalities

Still on a smooth compact Riemannian manifold $(\mathfrak{M}, g)$ we assume that $\mathrm{vol}_g(\mathfrak{M}) = 1$

For any $p \in (1,2) \cup (2, 2^*)$ or $p = 2^*$ if $d \geq 3$, consider the *interpolation inequality*

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})} \right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

What is the largest possible value of λ ?

- using $u = 1 + \varepsilon\,\varphi$ as a test function proves that $\lambda \leq \lambda_1$
- the minimum of $v \mapsto \|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} - \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})} \right]$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if λ is above the rigidity threshold

- the threshold case $p = 2$ is the *logarithmic Sobolev inequality*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \lambda \int_{\mathfrak{M}} u^2 \, \log\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\mathfrak{M})}} \right) dv_g \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})$$

Scope (3/4): flows

We shall consider a flow of porous media / fast diffusion type

$$u_t = u^{2-2\beta}\left(\Delta_g u + \kappa\,\frac{|\nabla u|^2}{u}\right)\,,\quad \kappa = 1+\beta\,(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt}\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2\,d\,v_g + \frac{\lambda}{p-2}\left[\int_{\mathfrak{M}} u^{2\,\beta}\,d\,v_g - \left(\int_{\mathfrak{M}} u^{\beta\,p}\,d\,v_g\right)^{2/p}\right]$$

is monotone decaying as long as λ is not too big. Hence, if the limit as $t\to\infty$ is 0 (convergence to the constants), we know that $\mathcal{F}[u]\ge 0$

Structure ? Link with computations in the rigidity approach

A collaboration mostly with M. Esteban and M. Loss

Scope (4/4): spectral estimates

1. Sharp interpolation inequalities are equivalent, by duality, to sharp estimates on the lowest eigenvalues of Schrödinger operators, the so-called *Keller-Lieb-Thirring inequalities*
2. These spectral estimate differ from semi-classical inequalities because they take into account *finite volume* effects. The semi-classical regime is recovered only in the limit of large potentials

A collaboration mostly with M. Esteban, A. Laptev, and M. Loss

Some references (1/2)

Some references (incomplete) and *goals*

1. rigidity results and elliptic PDEs: [Gidas-Spruck 1981], [Bidaut-Véron & Véron 1991], [Licois & Véron 1995]
   $\longrightarrow$ *systematize and clarify the strategy*
2. semi-group approach and Γ_2 or *carré du champ* method: [Bakry-Emery 1985], [Bakry & Ledoux 1996], [Bentaleb et al., 1993-2010], [Fontenas 1997], [Brouttelande 2003], [Demange, 2005 & 2008]
   $\longrightarrow$ *emphasize the role of the flow, get various improvements*
   $\longrightarrow$ *get rid of pointwise constraints on the curvature, discuss optimality*
3. harmonic analysis, duality and spectral theory: [Lieb 1983], [Beckner 1993]
   $\longrightarrow$ *apply results to get new spectral estimates*

Outline

1. The case of the sphere
   - Inequalities on the sphere
   - Flows on the sphere
   - Spectral consequences
   - Improved inequalities
2. The case of Riemannian manifolds
   - Flows
   - Spectral consequences
3. Inequalities on the line
   - Variational approaches
   - Mass transportation
   - Flows
4. The Moser-Trudinger-Onofri inequality... + another flow

Joint work with:

M.J. Esteban, G. Jankowiak, M. Kowalczyk, A. Laptev and M. Loss

The sphere

- The case of the sphere as a simple example

Inequalities on the sphere

A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere:

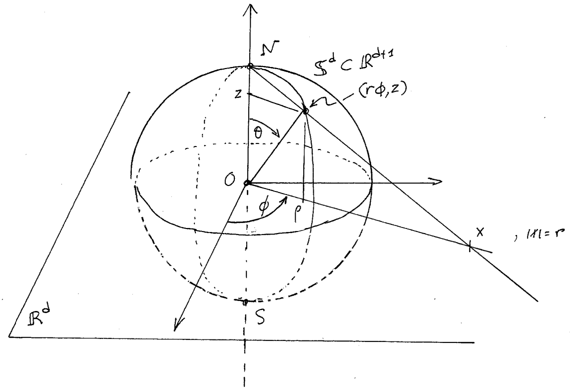
$$\frac{p-2}{d}\int_{\mathbb{S}^d}|\nabla u|^2\,d\,v_g+\int_{\mathbb{S}^d}|u|^2\,d\,v_g\geq\left(\int_{\mathbb{S}^d}|u|^p\,d\,v_g\right)^{2/p}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,dv_g)$$

- for any $p\in(2,2^*]$ with $2^*=\frac{2\,d}{d-2}$ if $d\geq3$
- for any $p\in(2,\infty)$ if $d=2$

Here dv_g is the uniform probability measure: $v_g(\mathbb{S}^d)=1$

- 1 is the optimal constant, equality achieved by constants
- $p=2^*$ corresponds to Sobolev's inequality...

Stereographic projection



Sobolev inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho\,\phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \geq 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2 - 1}{r^2 + 1} = 1 - \frac{2}{r^2 + 1}\,, \quad \rho = \frac{2\,r}{r^2 + 1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\, v(x)$$

- $p = 2^*$, $\mathsf{S}_d = \frac{1}{4}\, d\,(d-2)\, |\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \geq \mathsf{S}_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right]^{\frac{d-2}{d}} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

Extended inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq \frac{d}{p-2} \left[\left(\int_{\mathbb{S}^d} |u|^p \, d\,v_g \right)^{2/p} - \int_{\mathbb{S}^d} |u|^2 \, d\,v_g \right] \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

is valid

- for any $p \in (1,2) \cup (2,\infty)$ if $d = 1,\, 2$
- for any $p \in (1,2) \cup (2, 2^*]$ if $d \geq 3$

- Case $p = 2$: Logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 \, d\,v_g} \right) d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

- Case $p = 1$: Poincaré inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq d \int_{\mathbb{S}^d} |u - \bar{u}|^2 \, d\,v_g \quad \text{with} \quad \bar{u} := \int_{\mathbb{S}^d} u \, d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

A spectral approach when $p \in (1,2)$ – $1^{\rm st}$ step

[Dolbeault-Esteban-Kowalczyk-Loss] adapted from [Beckner] (case of Gaussian measures).

Nelson's hypercontractivity result. Consider the heat equation

$$\frac{\partial f}{\partial t} = \Delta_g f$$

with initial datum $f(t=0,\cdot) = u \in L^{2/p}(\mathbb{S}^d)$, for some $p \in (1,2]$, and let $F(t) := \|f(t,\cdot)\|_{\mathrm{L}^{p(t)}(\mathbb{S}^d)}$. The key computation goes as follows.

$$\frac{F'}{F} = \frac{p'}{p^2\,F^p}\left[\int_{\mathbb{S}^d} v^2 \log\left(\frac{v^2}{\int_{\mathbb{S}^d} v^2\,d\,v_g}\right) d\,v_g + 4\,\frac{p-1}{p'}\int_{\mathbb{S}^d} |\nabla v|^2\,d\,v_g\right]$$

with $v := |f|^{p(t)/2}$. With $4\,\frac{p-1}{p'} = \frac{2}{d}$ and $t_* > 0$ e such that $p(t_*) = 2$, we have

$$\|f(t_*,\cdot)\|_{\mathrm{L}^2(\mathbb{S}^d)} \le \|u\|_{\mathrm{L}^{2/p}(\mathbb{S}^d)} \quad \text{if} \quad \frac{1}{p-1} = e^{2\,d\,t_*}$$

A spectral approach when $p \in (1,2)$ – $2^{\rm nd}$ step

Spectral decomposition. Let $u = \sum_{k \in \mathbb{N}} u_k$ be a spherical harmonics decomposition, $\lambda_k = k\,(d+k-1)$, $a_k = \|u_k\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$ so that

$$\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k \in \mathbb{N}} a_k \text{ and } \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k \in \mathbb{N}} \lambda_k\, a_k$$

$$\|f(t_*,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = \sum_{k \in \mathbb{N}} a_k\, e^{-2\,\lambda_k\, t_*}$$

$$\begin{aligned}
\frac{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}{2-p} &\le \frac{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|f(t_*,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{2-p} \\
&= \frac{1}{2-p} \sum_{k \in \mathbb{N}^*} \lambda_k\, a_k\, \frac{1-e^{-2\,\lambda_k\, t_*}}{\lambda_k} \\
&\le \frac{1-e^{-2\,\lambda_1\, t_*}}{(2-p)\,\lambda_1} \sum_{k \in \mathbb{N}^*} \lambda_k\, a_k = \frac{1-e^{-2\,\lambda_1\, t_*}}{(2-p)\,\lambda_1}\, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}
\end{aligned}$$

The conclusion easily follows if we notice that $\lambda_1 = d$, and $e^{-2\,\lambda_1\, t_*} = p-1$ so that $\frac{1-e^{-2\,\lambda_1\, t_*}}{(2-p)\,\lambda_1} = \frac{1}{d}$

Optimality: a perturbation argument

- The optimality of the constant can be checked by a Taylor expansion of $u = 1 + \varepsilon\, v$ at order two in terms of $\varepsilon > 0$, small
- For any $p \in (1, 2^*]$ if $d \geq 3$, any $p > 1$ if $d = 1$ or 2, it is remarkable that

$$\mathcal{Q}[u] := \frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \geq \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)} \mathcal{Q}[u] = \frac{1}{d}$$

is achieved by $\mathcal{Q}[1 + \varepsilon\, v]$ as $\varepsilon \to 0$ and v is an eigenfunction associated with the first nonzero eigenvalue of Δ_g
- $p > 2$: no simple proof based on spectral analysis is available: [Beckner], an approach based on Lieb's duality, the Funk-Hecke formula and some (non-trivial) computations
- elliptic methods / Γ_2 formalism of Bakry-Emery / flow... they are the same (main contribution) and can be simplified (!) As a side result, you can go beyond these approaches and discuss optimality

Schwarz symmetry and the ultraspherical setting

$(\xi_0,\,\xi_1,\dots\xi_d)\in\mathbb{S}^d$, $\xi_d=z$, $\sum_{i=0}^d|\xi_i|^2=1$ [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on $\xi_d=z$

- Let $d\sigma(\theta):=\frac{(\sin\theta)^{d-1}}{Z_d}\,d\theta$, $Z_d:=\sqrt{\pi}\,\frac{\Gamma(\frac d2)}{\Gamma(\frac{d+1}{2})}$: $\forall\,v\in\mathrm{H}^1([0,\pi],d\sigma)$

$$\frac{p-2}{d}\int_0^\pi|v'(\theta)|^2\,d\sigma+\int_0^\pi|v(\theta)|^2\,d\sigma\ge\left(\int_0^\pi|v(\theta)|^p\,d\sigma\right)^{\frac2p}$$

- Change of variables $z=\cos\theta$, $v(\theta)=f(z)$

$$\frac{p-2}{d}\int_{-1}^1|f'|^2\,\nu\,d\nu_d+\int_{-1}^1|f|^2\,d\nu_d\ge\left(\int_{-1}^1|f|^p\,d\nu_d\right)^{\frac2p}$$

where $\nu_d(z)\,dz=d\nu_d(z):=Z_d^{-1}\,\nu^{\frac d2-1}\,dz$, $\nu(z):=1-z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$, consider the space $\mathrm{L}^2((-1,1), d\nu_d)$ with scalar product

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1\, f_2\, d\nu_d\,, \quad \|f\|_p = \left(\int_{-1}^{1} f^p\, d\nu_d\right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\,f := (1 - z^2)\,f'' - d\,z\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1, \mathcal{L}\,f_2 \rangle = -\int_{-1}^{1} f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p \in [1,2) \cup (2, 2^*]$, $d \geq 1$

$$-\langle f, \mathcal{L}\,f \rangle = \int_{-1}^{1} |f'|^2\,\nu\,d\nu_d \geq d\,\frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \quad \forall\,f \in \mathrm{H}^1([-1,1], d\nu_d)$$

Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents

Heat flow and the Bakry-Emery method

With $g=f^p$, *i.e.* $f=g^\alpha$ with $\alpha=1/p$

$$\text{(Ineq.)}\qquad -\langle f,\mathcal L\,f\rangle=-\langle g^\alpha,\mathcal L\,g^\alpha\rangle=:\mathcal I[g]\ge d\,\frac{\|g\|_1^{2\,\alpha}-\|g^{2\,\alpha}\|_1}{p-2}=:\mathcal F[g]$$

Heat flow

$$\frac{\partial g}{\partial t}=\mathcal L\,g$$

$$\frac{d}{dt}\,\|g\|_1=0\,,\quad \frac{d}{dt}\,\|g^{2\,\alpha}\|_1=-\,2\,(p-2)\,\langle f,\mathcal L\,f\rangle=2\,(p-2)\int_{-1}^1|f'|^2\,\nu\,d\nu_d$$

which finally gives

$$\frac{d}{dt}\mathcal F[g(t,\cdot)]=-\frac{d}{p-2}\,\frac{d}{dt}\,\|g^{2\,\alpha}\|_1=-\,2\,d\,\mathcal I[g(t,\cdot)]$$

$$\text{Ineq.}\iff\frac{d}{dt}\mathcal F[g(t,\cdot)]\le-\,2\,d\,\mathcal F[g(t,\cdot)]\Longleftarrow\frac{d}{dt}\mathcal I[g(t,\cdot)]\le-\,2\,d\,\mathcal I[g(t,\cdot)]$$

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\,\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} < \frac{2\,d}{d-2} = 2^*$$

... up to the critical exponent: a proof on two slides

$$\left[\frac{d}{dz},\mathcal{L}\right]u=(\mathcal{L}\,u)'-\mathcal{L}\,u'=-2\,z\,u''-d\,u'$$

$$\int_{-1}^{1}(\mathcal{L}\,u)^2\,d\nu_d=\int_{-1}^{1}|u''|^2\,\nu^2\,d\nu_d+d\int_{-1}^{1}|u'|^2\,\nu\,d\nu_d$$

$$\int_{-1}^{1}(\mathcal{L}\,u)\,\frac{|u'|^2}{u}\,\nu\,d\nu_d=\frac{d}{d+2}\int_{-1}^{1}\frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d-2\,\frac{d-1}{d+2}\int_{-1}^{1}\frac{|u'|^2\,u''}{u}\,\nu^2\,d\nu_d$$

On $(-1,1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t=u^{2-2\beta}\left(\mathcal{L}\,u+\kappa\,\frac{|u'|^2}{u}\,\nu\right)$$

If $\kappa=\beta\,(p-2)+1$, the L^p norm is conserved

$$\frac{d}{dt}\int_{-1}^{1}u^{\beta p}\,d\nu_d=\beta\,p\,(\kappa-\beta\,(p-2)-1)\int_{-1}^{1}u^{\beta(p-2)}\,|u'|^2\,\nu\,d\nu_d=0$$

$f = u^\beta$, $\|f'\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right) \geq 0$?

$$\mathcal{A} := -\frac{1}{2\,\beta^2}\,\frac{d}{dt}\int_{-1}^{1}\left(|(u^\beta)'|^2\,\nu + \frac{d}{p-2}\,\left(u^{2\beta} - \overline{u}^{2\beta}\right)\right)\,d\nu_d$$

$$= \int_{-1}^{1}\left(\mathcal{L}\,u + (\beta-1)\,\frac{|u'|^2}{u}\,\nu\right)\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)\,d\nu_d$$

$$+ \frac{d}{p-2}\,\frac{\kappa-1}{\beta}\int_{-1}^{1}|u'|^2\,\nu\;d\nu_d$$

$$= \int_{-1}^{1}|u''|^2\,\nu^2\;d\nu_d - 2\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\int_{-1}^{1}u''\,\frac{|u'|^2}{u}\,\nu^2\;d\nu_d$$

$$+ \left[\kappa\,(\beta-1) + \frac{d}{d+2}\,(\kappa+\beta-1)\right]\int_{-1}^{1}\frac{|u'|^4}{u^2}\,\nu^2\;d\nu_d$$

$$= \int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d \geq 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

$\mathcal{A}$ is nonnegative for some β if $\dfrac{8\,d^2}{(d+2)^2}\,(p-1)\,(2^*-p) \geq 0$

the rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$

$$-\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with the two-homogenous terms

Spectral consequences

- A quantitative deviation with respect to the semi-classical regime

Some references (2/2)

Consider the Schrödinger operator $H = -\Delta - V$ on $\mathbb{R}^d$ and denote by $(\lambda_k)_{k\geq 1}$ its eigenvalues

- Euclidean case [Keller, 1961]

$$|\lambda_1|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

[Lieb-Thirring, 1976]

$$\sum_{k\geq 1} |\lambda_k|^\gamma \leq \mathrm{L}_{\gamma,d} \int_{\mathbb{R}^d} V_+^{\gamma+\frac{d}{2}}$$

$\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$ [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [Dolbeault-Felmer-Loss-Paturel]... [Dolbeault-Laptev-Loss 2008]

- Compact manifolds: log Sobolev case: [Federbusch], [Rothaus]; case $\gamma = 0$ (Rozenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]

An interpolation inequality (I)

Lemma (Dolbeault-Esteban-Laptev)

Let $q \in (2, 2^*)$. *Then there exists a concave increasing function* $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ *with the following properties*

$$\mu(\alpha) = \alpha \quad \forall\, \alpha \in \left[0, \tfrac{d}{q-2}\right] \quad \textit{and} \quad \mu(\alpha) < \alpha \quad \forall\, \alpha \in \left(\tfrac{d}{q-2}, +\infty\right)$$

$$\mu(\alpha) = \mu_{\rm asymp}(\alpha)\,(1 + o(1)) \quad \textit{as} \quad \alpha \to +\infty\,, \quad \mu_{\rm asymp}(\alpha) := \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\,\alpha^{1-\vartheta}$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\alpha)\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

If $d \ge 3$ *and* $q = 2^*$, *the inequality holds with* $\mu(\alpha) = \min\{\alpha, \alpha_*\}$, $\alpha_* := \frac{1}{4}\,d\,(d-2)$

- $\mu_{\rm asymp}(\alpha) := \frac{\mathsf{K}_{q,d}}{\kappa_{q,d}}\,\alpha^{1-\vartheta}$, $\vartheta := d\,\frac{q-2}{2\,q}$ corresponds to the *semi-classical regime* and $\mathsf{K}_{q,d}$ is the optimal constant in the *Euclidean* Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}_{q,d}\,\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d)$$

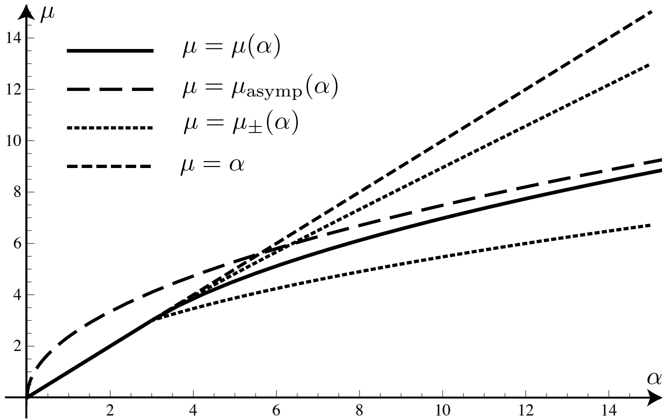
- Let φ be a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue

$$-\,\Delta\varphi = d\,\varphi$$

Consider $u = 1 + \varepsilon\,\varphi$ as $\varepsilon \to 0$ Taylor expand $\mathcal{Q}_\alpha$ around $u = 1$

$$\mu(\alpha) \le \mathcal{Q}_\alpha[1 + \varepsilon\,\varphi] = \alpha + \left[d + \alpha\,(2-q)\right]\varepsilon^2 \int_{\mathbb{S}^d} |\varphi|^2\,d\,v_g + o(\varepsilon^2)$$

By taking ε small enough, we get $\mu(\alpha) < \alpha$ for all $\alpha > d/(q-2)$
Optimizing on the value of $\varepsilon > 0$ (not necessarily small) provides an interesting test function...


μ
$\mu = \mu(\alpha)$
$\mu = \mu_{\rm asymp}(\alpha)$
$\mu = \mu_\pm(\alpha)$
$\mu = \alpha$
2
4
6
8
10
12
14
α

Consider the Schrödinger operator $-\Delta - V$ and the energy

$$\mathcal{E}[u] := \int_{\mathbb{S}^d} |\nabla u|^2 - \int_{\mathbb{S}^d} V\,|u|^2$$
$$\ge \int_{\mathbb{S}^d} |\nabla u|^2 - \mu\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \ge -\,\alpha(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{if } \mu = \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}$$

Theorem (Dolbeault-Esteban-Laptev)

Let $d \ge 1$, $p \in \big(\max\{1, d/2\}, +\infty\big)$. Then there exists a convex increasing function α s.t. $\alpha(\mu) = \mu$ if $\mu \in \big[0, \frac{d}{2}\,(p-1)\big]$ and $\alpha(\mu) > \mu$ if $\mu \in \big(\frac{d}{2}\,(p-1), +\infty\big)$

$$|\lambda_1(-\Delta - V)| \le \alpha\big(\|V\|_{\mathrm{L}^p(\mathbb{S}^d)}\big) \quad \forall\, V \in \mathrm{L}^p(\mathbb{S}^d)$$

For large values of μ, we have $\alpha(\mu)^{p-\frac{d}{2}} = \mathrm{L}^1_{p-\frac{d}{2},d}\,(\kappa_{q,d}\,\mu)^p\,(1+o(1))$ and the above estimate is optimal
If $p = d/2$ and $d \ge 3$, the inequality holds with $\alpha(\mu) = \mu$ iff $\mu \in [0, \alpha_*]$

A Keller-Lieb-Thirring inequality

Corollary (Dolbeault-Esteban-Laptev)

Let $d \geq 1, \gamma = p - d/2$

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim \mathrm{L}^1_{\gamma,d} \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}} \quad \text{as} \quad \mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)} \to \infty$$

if either $\gamma > \max\{0, 1 - d/2\}$ *or* $\gamma = 1/2$ *and* $d = 1$

However, if $\mu = \|V\|_{\mathrm{L}^{\gamma+\frac{d}{2}}(\mathbb{S}^d)} \leq \frac{1}{4}\, d\,(2\,\gamma + d - 2)$, *then we have*

$$|\lambda_1(-\Delta - V)|^{\gamma+\frac{d}{2}} \leq \int_{\mathbb{S}^d} V^{\gamma+\frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ *and this estimate is optimal*

$\mathrm{L}^1_{\gamma,d}$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq \mathrm{L}^1_{\gamma,d} \int_{\mathbb{R}^d} \phi_+^{\gamma+\frac{d}{2}}\, dx$$

Another interpolation inequality (II)

Let $d \geq 1$ and $\gamma > d/2$ and assume that $\mathrm{L}^1_{-\gamma,d}$ is the optimal constant in

$$\lambda_1(-\Delta+\phi)^{-\gamma} \leq \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{R}^d} \phi^{\frac{d}{2}-\gamma}\,dx$$

$$q = 2\,\frac{2\,\gamma-d}{2\,\gamma-d+2} \quad \text{and} \quad p = \frac{q}{2-q} = \gamma - \frac{d}{2}$$

Theorem (Dolbeault-Esteban-Laptev)

$$\big(\lambda_1(-\Delta+W)\big)^{-\gamma} \lesssim \mathrm{L}^1_{-\gamma,d} \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma} \quad \textit{as} \quad \beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{S}^d)} \to \infty$$

However, if $\gamma \geq \frac{d}{2}+1$ *and* $\beta = \|W^{-1}\|^{-1}_{\mathrm{L}^{\gamma-\frac{d}{2}}(\mathbb{S}^d)} \leq \frac{1}{4}\,d\,(2\,\gamma-d+2)$

$$\big(\lambda_1(-\Delta+W)\big)^{\frac{d}{2}-\gamma} \leq \int_{\mathbb{S}^d} W^{\frac{d}{2}-\gamma}$$

and this estimate is optimal

$\mathsf{K}^*_{q,d}$ is the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

$$\mathsf{K}^*_{q,d}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)} \quad \forall\, v \in \mathrm{H}^1(\mathbb{R}^d)$$

and $\mathcal{L}^{\,1}_{-\gamma,d} := \left(\mathsf{K}^*_{q,d}\right)^{-\gamma}$ with $q = 2\,\frac{2\,\gamma-d}{2\,\gamma-d+2}$, $\delta := \frac{2\,q}{2\,d-q\,(d-2)}$

Lemma (Dolbeault-Esteban-Laptev)

Let $q \in (0,2)$ and $d \ge 1$. There exists a concave increasing function ν

$$\nu(\beta) \le \beta \quad \forall\,\beta > 0 \quad \textit{and} \quad \nu(\beta) < \beta \quad \forall\,\beta \in \left(\tfrac{d}{2-q}, +\infty\right)$$

$$\nu(\beta) = \beta \quad \forall\,\beta \in \left[0, \tfrac{d}{2-q}\right] \quad \textit{if} \quad q \in [1,2)$$

$$\nu(\beta) = \mathsf{K}^*_{q,d}\,\left(\kappa_{q,d}\,\beta\right)^\delta\,(1+o(1)) \quad \textit{as} \quad \beta \to +\infty$$

such that

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \beta\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \ge \nu(\beta)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

The threshold case: $q = 2$

Lemma (Dolbeault-Esteban-Laptev)

Let $p > \max\{1, d/2\}$. *There exists a concave nondecreasing function* ξ

$$\xi(\alpha) = \alpha \quad \forall\, \alpha \in (0, \alpha_0) \quad \textit{and} \quad \xi(\alpha) < \alpha \quad \forall\, \alpha > \alpha_0$$

for some $\alpha_0 \in \left[\frac{d}{2}\,(p-1), \frac{d}{2}\,p\right]$, *and* $\xi(\alpha) \sim \alpha^{1-\frac{d}{2p}}$ *as* $\alpha \to +\infty$ *such that, for any* $u \in \mathrm{H}^1(\mathbb{S}^d)$ *with* $\|u\|_{\mathrm{L}^2(\mathbb{S}^d)} = 1$

$$\int_{\mathbb{S}^d} |u|^2 \log |u|^2 \, d\,v_g + p\,\log\left(\tfrac{\xi(\alpha)}{\alpha}\right) \le p\,\log\left(1 + \tfrac{1}{\alpha}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Corollary (Dolbeault-Esteban-Laptev)

$$e^{-\,\lambda_1(-\Delta - W)/\alpha} \le \frac{\alpha}{\xi(\alpha)}\left(\int_{\mathbb{S}^d} e^{-\,p\,W/\alpha}\, d\,v_g\right)^{1/p}$$

Improvements of the inequalities (subcritical range)

- as long as the exponent is either in the range $(1,2)$ or in the range $(2,2^*)$, on can establish *improved inequalities*

[Dolbeault-Esteban-Kowalczyk-Loss]

What does "improvement" mean ?

An *improved* inequality is

$$d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\Phi\left(\frac{\mathsf{e}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)\le \mathsf{i}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

for some function Φ such that $\Phi(0)=0$, $\Phi'(0)=1$, $\Phi'>0$ and $\Phi(s)>s$ for any s. With $\Psi(s):=s-\Phi^{-1}(s)$

$$\mathsf{i}-d\,\mathsf{e}\ge d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,(\Psi\circ\Phi)\left(\frac{\mathsf{e}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

Lemma (Generalized Csiszár-Kullback inequalities)

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\frac{d}{p-2}\left[\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right]$$

$$\ge d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,(\Psi\circ\Phi)\left(\mathsf{C}\,\frac{\|u\|^{2\,(1-r)}_{\mathrm{L}^s(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,\|u^r-\bar{u}^r\|^2_{\mathrm{L}^q(\mathbb{S}^d)}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

$s(p):=\max\{2,p\}$ and $p\in(1,2)$: $q(p):=2/p$, $r(p):=p$; $p\in(2,4)$: $q=p/2$, $r=2$; $p\ge4$: $q=p/(p-2)$, $r=p-2$

Linear flow: improved Bakry-Emery method

Cf. [Arnold, JD]

$$w_t = \mathcal{L}\, w + \kappa\, \frac{|w'|^2}{w}\, \nu$$

With $2^\sharp := \frac{2\,d^2+1}{(d-1)^2}$

$$\gamma_1 := \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p) \quad \text{if} \quad d > 1\,, \qquad \gamma_1 := \frac{p-1}{3} \quad \text{if} \quad d = 1$$

If $p \in [1,2) \cup (2, 2^\sharp]$ and w is a solution, then

$$\frac{d}{dt}\,(\mathsf{i} - \,d\,\mathsf{e}) \le -\,\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2}\; d\nu_d \le -\,\gamma_1\, \frac{|\mathsf{e}'|^2}{1-\,(p-2)\,\mathsf{e}}$$

Recalling that $\mathsf{e}' = -\,\mathsf{i}$, we get a differential inequality

$$\mathsf{e}'' + \,d\,\mathsf{e}' \ge \gamma_1\, \frac{|\mathsf{e}'|^2}{1-\,(p-2)\,\mathsf{e}}$$

After integration: $d\,\Phi(\mathsf{e}(0)) \le \mathsf{i}(0)$

Nonlinear flow: the Hölder estimate

$$w_t = w^{2-2\beta}\left(\mathcal{L}\,w + \kappa\,\frac{|w'|^2}{w}\right)$$

For all $p \in [1, 2^*]$, $\kappa = \beta\,(p-2)+1$, $\frac{d}{dt}\int_{-1}^{1} w^{\beta p}\,d\nu_d = 0$

$$-\frac{1}{2\beta^2}\,\frac{d}{dt}\int_{-1}^{1}\left(|(w^\beta)'|^2\,\nu + \frac{d}{p-2}\,\left(w^{2\beta}-\overline{w}^{2\beta}\right)\right)d\nu_d \ge \gamma\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d$$

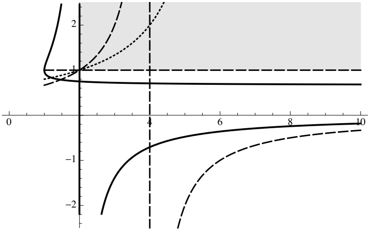
Lemma

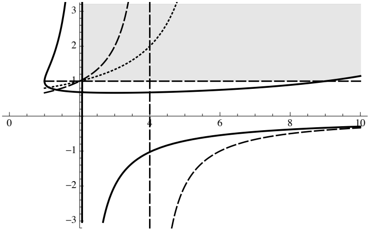
For all $w \in \mathrm{H}^1\big((-1,1), d\nu_d\big)$, *such that* $\int_{-1}^{1} w^{\beta p}\,d\nu_d = 1$

$$\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d \ge \frac{1}{\beta^2}\,\frac{\int_{-1}^{1}|(w^\beta)'|^2\,\nu\,d\nu_d\int_{-1}^{1}|w'|^2\,\nu\,d\nu_d}{\left(\int_{-1}^{1} w^{2\beta}\,d\nu_d\right)^{\delta}}$$

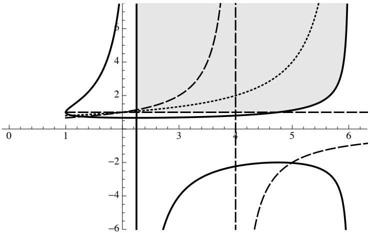
.... but there are conditions on β

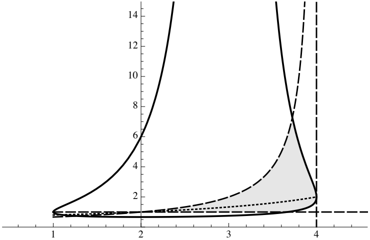
Admissible (p, β) for $d = 1, 2$



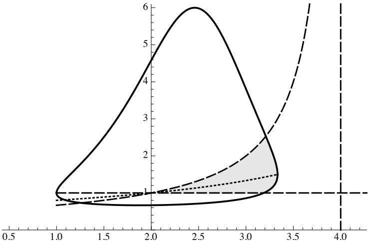


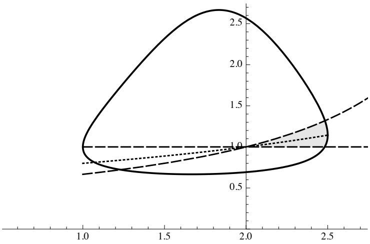
Admissible (p, β) for $d = 3, 4$





Admissible (p, β) for $d = 5, 10$





Riemannian manifolds

- no sign is required on the Ricci tensor and an improved integral criterion is established

- the flow explores the energy landscape... and shows the non-optimality of the improved criterion

Riemannian manifolds with positive curvature

$(\mathfrak{M}, g)$ is a smooth compact connected Riemannian manifold dimension d, no boundary, Δ_g is the Laplace-Beltrami operator $\mathrm{vol}(\mathfrak{M}) = 1$, $\mathfrak{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi, \xi)$$

Theorem (Licois-Véron, Bakry-Ledoux)

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \quad \textit{where} \quad \theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\Delta_g v + \frac{\lambda}{p-2}\,\left(v - v^{p-1}\right) = 0$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv 1$

Riemannian manifolds: first improvement

Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1,2) \cup (2,2^*)$

$$0 < \lambda < \lambda_\star = \inf_{u \in \mathrm{H}^2(\mathfrak{M})} \frac{\int_{\mathfrak{M}} \left[(1-\theta)\,(\Delta_g u)^2 + \frac{\theta\, d}{d-1}\,\mathfrak{R}(\nabla u, \nabla u) \right] d\, v_g}{\int_{\mathfrak{M}} |\nabla u|^2\, d\, v_g}$$

there is a unique positive solution in $C^2(\mathfrak{M})$: $u \equiv 1$

$\lim_{p \to 1_+} \theta(p) = 0 \Longrightarrow \lim_{p \to 1_+} \lambda_\star(p) = \lambda_1$ if ρ is bounded
$\lambda_\star = \lambda_1 = d\,\rho/(d-1) = d$ if $\mathfrak{M} = \mathbb{S}^d$ since $\rho = d-1$

$$(1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \le \lambda_\star \le \lambda_1$$

Riemannian manifolds: second improvement

$\mathrm{H}_g u$ denotes Hessian of u and $\theta = \dfrac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1}$

$$\mathrm{Q}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u - \frac{(d-1)\,(p-1)}{\theta\,(d+3-p)}\left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

$$\Lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{(1-\theta)\displaystyle\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g + \frac{\theta\,d}{d-1}\int_{\mathfrak{M}} \left[\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right]}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g}$$

Theorem (Dolbeault-Esteban-Loss)

Assume that $\Lambda_\star > 0$. For any $p \in (1,2) \cup (2,2^*)$, the equation has a unique positive solution in $C^2(\mathfrak{M})$ if $\lambda \in (0, \Lambda_\star)$: $u \equiv 1$

Optimal interpolation inequality

For any $p \in (1,2) \cup (2,2^*)$ or $p = 2^*$ if $d \geq 3$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})} \right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

Theorem (Dolbeault-Esteban-Loss)

*Assume $\Lambda_\star > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_\star, \lambda_1]$
If $\Lambda_\star < \lambda_1$, then the optimal constant Λ is such that*

$$\Lambda_\star < \Lambda \leq \lambda_1$$

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \varepsilon\,\varphi$ as a test function where φ we get $\lambda \leq \lambda_1$
A minimum of

$$v \mapsto \|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} - \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})} \right]$$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if $\lambda > \lambda_1$

The flow

The key tools the flow

$$u_t = u^{2-2\,\beta}\left(\Delta_g u + \kappa\,\frac{|\nabla u|^2}{u}\right)\,,\quad \kappa = 1 + \beta\,(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt}\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2\,d\,v_g + \frac{\lambda}{p-2}\left[\int_{\mathfrak{M}} u^{2\,\beta}\,d\,v_g - \left(\int_{\mathfrak{M}} u^{\beta\,p}\,d\,v_g\right)^{2/p}\right]$$

is monotone decaying

J. Demange, *Improved Gagliardo-Nirenberg-Sobolev inequalities on manifolds with positive curvature*, J. Funct. Anal., 254 (2008), pp. 593–611. Also see C. Villani, *Optimal Transport, Old and New*

Elementary observations (1/2)

Let $d \geq 2$, $u \in C^2(\mathfrak{M})$, and consider the trace free Hessian

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d} \Delta_g u$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, d v_g = \frac{d}{d-1} \int_{\mathfrak{M}} \| \mathrm{L}_g u \|^2 \, d v_g + \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) \, d v_g$$

Based on the Bochner-Lichnerovicz-Weitzenböck formula

$$\frac{1}{2} \Delta |\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u) \cdot \nabla u + \mathfrak{R}(\nabla u, \nabla u)$$

Elementary observations (2/2)

Lemma

$$\int_{\mathfrak{M}} \Delta_g u \,\frac{|\nabla u|^2}{u}\,d\,v_g$$
$$= \frac{d}{d+2}\int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2}\,d\,v_g - \frac{2\,d}{d+2}\int_{\mathfrak{M}} [\mathrm{L}_g u] : \left[\frac{\nabla u \otimes \nabla u}{u}\right] d\,v_g$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g \ge \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g \quad \forall\, u \in \mathrm{H}^2(\mathfrak{M})$$

and λ_1 is the optimal constant in the above inequality

The key estimates

$$\mathcal{G}[u] := \int_{\mathfrak{M}} \left[\theta\,(\Delta_g u)^2 + (\kappa+\beta-1)\,\Delta_g u\,\frac{|\nabla u|^2}{u} + \kappa\,(\beta-1)\,\frac{|\nabla u|^4}{u^2}\right] d\,v_g$$

Lemma

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta)\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g - \mathcal{G}[u] + \lambda\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g$$

$$\mathrm{Q}_g^\theta u := \mathrm{L}_g\,u - \tfrac{1}{\theta}\,\tfrac{d-1}{d+2}\,(\kappa+\beta-1)\left[\frac{\nabla u\otimes\nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

Lemma

$$\mathcal{G}[u] = \frac{\theta\,d}{d-1}\left[\int_{\mathfrak{M}} \|\mathrm{Q}_g^\theta u\|^2\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u,\nabla u)\,d\,v_g\right] - \mu\int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2}\,d\,v_g$$

with $\mu := \dfrac{1}{\theta}\left(\dfrac{d-1}{d+2}\right)^2(\kappa+\beta-1)^2 - \kappa\,(\beta-1) - (\kappa+\beta-1)\,\dfrac{d}{d+2}$

The end of the proof

Assume that $d \geq 2$. If $\theta = 1$, then μ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall\, p \in (1, 2^*)$$

where $\beta_\pm := \frac{\mathsf{b} \pm \sqrt{\mathsf{b}^2 - \mathsf{a}}}{2\,\mathsf{a}}$ with $\mathsf{a} = 2 - p + \left[\frac{(d-1)\,(p-1)}{d+2}\right]^2$ and $\mathsf{b} = \frac{d+3-p}{d+2}$
Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2) + p - 1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

Proposition

Let $d \geq 2$, $p \in (1,2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] \leq (\lambda - \Lambda_\star) \int_{\mathfrak{M}} |\nabla u|^2\, d\,v_g$$

The line

One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\theta}\,\|f\|_{\mathrm{L}^2(\mathbb{R})}^{1-\theta} \quad \text{if} \quad p\in(2,\infty)$$

$$\|f\|_{\mathrm{L}^2(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\eta}\,\|f\|_{\mathrm{L}^p(\mathbb{R})}^{1-\eta} \quad \text{if} \quad p\in(1,2)$$

with $\theta=\frac{p-2}{2\,p}$ and $\eta=\frac{2-p}{2+p}$

The threshold case corresponding to the limit as $p\to 2$ is the logarithmic Sobolev inequality

$$\int_{\mathbb{R}} u^2\,\log\left(\frac{u^2}{\|u\|_{\mathrm{L}^2(\mathbb{R})}^2}\right)dx \le \frac{1}{2}\,\|u\|_{\mathrm{L}^2(\mathbb{R})}^2\,\log\left(\frac{2}{\pi\,e}\,\frac{\|u'\|_{\mathrm{L}^2(\mathbb{R})}^2}{\|u\|_{\mathrm{L}^2(\mathbb{R})}^2}\right)$$

If $p>2$, $u_\star(x)=(\cosh x)^{-\frac{2}{p-2}}$ solves

$$-(p-2)^2\,u''+4\,u-\,2\,p\,|u|^{p-2}\,u=0$$

If $p\in(1,2)$ consider $u_*(x)=(\cos x)^{\frac{2}{2-p}}$, $x\in(-\pi/2,\pi/2)$

Mass transportation

Theorem (Dolbeault-Esteban-Laptev-Loss)

If $p \in (2,\infty)$, we have

$$\sup_G \frac{\int_{\mathbb{R}} G^{\frac{p+2}{3\,p-2}}\,dy}{\left(\int_{\mathbb{R}} G\,|y|^2\,dy\right)^{\frac{p-2}{3\,p-2}}\left(\int_{\mathbb{R}} G\,dy\right)^{\frac{4}{3\,p-2}}} = \mathsf{c}_p \inf_f \frac{\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{2\,(p-2)}{3\,p-2}}\,\|f\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{2\,(p+2)}{3\,p-2}}}{\|f\|_{\mathrm{L}^p(\mathbb{R})}^{\frac{4\,p}{3\,p-2}}}$$

and if $p \in (1,2)$, we obtain

$$\sup_G \frac{\int_{\mathbb{R}} G^{\frac{2}{4-p}}\,dy}{\left(\int_{\mathbb{R}} G\,|y|^2\,dy\right)^{\frac{2-p}{2\,(4-p)}}\left(\int_{\mathbb{R}} G\,dy\right)^{\frac{p+2}{2\,(4-p)}}} = \mathsf{c}_p \inf_f \frac{\|f'\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{2-p}{4-p}}\,\|f\|_{\mathrm{L}^p(\mathbb{R})}^{\frac{2\,p}{4-p}}}{\|f\|_{\mathrm{L}^2(\mathbb{R})}^{\frac{p+2}{4-p}}}$$

for some explicit numerical constant c_p

Flow

Let us define on $\mathrm{H}^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|^2_{\mathrm{L}^2(\mathbb{R})} + \frac{4}{(p-2)^2}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R})} - \mathsf{C}\,\|v\|^2_{\mathrm{L}^p(\mathbb{R})} \quad \text{s.t. } \mathcal{F}[u_\star] = 0$$

With $z(x) := \tanh x$, consider the *flow*

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1-z^2}}\left[v'' + \frac{2\,p}{p-2}\,z\,v' + \frac{p}{2}\,\frac{|v'|^2}{v} + \frac{2}{p-2}\,v\right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2,\infty)$. Then

$$\frac{d}{dt}\mathcal{F}[v(t)] \le 0 \quad \textit{and} \quad \lim_{t\to\infty}\mathcal{F}[v(t)] = 0$$

$$\tfrac{d}{dt}\mathcal{F}[v(t)] = 0 \quad \Longleftrightarrow \quad v_0(x) = u_\star(x - x_0)$$

Similar result for $p \in (1,2)$

The inequality ($p > 2$) and the ultraspherical operator

- *The problem on the line is equivalent to the critical problem for the ultraspherical operator*

$$\int_{\mathbb{R}} |v'|^2 \, dx + \frac{4}{(p-2)^2} \int_{\mathbb{R}} |v|^2 \, dx \geq \mathsf{C} \left(\int_{\mathbb{R}} |v|^p \, dx \right)^{\frac{2}{p}}$$

With

$$z(x) = \tanh x \,, \quad v_\star = (1-z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\star(x)\, f(z(x))$$

equality is achieved for $f = 1$ and, if we let $\nu(z) := 1 - z^2$, then

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{2\,p}{(p-2)^2} \int_{-1}^{1} |f|^2 \, d\nu_d \geq \frac{2\,p}{(p-2)^2} \left(\int_{-1}^{1} |f|^p \, d\nu_d \right)^{\frac{2}{p}}$$

where $d\nu_p$ denotes the probability measure $d\nu_p(z) := \frac{1}{\zeta_p} \, \nu^{\frac{2}{p-2}} \, dz$

$$d = \frac{2\,p}{p-2} \quad \Longleftrightarrow \quad p = \frac{2\,d}{d-2}$$

Change of variables = stereographic projection + Emden-Fowler

The Moser-Trudinger-Onofri inequality

Joint work with Maria J. Esteban and G. Jankowiak

Three equivalent forms

- The Euclidean (Moser-Trudinger-)Onofri inequality:

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right)-\int_{\mathbb{R}^2} u\,d\mu$$

$d\mu=\mu(x)\,dx$, $\mu(x)=\frac{1}{\pi}\,(1+|x|^2)^{-2}$, $x\in\mathbb{R}^2$

- The Onofri inequality on the two-dimensional sphere $\mathbb{S}^2$:

$$\frac{1}{4}\int_{\mathbb{S}^2}|\nabla v|^2\,d\sigma \ge \log\left(\int_{\mathbb{S}^2} e^v\,d\sigma\right)-\int_{\mathbb{S}^2} v\,d\sigma$$

$d\sigma$ is the uniform probability measure

- The Onofri inequality on the two-dimensional cylinder $\mathcal{C}=\mathbb{S}^1\times\mathbb{R}$:

$$\frac{1}{16\,\pi}\int_{\mathcal{C}}|\nabla w|^2\,dy \ge \log\left(\int_{\mathcal{C}} e^w\,\nu\,dy\right)-\int_{\mathcal{C}} w\,\nu\,dy$$

$y=(\theta,s)\in\mathcal{C}=\mathbb{S}^1\times\mathbb{R}$, $\nu(y)=\frac{1}{4\pi}\,(\cosh s)^{-2}$

[Moser (1971)], [Onofri (1982)]

The inequality seen as a limit case of the Gagliardo-Nirenberg inequalities

Proposition

[JD] *Assume that* $u \in \mathcal{D}(\mathbb{R}^2)$ *is such that* $\int_{\mathbb{R}^2} u \, d\mu = 0$ *and let*

$$f_p := F_p \left(1 + \frac{u}{2\,p}\right) \,, \quad F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^2$$

Then we have

$$1 \le \lim_{p\to\infty} \mathsf{C}_{p,2} \frac{\|\nabla f_p\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)} \, \|f_p\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f_p\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\, dx}}{\int_{\mathbb{R}^2} e^{u}\, d\mu}$$

Rigidity method in the symmetric case

Under an appropriate normalization, a critical point of

$$\mathsf{G}_\lambda[f] := \frac{1}{8}\int_{-1}^{1} |f'|^2\,\nu\,dz + \frac{\lambda}{2}\int_{-1}^{1} f\,dz \ge \log\left(\frac{1}{2}\int_{-1}^{1} e^f\,dz\right)$$

solves the Euler-Lagrange equation

$$-\frac{1}{2}\,\mathcal{L}f + \lambda = e^f$$

Theorem

For any $\lambda \in (0,1)$, the EL equation has a unique smooth solution $f = \log\lambda$. If $\lambda = 1$, f has to satisfy the differential equation $f'' = \frac{1}{2}\,|f'|^2$ and is either a constant or

$$f(z) = C_1 - 2\,\log(C_2 - z)$$

$$\frac{1}{8}\int_{-1}^{1} \nu^2\,\left|f'' - \tfrac{1}{2}\,|f'|^2\right|^2\,e^{-f/2}\,\nu\,dz + \frac{1-\lambda}{4}\int_{-1}^{1} \nu\,|f'|^2\,e^{-f/2}\,\nu\,dz = 0$$

Rigidity method in the symmetric case: proof

Multiply by $\mathcal{L}\big(e^{-f/2}\big)$ and integrate by parts

$$0=\int_{-1}^{1}\left(-\tfrac{1}{2}\,\mathcal{L}f+\lambda-e^{f}\right)\,\mathcal{L}\big(e^{-f/2}\big)\,\nu\,dz$$

$$=\frac{1}{4}\int_{-1}^{1}\nu^2\,|f''|^2\,e^{-f/2}\,\nu\,dz-\frac{1}{8}\int_{-1}^{1}\nu^2\,|f'|^2\,f''\,e^{-f/2}\,\nu\,dz$$

$$+\frac{1}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{-f/2}\,\nu\,dz-\frac{1}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{f/2}\,\nu\,dz$$

Multiply by $\frac{\nu}{2}\,|f'|^2\,e^{-f/2}$ and integrate by parts

$$0=\int_{-1}^{1}\left(-\tfrac{1}{2}\,\mathcal{L}f+\lambda-e^{f}\right)\left(\tfrac{\nu}{2}\,|f'|^2\,e^{-f/2}\right)\nu\,dz$$

$$=\frac{1}{8}\int_{-1}^{1}\nu^2\,|f'|^2\,f''\,e^{-f/2}\,\nu\,dz-\frac{1}{16}\int_{-1}^{1}\nu^2\,|f'|^4\,e^{-f/2}\,\nu\,dz$$

$$+\frac{\lambda}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{-f/2}\,\nu\,dz-\frac{1}{2}\int_{-1}^{1}\nu\,|f'|^2\,e^{f/2}\,\nu\,dz$$

A nonlinear flow method in the general case

On $\mathbb{S}^2$ let us consider the nonlinear evolution equation

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2}\left(e^{-f/2}\right) - \tfrac{1}{2}\,|\nabla f|^2\, e^{-f/2}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator. Let us define

$$\mathcal{R}_\lambda[f] := \frac{1}{2}\int_{\mathbb{S}^2}\|\mathrm{L}_{\mathbb{S}^2} f - \frac{1}{2}\,\mathrm{M}_{\mathbb{S}^2} f\|^2\, e^{-f/2}\, d\sigma + \frac{1}{2}\,(1-\lambda)\int_{\mathbb{S}^2}|\nabla f|^2\, e^{-f/2}\, d\sigma$$

where

$$\mathrm{L}_{\mathbb{S}^2} f := \mathrm{Hess}_{\mathbb{S}^2} f - \frac{1}{2}\,\Delta_{\mathbb{S}^2} f\,\mathrm{Id} \quad \text{and} \quad \mathrm{M}_{\mathbb{S}^2} f := \nabla f \otimes \nabla f - \frac{1}{2}\,|\nabla f|^2\,\mathrm{Id}$$

Theorem

Assume that f is a solution to with initial datum $v - \log\left(\int_{\mathbb{S}^2} e^v\, d\sigma\right)$, where $v \in \mathrm{L}^1(\mathbb{S}^2)$ is such that $\nabla v \in \mathrm{L}^2(\mathbb{S}^2)$. Then for any $\lambda \in (0,1]$ we have

$$\mathcal{G}_\lambda[v] \geq \int_0^\infty \mathcal{R}_\lambda[f(t,\cdot)]\, dt$$

Spectral consequences

Joint work with M.J. Esteban, A. Laptev, and M. Loss

- The same kind of results as for the sphere. However, estimates are not, in general, sharp.

Manifolds: the first interpolation inequality

Let us define

$$\kappa := \mathrm{vol}_g(\mathfrak{M})^{1-2/q}$$

Proposition

Assume that $q \in (2, 2^*)$ *if* $d \geq 3$, *or* $q \in (2, \infty)$ *if* $d = 1$ *or* 2. *There exists a concave increasing function* $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\mu(\alpha) = \kappa\,\alpha$ *for any* $\alpha \leq \frac{\Lambda}{q-2}$, $\mu(\alpha) < \kappa\,\alpha$ *for* $\alpha > \frac{\Lambda}{q-2}$ *and*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \mu(\alpha)\,\|u\|^2_{\mathrm{L}^q(\mathfrak{M})} \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})$$

The asymptotic behaviour of μ *is given by* $\mu(\alpha) \sim \mathsf{K}_{q,d}\,\alpha^{1-\vartheta}$ *as* $\alpha \to +\infty$, *with* $\vartheta = d\,\frac{q-2}{2\,q}$ *and* $\mathsf{K}_{q,d}$ *defined by*

$$\mathsf{K}_{q,d} := \inf_{v \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}}$$

Manifolds: the first Keller-Lieb-Thirring estimate

We consider $\|V\|_{\mathrm{L}^p(\mathfrak{M})} = \mu \mapsto \alpha(\mu)$

$$\int_{\mathfrak{M}} |\nabla u|^2 \, d\,v_g - \int_{\mathfrak{M}} V\,|u|^2 \, d\,v_g + \alpha(\mu) \int_{\mathfrak{M}} |u|^2 \, d\,v_g \\ \geq \|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} - \mu \, \|u\|^2_{\mathrm{L}^q(\mathfrak{M})} + \alpha(\mu) \, \|u\|^2_{\mathrm{L}^2(\mathfrak{M})}$$

p and $\frac{q}{2}$ are Hölder conjugate exponents

Theorem

Let $d \geq 1$, $p \in (1, +\infty)$ if $d = 1$ and $p \in (\frac{d}{2}, +\infty)$ if $d \geq 2$ and assume that $\Lambda_\star > 0$. With the above notations and definitions, for any nonnegative $V \in \mathcal{L}^p(\mathfrak{M})$, we have

$$|\lambda_1(-\Delta_g - V)| \leq \alpha\big(\|V\|_{\mathrm{L}^p(\mathfrak{M})}\big)$$

Moreover, we have $\alpha(\mu)^{p-\frac{d}{2}} = \mathcal{L}^{\,1}_{\gamma,d}\,\mu^p\,(1 + o(1))$ as $\mu \to +\infty$ with $\mathcal{L}^{\,1}_{\gamma,d} := (\mathsf{K}_{q,d})^{-p}$, $\gamma = p - \frac{d}{2}$

Manifolds: the second Keller-Lieb-Thirring estimate

Theorem

Let $d \geq 1$, $p \in (0, +\infty)$. There exists an increasing concave function $\nu : \mathbb{R}^+ \to \mathbb{R}^+$, satisfying $\nu(\beta) = \beta/\kappa$, for any $\beta \in (0, \frac{p+1}{2}\,\kappa\,\Lambda)$ if $p > 1$, such that for any positive potential W we have

$$\lambda_1(-\Delta + W) \geq \nu(\beta) \quad \textit{with} \quad \beta = \left(\int_{\mathfrak{M}} W^{-p}\,d\,v_g\right)^{1/p}$$

Moreover, for large values of β, we have
$\nu(\beta)^{-\,(p+\frac{d}{2})} = \mathcal{L}^{\,1}_{-(p+\frac{d}{2}),d}\,\beta^{-p}\,(1 + o(1))$ *as* $\beta \to +\infty$

The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

- Extension to compact Riemannian manifolds of dimension 2...

We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2$$

We define

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{\int_{\mathfrak{M}} \left[\, \| \mathrm{L}_g u - \frac{1}{2}\,\mathrm{M}_g u \|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] e^{-u/2}\, d v_g}{\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, d v_g}$$

Theorem

Assume that $d = 2$ and $\lambda_\star > 0$. If u is a smooth solution to

$$-\frac{1}{2}\,\Delta_g u + \lambda = e^u$$

then u is a constant function if $\lambda \in (0, \lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathfrak{M}$

$$\frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \lambda \int_{\mathfrak{M}} u\,d\,v_g \geq \lambda \log\left(\int_{\mathfrak{M}} e^u\,d\,v_g\right) \quad \forall\, u \in \mathrm{H}^1(\mathfrak{M})$$

for some constant $\lambda > 0$. Let us denote by λ_1 the first positive eigenvalue of $-\Delta_g$

Corollary

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4\,\pi, \lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than Λ

The flow

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

$$\begin{aligned}\mathcal{G}_\lambda[f] := \int_{\mathfrak{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla f, \nabla f)\,e^{-f/2}\,d\,v_g \\ - \lambda \int_{\mathfrak{M}} |\nabla f|^2\,e^{-f/2}\,d\,v_g\end{aligned}$$

Then for any $\lambda \le \lambda_\star$ we have

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_\lambda[f(t,\cdot)] = \int_{\mathfrak{M}} \left(-\tfrac{1}{2}\,\Delta_g f + \lambda\right)\left(\Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\right)d\,v_g \\ = -\,\mathcal{G}_\lambda[f(t,\cdot)]\end{aligned}$$

Since $\mathcal{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathcal{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathcal{F}_\lambda[u] \ge \int_0^\infty \mathcal{G}_\lambda[f(t,\cdot)]\,dt$$

Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure μ does the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \lambda\left[\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)-\int_{\mathbb{R}^2}u\,d\mu\right]$$

hold for some $\lambda > 0$? Let

$$\Lambda_\star := \inf_{x\in\mathbb{R}^2}\frac{-\,\Delta\log\mu}{8\,\pi\,\mu}$$

Theorem

Assume that μ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_\star$ and the inequality holds with $\lambda = \Lambda_\star$ if equality is achieved among radial functions

Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Joint work with G. Jankowiak

Preliminary observations

Legendre duality: Onofri and log HLS

Legendre's duality: $F^*[v] := \sup \left(\int_{\mathbb{R}^d} u\, v\, dx - F[u] \right)$

$$F_1[u] := \log \left(\int_{\mathbb{R}^2} e^u \, d\mu \right), \; F_2[u] := \frac{1}{16\,\pi} \int_0^\infty |\nabla u|^2 \, r^{d-1} \, dr + \int_0^\infty u \, \mu \, r^{d-1} \, dr$$

Onofri's inequality amounts to $F_1[u] \le F_2[u]$ with $d\mu(x) := \mu(x)\, dx$, $\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2}$

Proposition

For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ *with* $\int_0^\infty v \, r^{d-1} \, dr = 1$, *such that* $v \log v$ *and* $(1 + \log |x|^2)\, v \in \mathrm{L}^1(\mathbb{R}^2)$, *we have*

$$F_1^*[v] - F_2^*[v] =$$
$$\int_0^\infty v \, \log \left(\frac{v}{\mu} \right) r^{d-1} \, dr - 4\,\pi \int_0^\infty (v - \mu)\, (-\Delta)^{-1} (v - \mu) \; r^{d-1} \, dr \ge 0$$

[E. Carlen, M. Loss] [W. Beckner] [V. Calvez, L. Corrias]

A puzzling result of E. Carlen, J.A. Carrillo and M. Loss

[E. Carlen, J.A. Carrillo and M. Loss] The fast diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with exponent $m = d/(d+2)$, when $d \geq 3$, is such that

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{S}^d)}$$

obeys to

$$\frac{1}{2}\,\frac{d}{dt}\mathsf{H}_d[v(t,\cdot)] = \frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{S}^d)}\right]$$
$$= \tfrac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d\,\|u\|^{4/(d-1)}_{\mathrm{L}^{q+1}(\mathbb{S}^d)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^{2q}_{\mathrm{L}^{2q}(\mathbb{S}^d)}$$

with $u = v^{(d-1)/(d+2)}$ and $q = \frac{d+1}{d-1}$. If $\frac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d = (\mathsf{C}_{q,\,d})^{2q}$, the r.h.s. is nonnegative. Optimality is achieved simultaneously in both functionals (Barenblatt regime): the Hardy-Littlewood-Sobolev inequalities can be improved by an integral remainder term

... and the two-dimensional case

Recall that $(-\Delta)^{-1} v = G_d * v$ with

- $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$ if $d \ge 3$
- $G_2(x) = \frac{1}{2\,\pi}\,\log|x|$ if $d = 2$

Same computation in dimension $d = 2$ with $m = 1/2$ gives

$$\frac{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}{8}\,\frac{d}{dt}\left[\frac{4\,\pi}{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}\int_0^\infty v\,(-\Delta)^{-1} v\; r^{d-1}\,dr - \int_0^\infty v\,\log v\; r^{d-1}\,dr\right]$$
$$= \|u\|^4_{\mathrm{L}^4(\mathbb{R}^2)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} - \pi\,\|v\|^6_{\mathrm{L}^6(\mathbb{R}^2)}$$

The r.h.s. is one of the Gagliardo-Nirenberg inequalities ($d = 2$, $q = 3$): $\pi\,(\mathsf{C}_{3,2})^6 = 1$
The l.h.s. is bounded from below by the logarithmic Hardy-Littlewood-Sobolev inequality and achieves its minimum if $v = \mu$ with

$$\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2}\quad \forall\, x \in \mathbb{R}^2$$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{S}^d)} \geq \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v \, dx \quad \forall\, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$. Can we recover this using a nonlinear flow approach ? Can we improve it ?

Keller-Segel model: another motivation [J.A. Carrillo, E. Carlen and M. Loss] and [A. Blanchet, E. Carlen and J.A. Carrillo]

Using the Yamabe / Ricci flow

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{S}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\,dx + \mathsf{S}_d\left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\,dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2d}{d+2}$

A first statement

Proposition

[JD] *Assume that* $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{S}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\;dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

Theorem

[JD] *Assume that* $d \geq 5$ *and let* $q = \frac{d+2}{d-2}$. *There exists a positive constant* $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ *such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{S}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. del Pino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodriguez] *For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that*

$$\lim_{t\to T_-} (T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Improved inequality: proof (1/2)

The function $\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa\,T = \frac{2\,d}{d+2}\,\frac{T}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2}$$

Improved inequality: proof (2/2)

By the Cauchy-Schwarz inequality, we have

$$\frac{\mathsf{J}'^2}{(m+1)^2} = \|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{S}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2$$

$$\leq \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx = Cst\,\mathsf{J}''\,\mathsf{J}$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q}-1)\,,\quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \leq \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \leq 0$$

$$\mathsf{H}'' \leq -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \leq \mathsf{H}'(0)\,(1-e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \leq d/2$, the proof is completed □

$d = 2$: Onofri's and log HLS inequalities

$$\mathrm{H}_2[v] := \int_0^\infty (v-\mu)\,(-\Delta)^{-1}(v-\mu)\;r^{d-1}\,dr - \frac{1}{4\,\pi}\int_0^\infty v\,\log\left(\frac{v}{\mu}\right)\,r^{d-1}\,dr$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log\left(v/\mu\right) \quad t>0\,, \quad x \in \mathbb{R}^2$$

Proposition

If $v = \mu\,e^{u/2}$ is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_0^\infty v_0\;r^{d-1}\,dr = 1$, $v_0 \log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0 \log \mu \in \mathrm{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathrm{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_0^\infty |\nabla u|^2\;r^{d-1}\,dr - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

$$\geq \tfrac{1}{16\,\pi}\int_0^\infty |\nabla u|^2\;r^{d-1}\,dr + \int_{\mathbb{R}^2} u\,d\mu - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \geq 0$$

Improvements

Improved Sobolev inequality by duality

Theorem

[JD, G. Jankowiak] *Assume that* $d \geq 3$ *and let* $q = \frac{d+2}{d-2}$. *There exists a positive constant* $\mathcal{C} \leq 1$ *such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{S}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \leq \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Proof: the completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1} v|^2 \, dx = \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla(-\Delta)^{-1} v \, dx = \int_{\mathbb{R}^d} u \, v \, dx = \int_{\mathbb{R}^d} u^{2^*} \, dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \mathsf{S}_d \, \|u\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \nabla u - \nabla(-\Delta)^{-1} v \right|^2 dx$$

shows that

$$0 \le \mathsf{S}_d \, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \left[\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \right] - \left[\mathsf{S}_d \, \|u^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{S}^d)} - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} u^q \, dx \right]$$

The equality case

Equality is achieved if and only if

$$\mathsf{S}_d \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{S}^d)}^{\frac{4}{d-2}} \, u = (-\Delta)^{-1} \, v = (-\Delta)^{-1} \, u^q$$

that is, if and only if u solves

$$-\,\Delta u = \frac{1}{\mathsf{S}_d} \, \|u\|_{\mathrm{L}^{2^*}(\mathbb{S}^d)}^{-\frac{4}{d-2}} \, u^q$$

which means that u is an Aubin-Talenti extremal function

$$u_\star(x) := (1+|x|^2)^{-\frac{d-2}{2}} \quad \forall \, x \in \mathbb{R}^d$$

An identity

$$\begin{aligned}0 = \mathsf{S}_d\,\|u\|_{\mathrm{L}^{2^*}(\mathbb{S}^d)}^{\frac{8}{d-2}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}\right]\\ - \left[\mathsf{S}_d\,\|u^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{S}^d)} - \int_{\mathbb{R}^d} u^q\,(-\Delta)^{-1}\,u^q\,dx\right]\\ - \int_{\mathbb{R}^d}\left|\mathsf{S}_d\,\|u\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{S}^d)}\,\nabla u - \nabla(-\Delta)^{-1}\,u^q\right|^2 dx\end{aligned}$$

Another improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{S}^d)}$$

Theorem

Assume that $d \geq 3$. Then we have

$$0 \leq \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}\right]\right)$$

$$\forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d)\,,\quad v = u^{\frac{d+2}{d-2}}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ *for any* $x \geq 0$

Proof: $\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t))$ $\forall\, t \in [0, T)$, $\kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}$ and consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \leq \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \qquad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

... but $\mathcal{C} = 1$ is not optimal

Theorem

[JD, G. Jankowiak] *In the inequality*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{S}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \le \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} \right]$$

we have

$$\frac{d}{d+4} \le \mathsf{C}_d < 1$$

based on a (painful) linearization like the one used by Bianchi and Egnell

- Extensions: magnetic Laplacian [JD, Esteban, Laptev] or fractional Laplacian operator [Jankowiak, Nguyen]

Improved Onofri inequality

Theorem

Assume that $d = 2$. The inequality

$$\int_{\mathbb{R}^2} g \log\left(\frac{g}{M}\right) dx - \frac{4\pi}{M}\int_{\mathbb{R}^2} g\,(-\Delta)^{-1} g\,dx + M\,(1+\log\pi) \le M\left[\frac{1}{16\,\pi}\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \int_{\mathbb{R}^2} f\,d\mu - \log M\right]$$

holds for any function $f \in \mathcal{D}(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} e^f\,d\mu$ and $g = \pi\,e^f\,\mu$

Recall that

$$\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2} \quad \forall\, x \in \mathbb{R}^2$$

A summary

- the sphere: the flow tells us what to do, and provides a simple proof (*choice of the exponents / of the nonlinearity*) once the problem is reduced to the ultraspherical setting

- the spectral point of view on the inequality: how to measure the deviation with respect to the *semi-classical* estimates, a nice example of bifurcation (and *symmetry breaking*)

- *Riemannian manifolds:* no sign is required on the Ricci tensor and an improved integral criterion is established. We extend the theory from pointwise criteria to a non-local Schrödinger type estimate (Rayleigh quotient). The flow explores the energy landscape... and generically shows the non-optimality of the improved criterion

- the flow is a nice way of exploring an energy space. *Rigidity* result tell you that a local result is actually global because otherwise the flow would relate (far away) extremal points while keeping the energy minimal

Further considerations (1/2)

- other cases of application: bounded domains, weighted problems, interpolation inequalities on cylinders and weighted interpolation inequalities: solves conjecture by V. Felli and M. Schneider... a tool for the investigation of sharp qualitative properties that goes beyond standard tools for proving uniqueness and symmetry
- a gradient flow structure can be observed in some cases (with appropriate changes of variables) and under partial symmetry assumptions (*cf.* D. Bakry, I. Gentil and M. Ledoux, or G. Savaré *et al.*)). Formally, we can also use the flow to define a convenient notion of distance
- In some cases, the method formally enter in the *carré du champ* methods of D. Bakry and M. Emery, but it obeys to a very practical purpose: the explicit computation of the so-called $CD(\rho, N)$ condition. Moreover, this condition is always in a nonlocal form, which allows to relax the assumptions considerably

Further considerations (2/2)

- The *carré du champ* or Γ_2 methods are algebraic and very linear. For instance, in evolution problems, the heat flow or Fokker-Planck equations are the classical examples. The nonlinear flow approach is limited only by the compactness issues (critical exponents) and captures the nonlinear features of the functional inequalities
- Nonlinear improvements / correction terms are easy to obtain in the far from equilibrium range but the (entropy) method is still adapted to asymptotic regimes (and prescribes the adapted functional space for linearization): tis also opens a whole area of investigations like improved rates for well prepared initial data, correction terms (delays) for asymptotic profiles, etc.

http://www.ceremade.dauphine.fr/∼dolbeaul
▷ Preprints (or arxiv, or HAL)

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- J.D., Michal Kowalczyk. Uniqueness and rigidity in nonlinear elliptic equations, interpolation inequalities, and spectral estimates, Preprint, 2014

These slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/
▷ Lectures

Thank you for your attention !