Sobolev and Caffarelli-Kohn-Nirenberg inequalities for spinors

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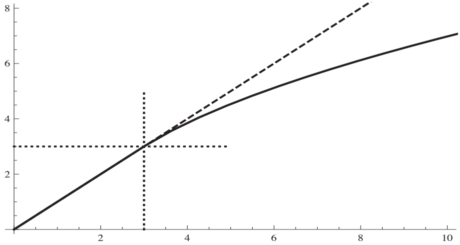
Outline

Bifurcation and phase transition in GNS inequalities



$\lambda \mapsto \mu(\lambda)$ *on* $\mathbb{S}^d$ *with* $d = 3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{\mu(\lambda)}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > d$$

▷ The inequality holds with $\mu(\lambda) = \lambda = d$ [Bakry, Emery, 1985] [Beckner, 1993], [Bidaut-Véron, Véron, 1991, Corollary 6.1]

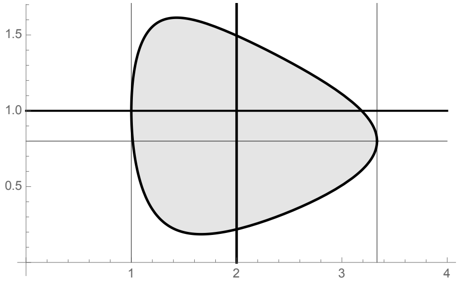
Carré du champ – admissible parameters on $\mathbb{S}^d$

[JD, Esteban, Kowalczyk, Loss] Monotonicity of the deficit

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{p-2}\left(\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

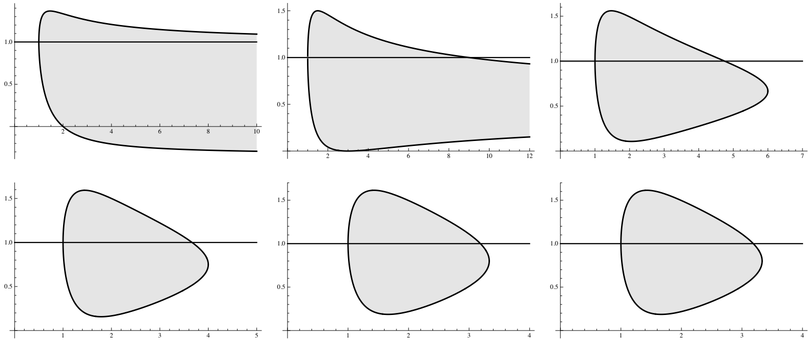
along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)}\left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u}\right)$$



Case $d=5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m). Improved inequalities inside !

Admissible parameters



$d = 1, 2, 3$ (first line) and $d = 4, 5$ and 10 (second line)
the curves $p \mapsto m_\pm(p)$ determine the admissible parameters (p, m)
[JD, Esteban, Kowalczyk, Loss 2014]

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p+2 \pm \sqrt{d\,(p-1)\,(2\,d-(d-2)\,p)}\right)$$

Another *Gagliardo-Nirenberg-Sobolev* inequality

$$\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac{\lambda}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^\theta\,\|u\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathbb{S}^d)}\ge\left(\frac{\mu(p,\theta,\lambda)}{p-2}\right)^\theta\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

- *Symmetry* holds if $\mu(p,\theta,\lambda)=\lambda$, optimal functions are constant
- *Symmetry breaking* if $\lambda>d\,\theta$: take $u_\varepsilon:=1+\varepsilon\,\varphi$, $\Delta\varphi+d\,\varphi=0$

Bakry-Emery exponent : $2^\#$. Take $p\in(2,2^\#]$

$$2^\#:=+\infty \text{ if } d=1,\quad 2^\#:=(2\,d^2+1)/(d-1)^2 \text{ if } d\ge2$$

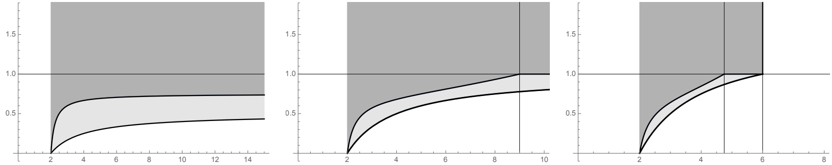
$$\theta^\#:=3\,\frac{p-2}{4\,p-7}\quad\text{if } d=1\,,\quad \frac{1}{\theta^\#}:=1+\frac{(p-1)\,(2^\#-p)}{p-2}\,\left(\frac{d-1}{d+2}\right)^2 \text{ if } d\ge2$$

Proposition

Let $d\ge1$, $p\in(2,2^\#)$, and $\theta\ge\theta^\#$. The function $\lambda\mapsto\mu(p,\theta,\lambda)$ is monotone increasing, concave and $\mu(p,\theta,\lambda)<\lambda$ if and only if $\lambda>d\,\theta$

[JD, Esteban, 2019]

Parameter range and qualitative results



horizontal axis: p, vertical axis: θ
in dimensions $d=1$, $d=2$ and $d=3$ (from left to right)

Theorem

Let $d\ge1$, $p\in(2,2^*)$ and $\theta>\theta_\star:=d\,(p-2)/(2\,p)$
The function $\lambda\mapsto\mu(p,\theta,\lambda)$ is monotone increasing, concave

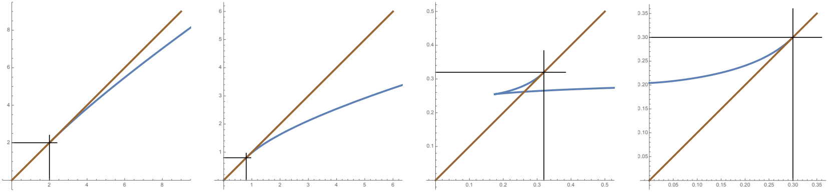
$$\mu(p,\theta,\lambda)\sim\kappa\,\lambda^{1-\theta_\star/\theta}\quad\text{as}\quad\lambda\to+\infty$$

$$\mu(p,\theta,\lambda)\le\lambda \text{ and } \mu(p,\theta,\lambda)<\lambda \text{ if } \lambda>d\,\theta$$

$$\mu(p,\theta,\lambda)=\lambda \text{ if } \lambda\le d\,\theta\,,\ \theta\ge\theta^\#\,,\ p\in(2,2^\#] \text{ or } p>2 \text{ if } d=1$$

Second and first order phase transitions

[Bou Dagher, JD, 2024]



$d = 1$, $p = 5$, $\theta = 2$: $\theta = 0.8$, $\theta = 0.32$ and $\theta = \theta_\star = 0.3$

Second and first order phase transitions

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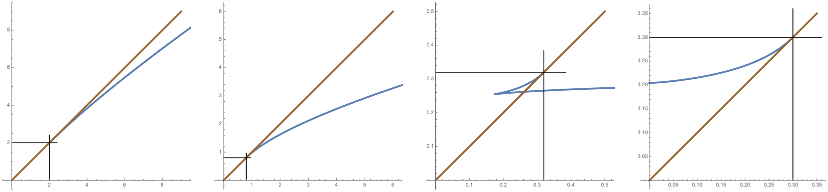

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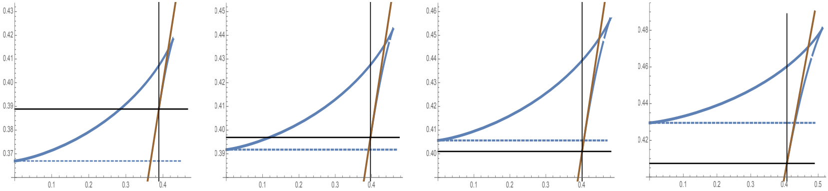
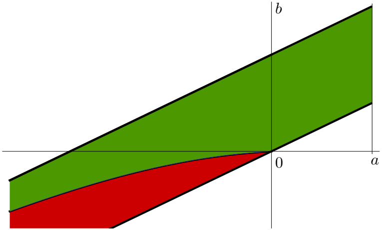


Critical case: $d = 1$, $\theta = \theta_\star$, for $p = 9.0$, 9.7, 10.1 and 10.8

The critical Caffarelli-Kohn-Nirenberg inequality



$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

$a \le b \le a+1$, $a < a_c$, $d \ge 3$

$p = \frac{2\,d}{d-2+2\,(b-a)} > 0$, $a_c = \frac{1}{2}\,(d-2)$

▷ *A radial optimal function:*
$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-2/(p-2)}$
among radially symmetric functions

Theorem (JD, Esteban, Loss, 2015)

There is *symmetry*, *i.e.*, $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$, and all optimal functions are radially symmetric if $b_{\rm FS(a)} \le b < a+1$. If $a < b < b_{\rm FS}(a)$, then there is *symmetry breaking*, $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$, and optimal functions are not radially symmetric.

[Caffarelli, Kohn, Nirenberg (1984)], [F. Catrina, Z.-Q. Wang (2001)]
[Smets, Willem], [Catrina, Wang], [Felli, Schneider]
[Bonforte, JD, Nazaret, Muratori], *etc.*

A new proof: rewriting of CKN

1) Change of variables: $v(r,\omega) = u(r^\alpha,\omega)$, $\mathsf{D}_\alpha u = (\alpha\,\partial_r u, \nabla_\omega u)$

$$\int_{\mathbb{R}^d} |\mathsf{D}_\alpha u|^2\,|x|^{n-d}\,dx \ge \mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d} |u|^p\,|x|^{n-d}\,dx\right)^{2/p}$$

with $n = 2\,p/(p-2)$. Symmetry means that the Aubin-Talenti function $u_*(x) := (1+|x|^2)^{-(n-2)/2}$ realizes the equality case

2) Relative measure: with $w = u/u_*$ and $d\mu_q(x) = |u_*(x)|^q\,|x|^{n-d}\,dx$

$$\int_{\mathbb{R}^d} |\mathsf{D}_\alpha w|^2\,d\mu_2\,dx + \tfrac14\,\alpha^2\,n\,(n-2)\int_{\mathbb{R}^d}|w|^2\,d\mu_p\,dx \ge \mathcal{C}_{\alpha,n}\left(\int_{\mathbb{R}^d}|w|^p\,d\mu_p\,dx\right)^{2/p}$$

3) Stereographic projection: $w(x) = f(z,\omega)$ with $z = \frac{1-|x|^2}{1+|x|^2}$, $\omega = \frac{2\,x}{1+|x|^2}$

$$\int_{\mathbb{S}^d}\left(\alpha^2(1-z^2)|f'|^2 + \frac{|\nabla_\omega f|^2}{1-z^2}\right)d\sigma_n + \frac{\alpha^2}{4}\,n\,(n-2)\int_{\mathbb{S}^d}|f|^2\,d\sigma_n$$
$$\ge \mathcal{K}_{\alpha,n}\left(\int_{\mathbb{S}^d}|f|^p\,d\sigma_n\right)^{2/p}$$

$$d\sigma_n = Z_n^{-1}\,(1-z^2)^{(n-2)/2}\,dz\,d\omega,\ z\in[-1,+1],\ \omega\in\mathbb{S}^{d-1}$$

Weighted fast diffusion on $\mathbb{S}^d$ and *carré du champ*

Let $'$ and ∇ denote the derivatives with respect to $z \in [-1,1]$ and $\omega \in \mathbb{S}^{d-1}$, Δ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ and

$$\mathsf{D}v := \left(\alpha\sqrt{1-z^2}\,v', \tfrac{1}{\sqrt{1-z^2}}\nabla v\right), \quad \mathsf{L}v := \mathsf{D}\cdot\mathsf{D}v$$

$$\mathsf{L}v = \alpha^2\,\mathcal{L}v + \tfrac{1}{1-z^2}\,\Delta v, \quad \mathcal{L}v := \left(1-z^2\right)v'' - n\,z\,v'$$

Weighted fast diffusion equation

$$\frac{\partial v}{\partial t} = \mathsf{L}v^m = -\,\mathsf{D}\cdot(v\,\mathsf{D}\mathsf{P}), \quad \mathsf{P} = \frac{m}{1-m}\,v^{m-1}, \quad m = \frac{n-1}{n}, \quad p = \frac{2\,n}{n-2}$$

$$v = u^p \quad \text{and} \quad \mathcal{D}(t) := \int_{\mathbb{S}^d}|\mathsf{D}u(t,\cdot)|^2\,d\sigma_n + \frac{n\,\alpha^2}{p-2}\int_{\mathbb{S}^d}|u(t,\cdot)|^2\,d\sigma_n$$

Proposition

$$\mathcal{D}'(t) \le 0 \text{ if } \alpha \le \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$$

- Nonlinear *carré du champ* techniques and Felli & Schneider (FS)
- [JD, Zhang, 2021]: weights $(1+\varepsilon-z^2)$ in ultrasperical coordinates

$\Longrightarrow$ First parabolic proof + stability results [Bou Dagher, JD, 2025]

Sobolev and Caffarelli-Kohn-Nirenberg inequalities for spinors

▷ Caffarelli-Kohn-Nirenberg inequalities for spinor valued functions in dimension $d = 3$

▷ Caffarelli-Kohn-Nirenberg inequalities for spinor valued functions in dimension $d = 2$
+ an equivalent problem for Aharonov-Bohm magnetic fields

▷ A Keller / Sobolev inequality for Dirac operators

Symmetry results for spinors in dimension $d = 3$

We consider *2-spinors*, which are $\mathbb{C}^2$-valued function

$$\mathbb{R}^3 \ni x \mapsto \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} \in \mathbb{C}^2$$

Caffarelli-Kohn-Nirenberg inequalities for spinors

$$\int_{\mathbb{R}^3} \frac{|\sigma \cdot \nabla \psi(x)|^2}{|x|^{2\,\alpha}}\,dx \ge \mathcal{C}_{\alpha,\beta} \left(\int_{\mathbb{R}^3} \frac{|\psi(x)|^p}{|x|^{\beta\,p}}\,dx \right)^{2/p} \qquad \text{(SCKN)}$$

where $\partial_j = \partial_{x_j}$ and the gradient term is defined by

$$\sigma \cdot \nabla \psi = \sum_{j=1}^{3} \sigma_j\,\partial_j \psi$$

and $\sigma = (\sigma_j)_{j=1,2,3}$ is the family of the *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\alpha \le \beta \le \alpha + 1$, $p = 6/(1 - 2\,\alpha + 2\,\beta)$, $\mathcal{C}_{\alpha,\beta} \ge 0$ is the best constant

Symmetry for spinors

Proposition (JD, Esteban, Frank, Loss)

Let $\Lambda := \left\{ k - \frac{1}{2} \,:\, k \in \mathbb{Z} \setminus \{0\} \right\}$
If $\alpha \in \Lambda$*, then* $\mathcal{C}_{\alpha,\beta} = 0$ *for all* $\alpha \le \beta \le \alpha + 1$
If $\alpha \notin \Lambda$*, then* $\mathcal{C}_{\alpha,\beta} > 0$ *for all* $\alpha \le \beta \le \alpha + 1$

Angular decomposition in eigenspaces of $\sigma \cdot L$

$$\mathrm{L}^2\left(\mathbb{S}, \mathbb{C}^2; d\omega\right) = \bigoplus_{k \in \mathbb{Z} \setminus \{-1\}} \mathcal{H}_k$$

where $L := \omega \wedge (-i\,\nabla)$ is the *angular momentum* operator

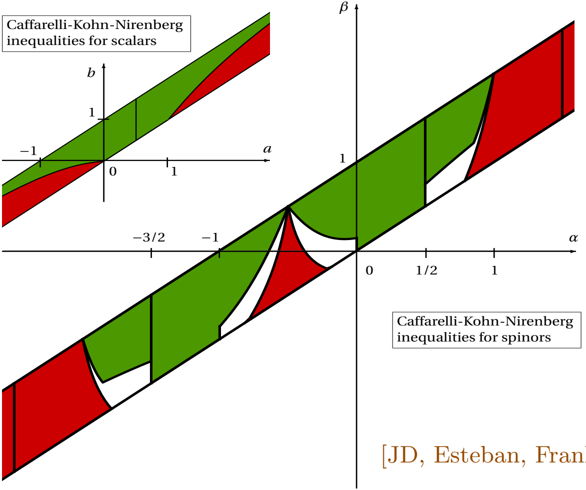
Definition

A spinor ψ on $\mathbb{R}^3$ is *symmetric* if there is a constant $\chi_0 \in \mathbb{C}^2$ and a complex-valued function f on $\mathbb{R}_+$ such that

$$\psi(x) = f(r)\,\chi_0 \quad \text{or} \quad \psi(x) = f(r)\,\sigma \cdot \omega\,\chi_0\,, \quad r = |x|\,, \quad \omega = x/r$$

i.e., $\psi \in \mathcal{H}_0$ or $\mathcal{H}_{-2}$

Results



[JD, Esteban, Frank, Loss, 2025]

Symmetry regions: green; symmetry breaking regions: red

The ingredients of the proof

- Existence of optimizers
- A Hardy inequality case: $\mathcal{C}_{\alpha,\alpha+1} = \min_{k\in\mathbb{Z}\setminus\{-1\}} \left(k - \alpha + \frac{1}{2}\right)^2$
- Passing to logarithmic variables

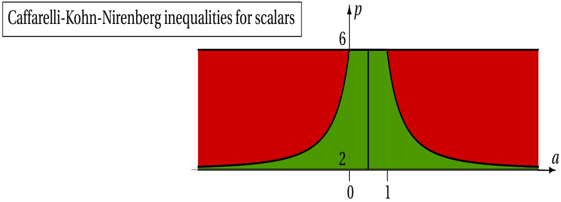
$$\iint_{\mathbb{R}\times\mathbb{S}} \left(|\partial_s\phi|^2 + \left|\left(\sigma\cdot L - \alpha + \tfrac{1}{2}\right)\phi\right|^2\right) ds\, d\omega \geq C_{\alpha,p} \left(\iint_{\mathbb{R}\times\mathbb{S}} |\phi|^p\, ds\, d\omega\right)^{2/p}$$

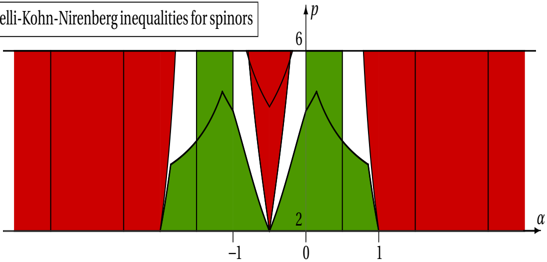
- Monotonicity properties: for some $\alpha_\star : (2,6) \to [-1/2, 0]$

$$\begin{aligned} C_{\alpha,p} &< C^\star_{\alpha,p} \quad \text{if} \quad -1/2 \leq \alpha < \alpha_\star(p) \\ C_{\alpha,p} &= C^\star_{\alpha,p} \quad \text{if} \quad \alpha_\star(p) \leq \alpha < 1/2 \end{aligned}$$

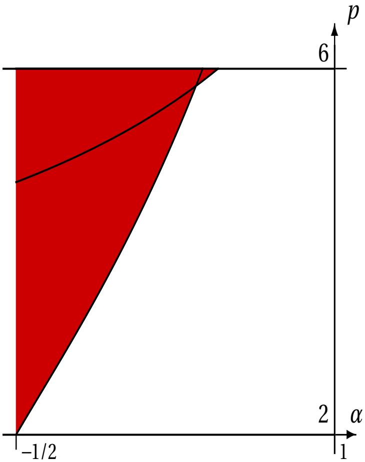
- A Gagliardo-Nirenberg interpolation inequality for spinors on the sphere based on tools of harmonic analysis
- A Keller-Lieb-Thirring estimate
- A chain of (optimal) estimates
- Instability: study of the quadratic form obtained by linearization and representation using spherical harmonics

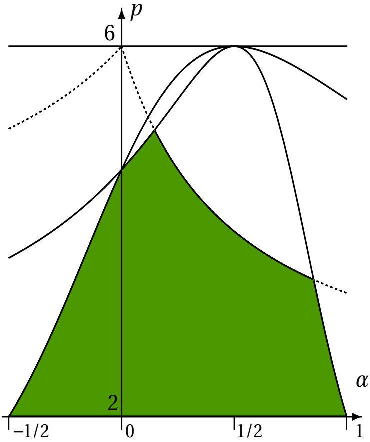
Logarithmic variables





Symmetry *versus* symmetry breaking (details)





Symmetry results for spinors in dimension $d = 2$

- the $d = 2$ *spinorial Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}^2} \frac{|\sigma \cdot \nabla \psi|^2}{|x|^{2\alpha}}\, dx \ge C_{\alpha,p} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^{\beta p}}\, dx \right)^{2/p} \qquad \text{(SCKN)}$$

for spinor valued functions $\psi : \mathbb{R}^2 \to \mathbb{C}^2$

▷ the *logarithmic Caffarelli-Kohn-Nirenberg inequality*

$$\int_{\mathbb{R}} \int_{\mathbb{S}^1} \left(|\partial_s \phi(s,\theta)|^2 + |(\alpha - i\sigma_3 \partial_\theta)\, \phi(s,\theta)|^2 \right) ds\, d\theta \ge C_{\alpha,p} \left(\int_{\mathbb{R}} \int_{\mathbb{S}^1} |\phi(s,\theta)|^p\, ds\, d\theta \right)^{2/p}$$

- *Interpolation inequalities for Aharonov–Bohm magnetic fields* $A(x) = (x_2, -x_1)/|x|^2$

$$\int_{\mathbb{R}^2} |(-i\nabla - \alpha A)\psi|^2\, dx \ge C^{\mathrm{AB}}_{\alpha,p} \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\, dx \right)^{2/p} \qquad \text{(AB)}$$

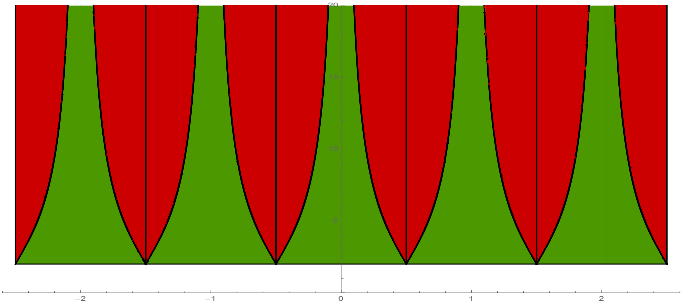
Theorem (JD, Frank, Weixler)

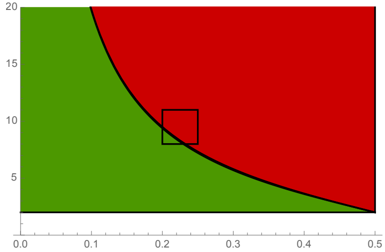
$C_{\alpha,p} = C^{\mathrm{AB}}_{\alpha,p}$ for any $(\alpha, p) \in (0, 1/2) \times (2, +\infty)$

Symmetry *versus* symmetry breaking

Theorem (JD, Frank, Weixler)

- For every $\alpha \in (0, 1/2)$ and $p > 2$, there is an optimizer with $C_{\alpha,p} > 0$ and $\lim_{\alpha \to 0_+} C_{\alpha,p} = 0$. Symmetry holds if and only if $\alpha \in \big(0, \alpha(p)\big]$ for some function $p \mapsto \alpha(p) : (2, \infty) \to (0, 1/2)$
- The symmetry and symmetry breaking regions are symmetric with respect to $\alpha = 0$ and 1-periodic





(SCKN) *with* $d = 2$. *Horizontal axis:* $\alpha \in (0, 1/2)$. *Vertical axis:* $p \in (2, \infty)$

- *Symmetry range: green, by the equivalence with Aharonov-Bohm problem and entropy methods for flows associated to (CKN) inequalities*
- *Symmetry breaking range: red and blue; Undecided in the tiny white gap*

- magnetic ring: an interpolation inequality on $\mathbb{S}^1$ [JD, Esteban, Laptev, Loss]
- Aharonov-Bohm and Caffarelli-Kohn-Nirenberg inequalities [Bonheure, JD, Esteban, Laptev, Loss]
- a Gegenbauer polynomial basis to study linear instability numerically

The Keller-Lieb-Thirring inequality (Schrödinger) on $\mathbb{R}^d$

- With $q<2^*:=2\,d/(d-2)$ if $d\ge3$, and $\vartheta=d\,(q-2)/(2\,q)$
The *Gagliardo-Nirenberg-Sobolev inequality*

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\vartheta}\,\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^{1-\vartheta}\ge\mathscr{C}_q\,\|u\|_{\mathrm{L}^q(\mathbb{R}^d)}$$

can be rewritten as

$$\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2+\lambda\,\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2\ge\mathsf{C}_q\,\lambda^{1-\vartheta}\,\|u\|_{\mathrm{L}^q(\mathbb{R}^d)}^2$$

for any $(\lambda,u)\in(0,+\infty)\times\mathrm{H}^1(\mathbb{R}^d)$, with $\mathscr{C}_q^2=\vartheta^{\vartheta}(1-\vartheta)^{1-\vartheta}\,\mathsf{C}_q$

- Let λ be such that $\mathsf{C}_q\,\lambda^{1-\vartheta}=\|V\|_{\mathrm{L}^p(\mathbb{R}^d)}$. The Schrödinger energy is

$$\int_{\mathbb{R}^d}|\nabla u|^2\,dx-\int_{\mathbb{R}^d}V\,|u|^2\,dx\ge\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2-\|V\|_{\mathrm{L}^p(\mathbb{R}^d)}\,\|u\|_{\mathrm{L}^q(\mathbb{R}^d)}^2$$
$$\ge-\left(\mathsf{C}_q^{-1}\,\|V\|_{\mathrm{L}^p(\mathbb{R}^d)}\right)^{1/(1-\vartheta)}\,\|u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

Keller-Lieb-Thirring inequality : with $\eta=1/(1-\vartheta)=2\,p/(2\,p-d)$

$$\forall\,V\in\mathrm{L}^p(\mathbb{R}^d)\,,\quad\big(\lambda_1-(-\Delta-V)\big)_-\ge-\mathsf{K}_p\,\|V\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\eta}$$

Hölder duality and interpolation inequalities on $\mathbb{S}^d$

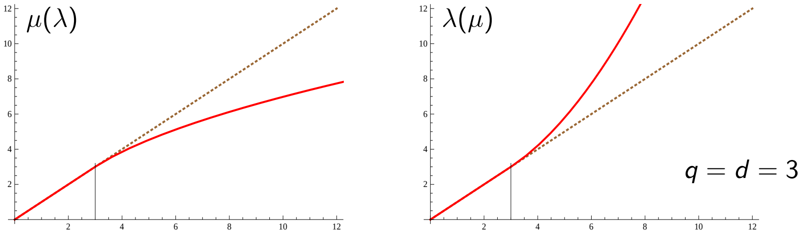
Same on $\mathbb{S}^d$ as on $\mathbb{R}^d$... Let $p = \frac{q}{q-2}$. Consider the Schrödinger energy

$$\int_{\mathbb{S}^d} |\nabla u|^2 - \int_{\mathbb{S}^d} V\,|u|^2 \geq \int_{\mathbb{S}^d} |\nabla u|^2 - \mu\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}$$

$$\geq -\lambda(\mu)\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \quad \text{if } \mu = \|V_+\|_{\mathrm{L}^p(\mathbb{S}^d)}$$

From (GNS) $\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq \mu(\lambda)\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)}$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \mu(\lambda)\,\|u\|^2_{\mathrm{L}^q(\mathbb{S}^d)} \geq -\lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \qquad \text{(Keller)}$$



[JD, Esteban, Laptev, 2014], [JD, Esteban, Laptev, Loss, 2013]

The interpolation inequality

Aharonov-Bohm vector potential

$$\mathbf{A}(x)=\frac{a}{|x|^2}\,(x_2,-x_1)\;,\quad x=(x_1,x_2)\in\mathbb{R}^2\setminus\{0\}\,,\quad a\in\mathbb{R}$$

Magnetic Hardy inequality [Laptev, Weidl, 1999]

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx\ge\min_{k\in\mathbb{Z}}\,(a-k)^2\int_{\mathbb{R}^2}\frac{|\psi|^2}{|x|^2}\,dx \tag{1}$$

where $\nabla_{\mathbf{A}}\,\psi:=\nabla\psi+\,i\,\mathbf{A}\,\psi$, so that, with $\psi=|\psi|\,e^{iS}$

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx=\int_{\mathbb{R}^2}\left[(\partial_r\,|\psi|)^2+(\partial_r S)^2\,|\psi|^2+\frac{1}{r^2}\,(\partial_\theta S+A)^2\,|\psi|^2\right]dx$$

Magnetic interpolation inequality

$$\int_{\mathbb{R}^2}|\nabla_{\mathbf{A}}\,\psi|^2\,dx+\lambda\int_{\mathbb{R}^2}\frac{|\psi|^2}{|x|^2}\,dx\ge\mu(\lambda)\left(\int_{\mathbb{R}^2}\frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p}$$

A magnetic Keller-Lieb-Thirring estimate

Let $q \in (1,+\infty)$ and denote by $\mathrm{L}^q_\star(\mathbb{R}^2)$ the space defined using the weighted norm $|||\phi|||_q := \left(\int_{\mathbb{R}^2} |\phi|^q\,|x|^{2\,(q-1)}\,dx\right)^{1/q}$

Theorem

Let $a \in (0,1/2)$, $q \in (1,\infty)$ and $\phi \in \mathrm{L}^q_\star(\mathbb{R}^2)$: $\mu \mapsto \lambda(\mu)$ is a convex monotone increasing function such that $\lim_{\mu\to0^+} \lambda(\mu) = -\min_{k\in\mathbb{Z}}(a-k)^2$ and

$$\lambda_1(-\Delta_{\mathbf{A}} - \phi) \geq -\lambda\left(|||\phi|||_q\right)$$

For some $\mu_\star > 0$, the equality case is achieved for any $\mu \leq \mu_\star$ by

$$\phi(x) = \left(|x|^\alpha + |x|^{-\alpha}\right)^{-2} \quad \forall\, x \in \mathbb{R}^2\,, \quad \text{with} \quad \alpha = \frac{p-2}{2}\sqrt{\lambda(\mu)+a^2}$$

The equality case is achieved only by non-radial functions if $\mu > \mu_\bullet > \mu_\star$

[Bonheure, JD, Esteban, Laptev, Loss, 2019]

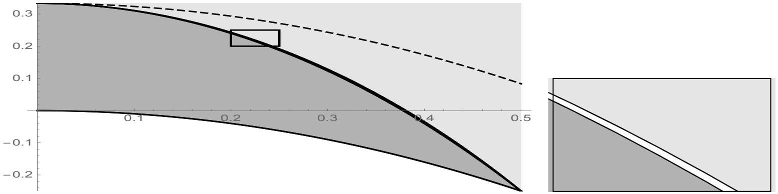


Figure: Case $p = 4$, $a \in (0, 1/2)$
Symmetry breaking region: $\lambda > \lambda_\bullet(a)$; Symmetry breaking region: $\lambda < \lambda_\star$

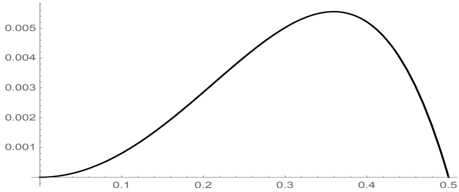


Figure: The curve $a \mapsto \lambda_\bullet(a) - \lambda_\star(a)$

Dirac operator and a notion of ground state

The ***free Dirac operator*** in dimension d is defined by

$$\not{D}_m := \sum_{j=1}^{d} \alpha_j\,(-\mathrm{i}\,\partial_j) + m\,\beta = \boldsymbol{\alpha}\cdot(-\mathrm{i}\,\nabla) + m\,\beta$$

- $d=1$, $\boldsymbol{\alpha}=\sigma_2$ and $\beta=\sigma_3$: $\not{D}_m=\sigma_2\,(-\mathrm{i}\,\partial_1)+m\,\sigma_3$
- $d=2$, $\boldsymbol{\alpha}=(\sigma_j)_{j=1,2}$ and $\beta=\sigma_3$: $\not{D}_m=\sum_{j=1}^2\sigma_j(-\mathrm{i}\,\partial_j)+m\,\sigma_3$
- $d=3$, $\boldsymbol{\alpha}=(\alpha_k)_{k=1,2,3}$ and β such that

$$\alpha_k := \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}$$

Here $(\sigma_j)_{j=1,2,3}$ are the Pauli matrices

Ground state : $\lambda_D(V)$ = lowest eigenvalue in $(-m,m)$ of $\not{D}_m - V$

$$\Lambda_D(\alpha,p) := \inf\left\{\lambda_D(V)\,:\,V\in \mathrm{L}^p(\mathbb{R}^d,\mathbb{R}^+)\ \text{and}\ \|V\|_{\mathrm{L}^p(\mathbb{R}^d)}=\alpha\right\}$$

A Sobolev/Keller inequality for a Dirac operator

$$\lambda_D(V) \geq \Lambda_D(\alpha, p) \quad \forall\, V \in \mathrm{L}^p(\mathbb{R}^d, \mathbb{R}^+) \text{ such that } \|V\|_{\mathrm{L}^p(\mathbb{R}^d)} = \alpha$$

[JD, Gontier, Pizzichillo, van den Bosch, 2025]

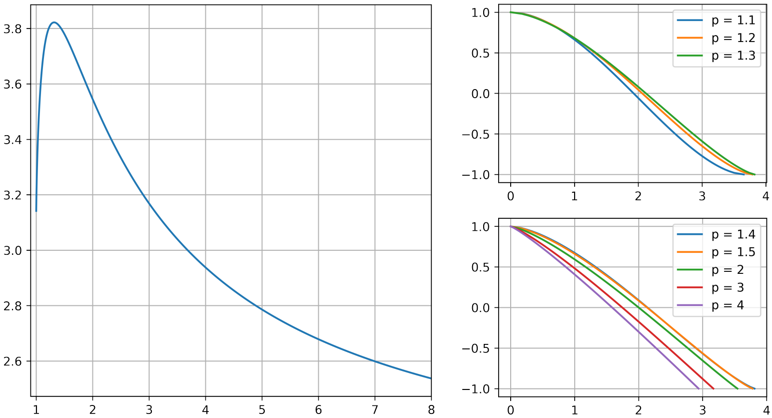
Theorem

Let $p \geq d \geq 1$. There exists $\alpha_\star(p) > 0$ such that the map $\alpha \mapsto \Lambda_D(\alpha, p)$ defined on $\big[0, \alpha_\star(p)\big)$ is continuous, strictly decreasing, takes values in $(-m, m]$, and such that

$$\lim_{\alpha \to 0_+} \Lambda_D(\alpha, p) = m \quad \text{and} \quad \lim_{\alpha \to \alpha_\star(p)} \Lambda_D(\alpha, p) = -m$$

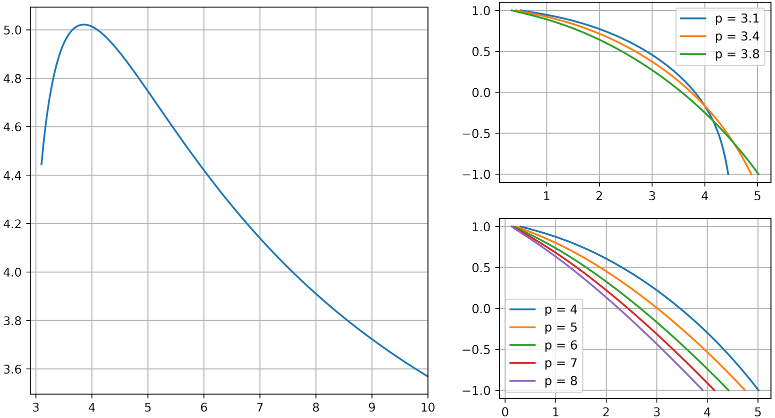
If $(p, d) \neq (1, 1)$, then $\Lambda_D(\alpha, p)$ is attained on $\big(0, \alpha_\star(p)\big)$

The case of dimension $d=1$



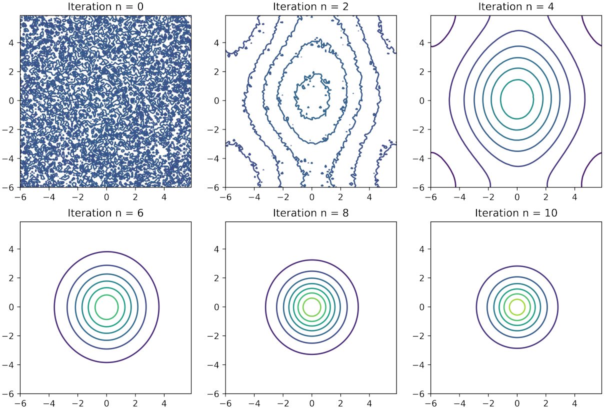
With $m=1$*. The function* $p\mapsto\alpha_\star(p)$ *(left), has a maximum at* $p\approx 1.32$
and $\lim_{p\to 1_+}\alpha_\star(p)=\pi$ *and* $\lim_{p\to+\infty}\alpha_\star(p)=2$
The function (right) $\Lambda_D(\alpha,p)$ *for various* p *is such that* $\Lambda_D\big(\alpha_\star(p),p\big)=-1$

The radial case in dimension $d = 3$



Radial case with $d = 3$ *and* $m = 1$
(Left) The function $p \mapsto \alpha_\star^{\rm rad}(p)$ *reaches its maximum at* $p \approx 3.86$
(Right) The maps $\alpha \mapsto \Lambda_D^{{\rm rad},(\kappa=1)}(\alpha, p)$

Is the optimal potential radial?



A numerical answer on $\mathbb{R}^2$*... Contour lines of the potential (by a fixed point method) for* $p = 3$ *and* $\lambda = 1/2$*, for some initial potential chosen at random*

Conjecture. The optimal potential at $\alpha = \alpha_\star(p)$ is an ***Aubin-Talenti*** profile up to an angular spinor

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Thank you for your attention...

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