Stability in Gagliardo-Nirenberg inequalities

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Outline

- Introduction

▷ (almost) a first *constructive result* on the ***stability of Gagliardo-Nirenberg inequalities***

- The ***fast diffusion flow*** and ***entropy methods***

▷ *Rényi entropy powers:* a word on the *carré du champ* method
▷ the *entropy-entropy production inequality*
▷ spectral gap: the *asymptotic time layer*
▷ the *initial time layer*, a backward nonlinear estimate
▷ the *threshold time*

- The ***uniform convergence in relative error***

▷ from local to global estimates: a quantitative version of Harnack's inequality and Hölder regularity
▷ the stability result in the entropy framework

Main results of this lecture have been obtained in collaboration with

Matteo Bonforte
▷ *Universidad Autónoma de Madrid and ICMAT*

Bruno Nazaret
▷ *Université Paris 1 Panthéon-Sorbonne and Mokaplan team*

Nikita Simonov
▷ *Ceremade, Université Paris-Dauphine (PSL)*

Introduction

- A special family of Gagliardo-Nirenberg inequalities
- Optimal functions
- A stability result

Gagliardo-Nirenberg inequalities

For any smooth f on $\mathbb{R}^d$ with compact support

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathscr{C}_{\mathrm{GN}} \, \|f\|_{2p} \tag{1}$$

[Gagliardo, 1958] [Nirenberg, 1959] $\quad \theta = \dfrac{d\,(p-1)}{p\,(d+2-p\,(d-2))}$

- if $d \geq 3$, the exponent p is in the range $1 < p \leq \frac{d}{d-2}$ and $2p = \frac{2\,d}{d-2} = 2^* =: 2\,p^*$ is the critical Sobolev exponent, corresponding to *Sobolev's inequality* with ($\theta = 1$) [Rodemich, 1968] [Aubin & Talenti, 1976]

$$\|\nabla f\|_2^2 \geq \mathsf{S}_d \, \|f\|_{2^*}^2$$

▷ if $d = 1$ or 2, the exponent p is in the range $1 < p < +\infty =: p^*$

- the limit case as $p \to 1_+$ is *Euclidean logarithmic Sobolev inequality in scale invariant form* [Blachman, 1965] [Stam, 1959] [Weissler, 1978]

$$\frac{d}{2} \log\left(\frac{2}{\pi\,d\,e} \int_{\mathbb{R}^d} |\nabla f|^2 \, dx\right) \geq \int_{\mathbb{R}^d} |f|^2 \log|f|^2 \, dx$$

for any function $f \in \mathrm{H}^1(\mathbb{R}^d, dx)$ such that $\|f\|_2 = 1$

Optimal functions and scalings

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\rm GN} \, \|f\|_{2p} \tag{1}$$

[del Pino, JD, 2002] Equality is achieved by the *Aubin-Talenti* type function

$$\mathrm{g}(x) = \left(1+|x|^2\right)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

By homogeneity, translation, scalings, equality is also achieved by

$$\mathrm{g}_{\lambda,\mu,y}(x) := \mu\,\lambda^{-\frac{d}{2p}}\,\mathrm{g}\left(\frac{x-y}{\lambda}\right) \quad (\lambda,\mu,y) \in (0,+\infty)\times\mathbb{R}\times\mathbb{R}^d$$

▷ A non-scale invariant form of the inequality

$$a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} \ge \mathscr{K}_{\rm GN}\,\|f\|_{2p}^{2p\gamma}$$

$a = \frac{1}{2}\,(p-1)^2$, $b = 2\,\frac{d-p\,(d-2)}{p+1}$, $\mathscr{K}_{\rm GN} = \|\mathrm{g}\|_{2p}^{2p\,(1-\gamma)}$ and $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$

If $p=1$: standard *Euclidean logarithmic Sobolev inequality* [Gross, 1975]

$$\int_{\mathbb{R}^d} |\nabla f|^2\,dx \ge \frac{1}{2}\int_{\mathbb{R}^d} |v|^2 \log\left(\frac{|f|^2}{\|f\|_2^2}\right) dx + \frac{d}{4}\,\log\left(2\,\pi\,e^2\right)\|f\|_2^2$$

The stability issue

What kind of distance to the manifold $\mathfrak{M}$ of the Aubin-Talenti type functions is measured by the *deficit functional* δ ?

$$\delta[f] := a\,\|\nabla f\|_2^2 + b\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}}\,\|f\|_{2p}^{2p\gamma}$$

Some (not completely satisfactory) answers:

▷ In the critical case $p = d/(d-2)$, $d \geq 3$, [Bianchi, Egnell, 1991]: there is a positive constant $\mathscr{C}$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq \mathscr{C}\,\inf_{\mathfrak{M}}\|\nabla f - \nabla g\|_2^2$$

▷ [JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a constant $\mathscr{C}$ with $1 < \mathscr{C} \leq 1 + \frac{4}{d}$ such that

$$\|\nabla f\|_2^2 - \mathsf{S}_d\,\|f\|_{2^*}^2 \geq \frac{\mathscr{C}}{\mathsf{S}_d\,\|f\|_{2^*}^{2(2^*-2)}}\left(\mathsf{S}_d\,\|f^q\|_{\frac{2d}{d+2}}^2 - \int_{\mathbb{R}^d}|f|^q\,(-\Delta)^{-1}|f|^q\,\mathrm{d}x\right)$$

▷ [Blanchet, Bonforte, JD, Grillo, Vázquez] [JD, Toscani]... various improvements based on entropy methods and fast diffusion flows

A stability result

The *relative entropy*

$$\mathcal{E}[f] := \frac{2p}{1-p} \int_{\mathbb{R}^d} \left(|f|^{p+1} - \mathrm{g}^{p+1} - \frac{1+p}{2p}\, \mathrm{g}^{1-p} \left(|f|^{2p} - \mathrm{g}^{2p} \right) \right) dx$$

Theorem

Let $d \ge 1$, $p \in (1, p^*)$, $A > 0$ and $G > 0$. There is a $\mathscr{C} > 0$ such that

$$\delta[f] \ge \mathscr{C}\, \mathcal{E}[f]$$

for any $f \in \mathscr{W} := \{ f \in \mathrm{L}^1(\mathbb{R}^d, (1+|x|)^2\, dx) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d, dx) \}$ such that

$$\int_{\mathbb{R}^d} |f|^{2p}\, dx = \int_{\mathbb{R}^d} |\mathrm{g}|^{2p}\, dx\,, \quad \int_{\mathbb{R}^d} x\, |f|^{2p}\, dx = 0$$

$$\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f|^{2p}\, dx \le A \quad \text{and} \quad \mathcal{E}[f] \le G$$

Reminder: $\delta[f] := a\, \|\nabla f\|_2^2 + b\, \|f\|_{p+1}^{p+1} - \mathscr{K}_{\mathrm{GN}}\, \|f\|_{2p}^{2p\gamma}$

Some comments

▷ *The constant $\mathscr{C}$ is explicit*

▷ *A Csiszár-Kullback inequality*. There exists a constant $C_p > 0$ such that

$$\left\| |f|^{2p} - g^{2p} \right\|_{\mathrm{L}^1(\mathbb{R}^d)} \le C_p \sqrt{\mathscr{E}[f]} \quad \text{if} \quad \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} = \|g\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}$$

▷ Literature on stability of Sobolev type inequalities is huge:
– Weak $\mathrm{L}^{2^*/2}$-remainder term in bounded domains [Brezis, Lieb, 1985]
– Fractional versions and $(-\Delta)^s$ [Lu, Wei, 2000] [Gazzola, Grunau, 2001] [Bartsch, Weth, Willem, 2003] [Chen, Frank, Weth, 2013]
– Inverse stereographic projection (eigenvalues): [Ding, 1986] [Beckner, 1993] [Morpurgo, 2002] [Bartsch, Schneider, Weth, 2004]
– Symmetrization [Cianchi, Fusco, Maggi, Pratelli, 2009] and [Figalli, Maggi, Pratelli, 2010]

... to be continued

▷ On stability and flows (continued)

– Many other papers by Figalli and his collaborators, among which (most recent ones): [Figalli, Neumayer, 2018] [Neumayer, 2020] [Figalli, Zhang, 2020] [Figalli, Glaudo, 2020]

– Stability for Gagliardo-Nirenberg inequalities [Carlen, Figalli, 2013] [Seuffert, 2017] [Nguyen, 2019]

– Gradient flow issues [Otto, 2001] and many subsequent papers

– Carré du champ applied to the fast diffusion equation [Carrillo, Toscani, 2000] [Carrillo and Vázquez, 2003] [CJMTU, 2001] [Jüngel, 2016]

– Spectral gap properties [Scheffer, 2001] [Denzler, McCann, 2003 & 2005]

▷ On entropy methods

– Carré du champ: the semi-group and Markov precesses point of view [Bakry, Gentil, Ledoux, 2014]

– The PDE point of view (+ some applications to numerical analysis) [Jüngel, 2016]

▷ Global Harnack principle: [Vázquez, 2003] [Bonforte, Vázquez, 2006] [Vázquez, 2006] [Bonforte, Simonov, 2020]

$\Longrightarrow$ Our tool: the fast diffusion equation

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy-entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \ge 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with initial datum $u(t=0,x) = u_0(x) \ge 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathscr{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$\mathscr{U}(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$, $\kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$ and $\mathscr{B}$ is the Barenblatt profile

$$\mathscr{B}(x) := \left(C + |x|^2\right)^{-\frac{1}{1-m}}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} u^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx \quad \text{with} \quad \mathsf{P} = \frac{m}{m-1} \, u^{m-1} \text{ is the } \textit{pressure variable}$$

If u solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

The *Rényi entropy power*

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{2}{d} \, \frac{1}{1-m} - 1$$

applied to self-similar Barenblatt solutions has a linear growth in t

The variation of the Fisher information

Lemma

If u solves $\frac{\partial u}{\partial t} = \Delta u^m$ with $\frac{d-1}{d} =: m_1 \le m < 1$, then

$$\mathrm{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} u \, |\nabla \mathrm{P}|^2 \, \mathrm{d}x = -2 \int_{\mathbb{R}^d} u^m \left(\left\| \mathrm{D}^2 \mathrm{P} - \frac{1}{d} \Delta \mathrm{P} \, \mathrm{Id} \right\|^2 + (m - m_1) \, (\Delta \mathrm{P})^2 \right) \mathrm{d}x$$

▷ This is where the limitation $m \ge m_1 := \frac{d-1}{d}$ appears

.... there are no boundary terms in the integrations by parts ?

The concavity property

Theorem

[Toscani, Savaré, 2014] *Assume that $m_1 \le m < 1$ if $d > 1$ and $m > 1/2$ if $d = 1$. Then $F(t)$ is increasing, $(1-m)\,\mathsf{F}''(t) \le 0$ and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I}$$

[Dolbeault, Toscani, 2016] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \ge \mathsf{E}[\mathscr{B}]^{\sigma-1}\,\mathsf{I}[\mathscr{B}]$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p} \tag{1}$$

$u^{m-1/2} = \frac{f}{\|f\|_{2p}}$ and $p = \frac{1}{2m-1} \in (1, p^*) \Longleftrightarrow \max\left\{\frac{1}{2}, m_1\right\} < m < 1$

Self-similar variables: entropy-entropy production method

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \Delta u^m$$

has a self-similar solution

$$\mathscr{U}(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}}\,\mathscr{B}\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mathscr{B}(x) := \left(1+|x|^2\right)^{-\frac{1}{1-m}}$$

A time-dependent rescaling based on *self-similar variables*

$$u(t,x) = \frac{1}{\kappa^d\, R^d}\, v\left(\tau, \frac{x}{\kappa\, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \tfrac{1}{2}\log\left(\frac{R(t)}{R_0}\right)$$

Then the function v solves *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau} + \nabla\cdot\left[v\left(\nabla u^{m-1} - 2\,x\right)\right] = 0$$

with *same initial datum* $v_0 = u_0$ if $R_0 = R(0) = 1$

Free energy and Fisher information

The function v and $\mathscr{B}$ (same mass) solve the Fokker-Planck type equation

$$\frac{\partial v}{\partial t}+\nabla\cdot\left[v\left(\nabla v^{m-1}-2\,x\right)\right]=0 \tag{3}$$

A Lyapunov functional [Ralston, Newman, 1984]

Generalized entropy or *Free energy*

$$\mathscr{F}[v]:=-\frac{1}{m}\int_{\mathbb{R}^d}\left(v^m-\mathscr{B}^m-m\,\mathscr{B}^{m-1}\,(v-\mathscr{B})\right)\mathrm{d}x$$

Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathscr{F}[v]=-\mathscr{I}[v]\,,\quad \mathscr{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}+2\,x\right|^2\mathrm{d}x$$

The entropy - entropy production inequality

$$\mathscr{B}(x) := \left(1+|x|^2\right)^{-\frac{1}{1-m}}$$

Theorem

[del Pino, JD, 2002] $d \ge 3$, $m \in [m_1, 1)$, $m > \frac{1}{2}$, $\int_{\mathbb{R}^d} v_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x$

$$\int_{\mathbb{R}^d} v \left|\nabla v^{m-1} + 2\,x\right|^2 \mathrm{d}x = \mathscr{I}[v] \ge 4\,\mathscr{F}[v] = 4 \int_{\mathbb{R}^d} \left(\frac{\mathscr{B}^m}{m} - \frac{v^m}{m} + |x|^2 (v - \mathscr{B})\right) \mathrm{d}x$$

$p = \frac{1}{2m-1}$, $v = f^{2p}$

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathscr{C}_{\mathrm{GN}} \, \|f\|_{2p} \iff \delta[f] = \mathscr{I}[v] - 4\,\mathscr{F}[v] \ge 0$$

Corollary

[del Pino, JD, 2002] *A solution v of (3) with initial data $v_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2 v_0 \in L^1(\mathbb{R}^d)$, $v_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\mathscr{F}[v(t,\cdot)] \le \mathscr{F}[v_0] \, e^{-4\,t}$$

A computation on a large ball, with boundary terms

Carré du champ method [Carrillo, Toscani] [Carrillo, Vázquez] [Carrillo, Jüngel, Toscani, Markowich, Unterreiter]

$$\frac{\partial u}{\partial t} + \nabla\cdot\Big[v\Big(\nabla v^{m-1} - 2\,x\Big)\Big] = 0 \quad t>0\,,\quad x\in B_R$$

$$\Big(\nabla v^{m-1} - 2\,x\Big)\cdot\frac{x}{|x|} = 0 \quad t>0\,,\quad x\in\partial B_R$$

$$\frac{d}{dt}\int_{B_R} v\,|\nabla v^{m-1} - 2\,x|^2\,dx + 4\int_{B_R} v\,|\nabla v^{m-1} - 2\,x|^2\,dx$$

$$+\,2\,\frac{1-m}{m}\int_{B_R} v^m\left(\left\|\mathrm{D}^2\big(v^{m-1} - \mathscr{B}^{m-1}\big)\right\|^2 - (1-m)\left|\Delta\big(v^{m-1} - \mathscr{B}^{m-1}\big)\right|^2\right)dx$$

$$= \int_{\partial B_R} v^m\left(\omega\cdot\nabla|\big(v^{m-1} - \mathscr{B}^{m-1}\big)|^2\right)d\sigma \le 0 \text{ (by Grisvard's lemma)}$$

Improvement: $\exists\,\phi$ such that $\phi''>0$, $\phi(0)=0$ and $\phi'(0)=4$ [Toscani, JD]

$$\mathscr{I}[v|\mathscr{B}_\sigma] \ge \phi\big(\mathscr{F}[v|\mathscr{B}_\sigma]\big) \quad\Longleftarrow\quad \text{idea: } \frac{d\mathscr{I}}{dt} + 4\,\mathscr{I} \le -\frac{\mathscr{I}}{\mathscr{F}^2}$$

Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H_1) $\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}}$

(H_2) *if* $d \ge 3$ *and* $m \le m_* := \frac{d-4}{d-2}$, *then* $\left(v_0-\mathscr{B}\right)$ *is integrable*

Theorem

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009] *If* $m<1$ *and* $m \neq m_*$, *then*

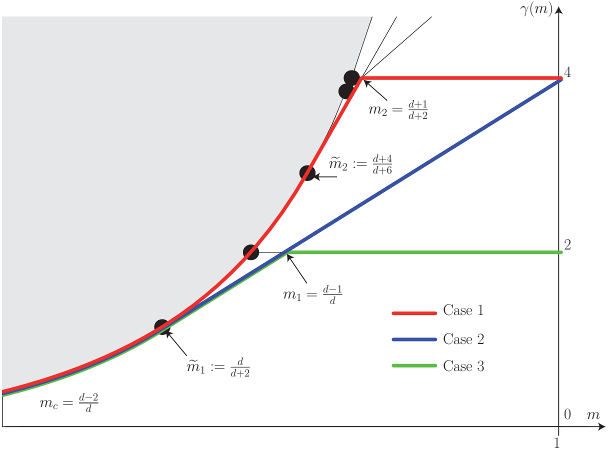
$$\mathscr{F}[v(t,\cdot)] \le C\, e^{-2\gamma(m)\,t} \quad \forall\, t \ge 0\,, \quad \gamma(m) := (1-m)\,\Lambda_{\alpha,d}$$

where $\Lambda_{\alpha,d}>0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in \mathrm{H}^1(d\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$$

with $\alpha := \frac{1}{m-1} < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gap and the asymptotic time layer



$$\mathscr{F}[v(t,\cdot)] \le C\, e^{-2\gamma(m)\,t} \quad \forall\, t \ge 0$$

[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

Spectral gap and improvements... the details

▷ *Asymptotic time layer* [BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2015]

Corollary

Assume that v solves (3): $\partial_t v + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0$ with initial datum $v_0 \ge 0$ such that $\int_{\mathbb{R}^d} v_0 \,\mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \,\mathrm{d}x$

(i) *there is a constant $\mathscr{C}_1 > 0$ such that $\mathscr{F}[v(t,\cdot)] \le \mathscr{C}_1\, e^{-2\gamma(m)\,t}$ with $\gamma(m) = 2$ if $m_1 \le m < 1$*

(ii) *if $m_1 \le m < 1$ and $\int_{\mathbb{R}^d} x\, v_0 \,\mathrm{d}x = 0$, there is a constant $\mathscr{C}_2 > 0$ such that $\mathscr{F}[v(t,\cdot)] \le \mathscr{C}_2\, e^{-2\gamma(m)\,t}$ with $\gamma(m) = 4 - 2\,d\,(1-m)$*

(iii) *Assume that $\frac{d+1}{d+2} \le m < 1$ and $\int_{\mathbb{R}^d} x\, v_0 \,\mathrm{d}x = 0$ and let*

$$\mathscr{B}_\sigma := \sigma^{-\frac{d}{2}}\, \mathscr{B}\big(x/\sqrt{\sigma}\big)$$

be such that $\int_{\mathbb{R}^d} |x|^2\, u(t,x) \,\mathrm{d}x = \int_{\mathbb{R}^d} |x|^2\, \mathscr{B}_\sigma(x) \,\mathrm{d}x$. Then there is a constant $\mathscr{C}_3 > 0$ such that $\mathscr{F}[v(t,\cdot)|\mathscr{B}_\sigma] \le \mathscr{C}_3\, e^{-4\,t}$

▷ *Initial time layer* $\mathscr{I}[v|\mathscr{B}_\sigma] \ge \phi(\mathscr{F}[v|\mathscr{B}_\sigma]) \Rightarrow$ faster decay for $t \sim 0$

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} |g|^2\,\mathscr{B}^{2-m}\,\mathrm{d}x \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,\mathrm{d}x$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,\mathrm{d}x = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,\mathrm{d}x = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,d\,(m - m_1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,\mathrm{d}x = 0$ and

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathscr{Q}[v] \ge 4 + \eta$$

The initial time layer improvement: backward estimate

Rephrasing the *carré du champ* method, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q} - 4)$$

Lemma

Assume that $m > m_1$ and v is a solution to (3) with nonnegative initial datum v_0. If for some $\eta > 0$ and $T > 0$, we have $\mathcal{Q}[v(T,\cdot)] \ge 4 + \eta$, then

$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,T}}{4 + \eta - \eta\,e^{-4\,T}} \quad \forall\, t \in [0, T]$$

Regularity and stability

Our strategy

Choose $\varepsilon > 0$, *small enough*

Get a threshold time $t_\star(\varepsilon)$

0 $\quad$ $t_\star(\varepsilon)$ $\quad$ t

Backward estimate $\quad$ *Forward estimate*

by entropy methods $\quad$ *based on a spectral gap*

Initial time layer $\quad$ Asymptotic time layer

Uniform convergence in relative error: statement

Theorem

Assume that $m \in (m_1, 1)$ *if* $d \geq 2$, $m \in (1/3, 1)$ *if* $d = 1$ *and let* $\varepsilon \in (0, 1/2)$, *small enough,* $A > 0$, *and* $G > 0$ *be given. There exists an explicit time* $t_\star \geq 0$ *such that, if* u *is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{2}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ *satisfying*

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u \, dx \leq A < \infty \tag{H$_A$}$$

$\int_{\mathbb{R}^d} u_0 \, \mathrm{d}x = \int_{\mathbb{R}^d} \mathscr{B} \, \mathrm{d}x = \mathscr{M}$ *and* $\mathscr{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq t_\star$$

Interpolation inequalities

Our regularity theory relies on the inequality

$$\|f\|^2_{\mathrm{L}^{p_m}(B)} \le \mathscr{K} \left(\|\nabla f\|^2_{\mathrm{L}^2(B)} + \|f\|^2_{\mathrm{L}^2(B)} \right)$$

where $B \subset \mathbb{R}^d$ is the unit ball

	p_m	$\mathscr{K}$	q	β
$d \ge 3$	$\frac{2d}{d-2}$	$\frac{2}{\pi}\,\Gamma\big(\frac{d}{2}+1\big)^{2/d}$	$\frac{d}{2}$	α
$d = 2$	4	$0.0564922... < 2/\frac{2}{\sqrt{\pi}} \approx 1.12838$	2	$2\,(\alpha-1)$
$d = 1$	$\frac{4}{m}$	$2^{1+\frac{m}{2}}\,\max\left(\frac{2\,(2-m)}{m\,\pi^2}, \frac{1}{4}\right)$	$\frac{2}{2-m}$	$\frac{2\,m}{2-m}$

Local estimates (1/2): a Herrero-Pierre type lemma

Lemma

For all $(t,R)\in(0,+\infty)^2$, any solution u of (2) satisfies

$$\sup_{y\in B_{R/2}(x)} u(t,y)\le\overline{\kappa}\left(\frac{1}{t^{d/\alpha}}\left(\int_{B_R(x)}u_0(y)\,dy\right)^{2/\alpha}+\left(\frac{t}{R^2}\right)^{\frac{1}{1-m}}\right)$$

$$\overline{\kappa}=\mathsf{k}\,\mathscr{K}^{\frac{2q}{\beta}}$$

$$\mathsf{k}^\beta=\left(\tfrac{4\beta}{\beta+2}\right)^\beta\left(\tfrac{4}{\beta+2}\right)^2\pi^{8(q+1)}\,e^{8\sum_{j=0}^\infty\log(j+1)\left(\frac{q}{q+1}\right)^j}\,2^{\frac{2m}{1-m}}\left(1+\mathfrak{a}\,\omega_d\right)^2\mathfrak{b}$$

$$\omega_d:=|\mathbb{S}^{d-1}|=\frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$\mathfrak{a}=\frac{3\,(16\,(d+1)\,(3+m))^{\frac{1}{1-m}}}{(2-m)\,(1-m)^{\frac{m}{1-m}}}+\frac{2^{\frac{d-m(d+1)}{1-m}}}{3^d\,d}\quad\text{and}\quad\mathfrak{b}=\frac{38^{2(q+1)}}{\left(1-(2/3)^{\frac{\beta}{4(q+1)}}\right)^{4(q+1)}}$$

Local estimates (2/2)

Lemma

For all $R>0$, any solution u of (2) satisfies

$$\inf_{|x-x_0|\le R} u(t,x) \ge \kappa \left(R^{-2}\,t\right)^{\frac{1}{1-m}} \quad \textit{if} \quad t \ge \kappa_\star\, \|u_0\|_{\mathrm{L}^1(B_R(x_0))}^{1-m}\, R^\alpha =: 2\,\underline{t}$$

$$\kappa_\star = 2^{3\,\alpha+2}\,d^{\,\alpha} \quad \text{and} \quad \kappa = \alpha\,\omega_d \left(\frac{(1-m)^4}{2^{38}\,d^{\,4}\,\pi^{16\,(1-m)\,\alpha}\,\overline{\kappa}\,\alpha^2\,(1-m)}\right)^{\frac{2}{(1-m)^2\,\alpha\,d}}$$

Global Harnack Principle

Proposition

Any solution u of (2) satisfies

$$u(t,x) \le B\left(t+\overline{t}-\tfrac{1}{\alpha},x;\overline{M}\right) \quad \forall\,(t,x)\in[\overline{t},+\infty)\times\mathbb{R}^d$$

$$u(t,x) \ge B\left(t-\underline{t}-\tfrac{1}{\alpha},x;\underline{M}\right) \quad \forall\,(t,x)\in[2\,\underline{t},+\infty)\times\mathbb{R}^d$$

$$c:=\max\left\{1,2^{5-m}\,\overline{\kappa}^{1-m}\,\mathrm{b}^{\alpha}\right\},\quad \overline{t}:=c\,t_0\,A^{1-m}$$

$$\overline{M}:=2^{\frac{\alpha}{2(1-m)}}\,\overline{\kappa}^{\frac{\alpha}{2}}\,(1+c)^{\frac{d}{2}}\,\mathrm{b}^{-\frac{d\,\alpha}{2}}\,\mathcal{M}^2$$

$$\underline{M}:=\min\left\{2^{-d/2}\left(\frac{\kappa}{\mathrm{b}^d}\right)^{\alpha/2},\frac{\kappa}{(d\,(1-m))^{d/2}\,\alpha^{\frac{\alpha}{2(1-m)}}}\right\}\kappa_\star^{\frac{1}{1-m}}\,\mathcal{M}^2$$

The outer estimates

Corollary

For any $\varepsilon \in (0, \underline{\varepsilon})$, *any solution* u *of* (2) *satisfies*

$$u(t,x) \ge (1-\varepsilon)\, B(t,x) \quad \text{if} \quad |x| \ge R(t)\, \underline{\rho}(\varepsilon) \quad \text{and} \quad t \ge \underline{T}(\varepsilon)$$

$$u(t,x) \le (1+\varepsilon)\, B(t,x) \quad \text{if} \quad |x| \ge R(t)\, \overline{\rho}(\varepsilon) \quad \text{and} \quad t \ge \overline{T}(\varepsilon)$$

The inner estimate

Proposition

There exist a numerical constant $\mathsf{K}>0$ *and an exponent* $\vartheta\in(0,1)$ *such that, for any* $\varepsilon\in(0,\varepsilon_{m,d})$ *and for any* $t\ge 4\,T(\varepsilon)$*, any solution* u *of* (2) *satisfies*

$$\left|\frac{u(t,x)}{B(t,x)}-1\right|\le\frac{\mathsf{K}}{\varepsilon^{\frac{1}{1-m}}}\left(\frac{1}{t}+\frac{\sqrt{G}}{R(t)}\right)^{\vartheta}\quad\text{if}\quad|x|\le 2\,\rho(\varepsilon)\,R(t)$$

$$\mathsf{K}:=2^{\frac{3d}{\alpha}+\frac{3+6\alpha}{\alpha(1-m)}+\vartheta+10}\,\frac{(\alpha+\mathscr{M})^{\vartheta}}{m^{\vartheta}(1-m)^{2(1+\vartheta)+\frac{2}{1-m}}}$$

$$\times\left[1+\mathsf{b}^{d}\,C_{d,\nu,1}\left(\left(\overline{\kappa}\,\mathscr{M}^{\frac{2}{\alpha}}\,\frac{2^{\nu}}{2^{\nu}-1}+c\right)^{\frac{d}{d+\nu}}+\frac{\mu^{2d}}{\alpha^{\frac{d}{\alpha}}}\,\mathscr{M}^{\frac{d}{d+\nu}}\right)\right]$$

b and c are numerical constants, $\vartheta=\nu/(d+\nu)$ and ν is a Hölder regularity exponent which arises from Harnack's inequality in J. Moser's proof

An L^p-C^ν interpolation

Let $\lfloor u \rfloor_{C^\nu(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\nu}$

Lemma

Let $p \ge 1$ and $\nu \in (0,1)$

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le C_{d,\nu,p}\, \lfloor u \rfloor_{C^\nu(\mathbb{R}^d)}^{\frac{d}{d+p\nu}}\, \|u\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{p\nu}{d+p\nu}} \quad \forall\, u \in \mathrm{L}^p(\mathbb{R}^d) \cap C^\nu(\mathbb{R}^d)$$

$$C_{d,\nu,p} = 2^{\frac{(p-1)(d+p\nu)+dp}{p(d+p\nu)}} \left(1+\frac{d}{\omega_d}\right)^{\frac{1}{p}} \left(1+\left(\frac{d}{p\nu}\right)^{\frac{1}{p}}\right)^{\frac{d}{d+p\nu}} \left(\left(\frac{d}{p\nu}\right)^{\frac{p\nu}{d+p\nu}} + \left(\frac{p\nu}{d}\right)^{\frac{d}{d+p\nu}}\right)^{1/p}$$

The threshold time

Proposition

Let $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/3,1)$ *if* $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ *and* $G>0$

$$t_\star=\mathsf{c}_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a}=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$ *and* $\vartheta=\nu/(d+\nu)$

$$\mathsf{c}_\star=\mathsf{c}_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\,\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\,\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy-entropy production inequality

Theorem

Let $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/2,1)$ *if* $d=1$, $A>0$ *and* $G>0$. *Then there is a positive number* ζ *such that*

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ *such that* $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,\mathrm{d}x=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,\mathrm{d}x=0$ *and* v *satisfies* (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\varepsilon_\star)\,,\quad \varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad T=\frac12\log R(t_\star)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge T$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,T]\,,\quad\text{where}\quad \zeta=\frac{4\,\eta\,e^{-4\,T}}{4+\eta-\eta\,e^{-4\,T}}$$

Two consequences

$$\zeta = \mathrm{Z}(A, \mathscr{F}[u_0])\,, \quad \mathrm{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\,\frac{2}{\alpha}} + G}\,, \quad \zeta_\star := \frac{4\,\eta}{4+\eta}\left(\frac{\varepsilon_\star^{\mathrm{a}}}{2\,\alpha\,\mathrm{c}_\star}\right)^{\frac{2}{\alpha}} c_\alpha$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (3) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,\mathrm{d}x = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v_0\,\mathrm{d}x = 0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)] \leq \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \geq 0$$

▷ The stability in the entropy - entropy production estimate $\mathscr{I}[v] - 4\,\mathscr{F}[v] \geq \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

A general stability result

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2} \right)^{\frac{2p}{d-p(d-4)}}$$

$$\mathsf{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}} \int_{|x|>r} |f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,|f|^{p+1} - \mathsf{g}^{p+1} - \frac{1+p}{2p}\,\mathsf{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,|f|^{2p} - \mathsf{g}^{2p}\right)\right)\mathrm{d}x$$

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}(\mathsf{A}[f],\mathsf{E}[f])$$

Theorem

Let $d\geq 1$ and $p\in(1,p^*)$. For any $f\in\mathscr{W}$, we have

$$\left(\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\right)^{2p\gamma} \geq \mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

References

- M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov. *Stability in Gagliardo-Nirenberg inequalities.* Preprint https://hal.archives-ouvertes.fr/hal-02887010

- M. Bonforte, J. Dolbeault, B. Nazaret, and N. Simonov. *Explicit constants in Harnack inequalities and regularity estimates, with an application to the fast diffusion equation* (supplementary material). Preprint https://hal.archives-ouvertes.fr/hal-02887013

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Thank you for your attention !

Appendix: linear tools and corresponding constants

Linear parabolic equations

$$\frac{\partial v}{\partial t} = \nabla \cdot \big(A(t,x)\, \nabla v \big) \tag{4}$$

on $\Omega_T := (0,T) \times \Omega$, where Ω is an open domain, $A(t,x)$ is a real symmetric matrix such that

$$\lambda_0\, |\xi|^2 \le \sum_{i,j=1}^{d} A_{i,j}(t,x)\, \xi_i\, \xi_j \le \lambda_1\, |\xi|^2 \quad \forall\, (t,x,\xi) \in \mathbb{R}^+ \times \Omega_T \times \mathbb{R}^d$$

for some positive constants λ_0 and λ_1

$$D_R^+(t_0,x_0) := \big(t_0 + \tfrac{3}{4}\, R^2, t_0 + R^2\big) \times B_{R/2}(x_0)$$

$$D_R^-(t_0,x_0) := \left(t_0 - \tfrac{3}{4}\, R^2, t_0 - \tfrac{1}{4}\, R^2\right) \times B_{R/2}(x_0)$$

$$\mathsf{h} := \exp\left[2^{d+4}\, 3^d\, d + c_0^3\, 2^{2\,(d+2)+3} \left(1 + \frac{2^{d+2}}{(\sqrt{2}-1)^{2\,(d+2)}} \right) \sigma \right]$$

$$c_0 = 3^{\frac{2}{d}}\, 2^{\frac{(d+2)\,(3\,d^2+18\,d+24)+13}{2\,d}} \left(\frac{(2+d)^{1+\frac{4}{d^2}}}{d^{1+\frac{2}{d^2}}} \right)^{(d+1)(d+2)} \mathcal{K}^{\frac{2\,d+4}{d}}$$

Harnack inequality (Moser)

Theorem

Let $T>0$, $R\in(0,\sqrt{T})$, and take $(t_0,x_0)\in(0,T)\times\Omega$ such that $(t_0-R^2,t_0+R^2)\times B_{2R}(x_0)\subset\Omega_T$. If u is a weak solution of (4), then

$$\sup_{D_R^-(t_0,x_0)} v\le\overline{\mathsf{h}}\inf_{D_R^+(t_0,x_0)} v$$

with $\overline{\mathsf{h}}:=\mathsf{h}^{\lambda_1+1/\lambda_0}$

This Harnack inequality goes back to [Moser, 1964] [Moser1971]
The dependence of the constant on λ_0 and λ_1 is optimal [Moser, 1971]
The dependence of h on d is pointed out in [Gutierrez, Wheeden, 1990]
To our knowledge, this is the first explicit expression of h

Harnack inequality implies Hölder continuity

Let $\Omega_1 \subset \Omega_2 \subset \mathbb{R}^d$ two bounded domains and let us consider
$Q_1 := (T_2, T_3) \times \Omega_1 \subset (T_1, T_4) \times \Omega_2 =: Q_2$, where $0 \le T_1 < T_2 < T_3 < T < 4$

We define the *parabolic distance* between Q_1 and Q_2 as

$$d(Q_1, Q_2) := \inf_{\substack{(t,x) \in Q_1 \\ (s,y) \in [T_1, T_4] \times \partial\Omega_2 \cup \{T_1, T_4\} \times \Omega_2}} |x-y| + |t-s|^{\frac{1}{2}}$$

Theorem

If v is a nonnegative solution of (4) on Q_2, then

$$\sup_{(t,x),(s,y) \in Q_1} \frac{|v(t,x) - v(s,y)|}{\left(|x-y| + |t-s|^{1/2}\right)^{\nu}} \le 2 \left(\frac{128}{d(Q_1, Q_2)} \right)^{\nu} \|v\|_{\mathrm{L}^\infty(Q_2)}$$

where $\nu := \log_4 \left(\frac{\overline{\mathsf{h}}}{\overline{\mathsf{h}} - 1} \right)$

Appendix: still another interpolation inequality

An L^p-C^ν interpolation

Let $\lfloor u \rfloor_{C^\nu(\Omega)} := \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u(x)-u(y)|}{|x-y|^\nu}$

Lemma

Let $p \ge 1$ and $\nu \in (0,1)$. Then there exists a positive constant $C_{d,\nu,p}$ such that, for any $R>0$ and $x \in \mathbb{R}^d$

$$\|u\|_{\mathrm{L}^\infty(B_R(x))} \le C_{d,\nu,p} \left(\lfloor u \rfloor_{C^\nu(B_{2R}(x))}^{\frac{d}{d+p\nu}} \|u\|_{\mathrm{L}^p(B_{2R}(x))}^{\frac{p\nu}{d+p\nu}} + R^{-\frac{d}{p}} \|u\|_{\mathrm{L}^p(B_{2R}(x))} \right)$$

for any $u \in \mathrm{L}^p(B_{2R}(x)) \cap C^\nu(B_{2R}(x))$, and

$$\|u\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \le C_{d,\nu,p} \lfloor u \rfloor_{C^\nu(\mathbb{R}^d)}^{\frac{d}{d+p\nu}} \|u\|_{\mathrm{L}^p(\mathbb{R}^d)}^{\frac{p\nu}{d+p\nu}} \quad \forall\, u \in \mathrm{L}^p(\mathbb{R}^d) \cap C^\nu(\mathbb{R}^d)$$

$$C_{d,\nu,p} = 2^{\frac{(p-1)(d+p\nu)+dp}{p(d+p\nu)}} \left(1+\frac{d}{\omega_d}\right)^{\frac{1}{p}} \left(1+\left(\frac{d}{p\nu}\right)^{\frac{1}{p}}\right)^{\frac{d}{d+p\nu}} \left(\left(\frac{d}{p\nu}\right)^{\frac{p\nu}{d+p\nu}} + \left(\frac{p\nu}{d}\right)^{\frac{d}{d+p\nu}}\right)^{1/p}$$