Symmetry in interpolation inequalities

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Outline

- **Caffarelli-Kohn-Nirenberg inequalities**
  ▷ the symmetry issue
  ▷ the result
  ▷ the critical and the subcritical regime
- **The proof of the symmetry result**
  ▷ Rényi entropy powers *vs.* the Bakry-Emery method
  ▷ A rigidity result
  ▷ The strategy of the proof
- **Inequalities and flows on compact manifolds: the sphere**
  ▷ Flows on the sphere, improvements in the subcritical range
  ▷ Can one prove Sobolev's inequalities with a heat flow ?
  ▷ The *bifurcation* viewpoint; constraints, improved inequalities
- **Fast diffusion equations with weights: large time asymptotics**
  ▷ Relative uniform convergence, asymptotic rates, asymptotic and global estimates
- **Entropy methods, gradient flows and rates of convergence**

Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)

and also: S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk

Some features of the method

$\triangleright$ Characterize bubble-like functions (in presence of weights)

$\triangleright$ A new way of testing elliptic equations, similar to Pohozaev's method ?

$\triangleright$ Uniqueness (rigidity) and Bakry-Emery methods unified through a non-linear flow interpretation

$\triangleright$ The non-linear parabolic flow interpretation as a tool to extend local (bifurcation) results to global (uniqueness) results

$\triangleright$ Linearization: why is the method optimal in case of Caffarelli-Kohn-Nirenberg inequalities ?

Background references (entropy methods)

- Rigidity methods, uniqueness in nonlinear elliptic PDE's: [B. Gidas, J. Spruck, 1981], [M.-F. Bidaut-Véron, L. Véron, 1991]
- Probabilistic methods (Markov processes), semi-group theory and *carré du champ* methods (Γ_2 theory): [D. Bakry, M. Emery, 1984], [Bakry, Ledoux, 1996], [Demange, 2008], [JD, Esteban, Loss, 2014 & 2015] → *D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs

  ▷ Entropy-entropy production inequalities: Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Unterreiter, Villani..., [del Pino, JD, 2001], [Blanchet, Bonforte, JD, Grillo, Vázquez] → *A. Jüngel, Entropy Methods for Diffusive Partial Differential Equations (2016)*

  ▷ Mass transportation: [Otto] → *C. Villani, Optimal transport. Old and new (2009)*

  ▷ Rényi entropy powers (information theory) [Savaré, Toscani, 2014], [Dolbeault, Toscani]

Caffarelli-Kohn-Nirenberg inequalities and symmetry issues

Critical Caffarelli-Kohn-Nirenberg inequalities

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

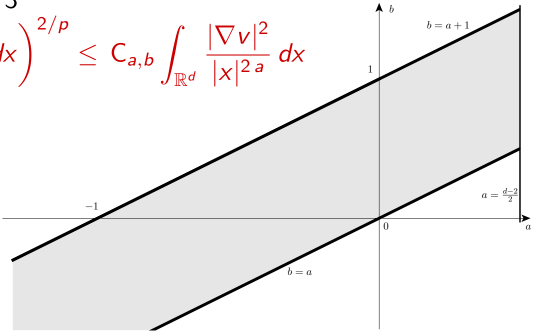
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

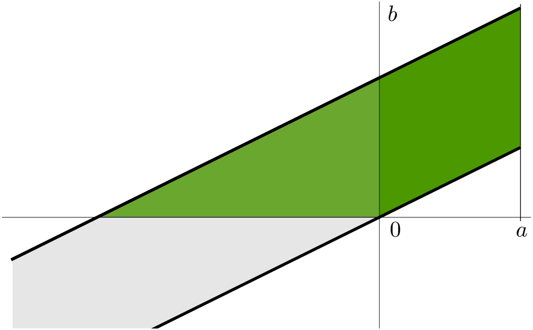
$a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$

and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]
[F. Catrina, Z.-Q. Wang (2001)]

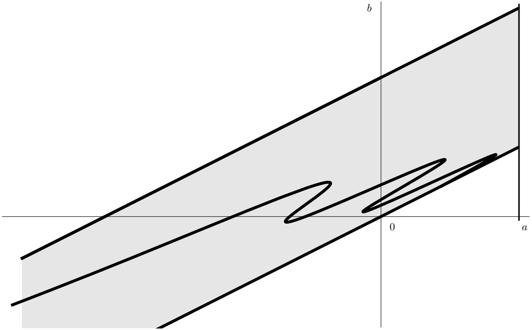
Symmetry 1: moving planes and symmetrization



[Chou, Chu], [Horiuchi]
[Betta, Brock, Mercaldo, Posteraro]
+ Perturbation results: [CS Lin, ZQ Wang], [Smets, Willem], [JD, Esteban, Tarantello 2007], [JD, Esteban, Loss, Tarantello, 2009]

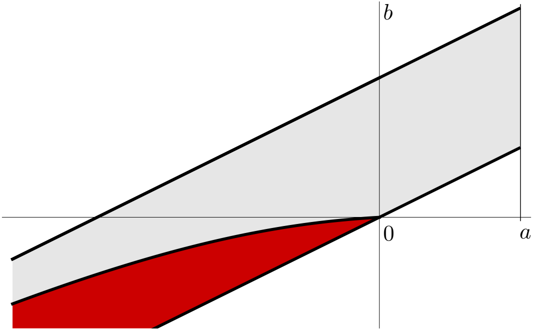
The threshold between symmetry and symmetry breaking

[F. Catrina, Z.-Q. Wang]



[JD, Esteban, Loss, Tarantello, 2009] There is a curve which separates the symmetry region from the symmetry breaking region, which is parametrized by a function $p \mapsto a + b$

Linear instability of radial minimizers: the Felli-Schneider curve

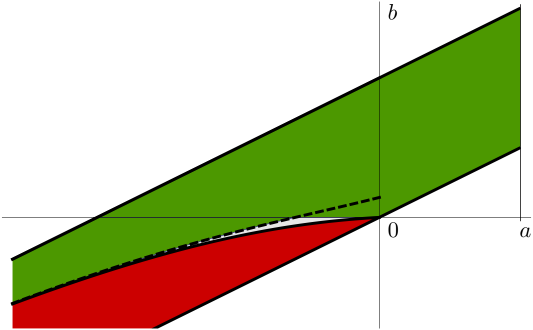


[Catrina, Wang], [Felli, Schneider] The functional

$$C^\star_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx-\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v=v_\star$

Direct spectral estimates



[JD, Esteban, Loss, 2011]: sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

The Emden-Fowler transformation and the cylinder

▷ [F. Catrina, Z.-Q. Wang (2001)] *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r=|x|\,, \quad s=-\log r \quad \text{and} \quad \omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}\quad\forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda := (a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a=a_c\pm\sqrt{\Lambda} \quad \text{and} \quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

The bifurcation point of view (p or $b-a$ fixed)

Define

$$\mathcal{Q}_\Lambda[\varphi] := \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

and look for

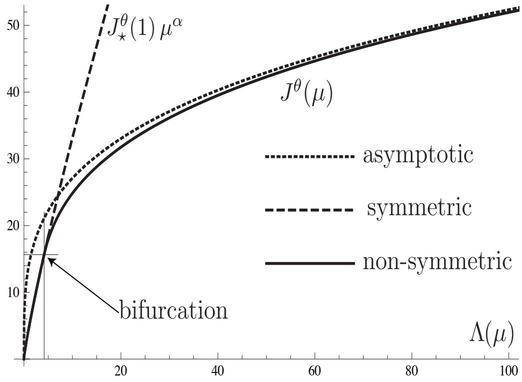
$$\mu(\Lambda) := \inf_{\varphi\in\mathrm{H}^1(\mathcal{C})} \mathcal{Q}_\Lambda[\varphi]$$

compared to

$$\mu_\star(\Lambda) := \inf_{\varphi\in\mathrm{H}^1(\mathbb{R})} \mathcal{Q}_\Lambda[\varphi] = K_\star\,\Lambda^{\frac{1}{\alpha}}$$

As $\Lambda > 0$ increases, *symmetry breaking* occurs when $\mu(\Lambda) < \mu_\star(\Lambda)$

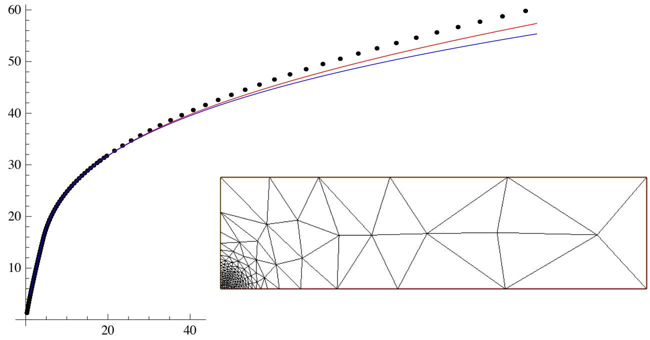
Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

Other evidences

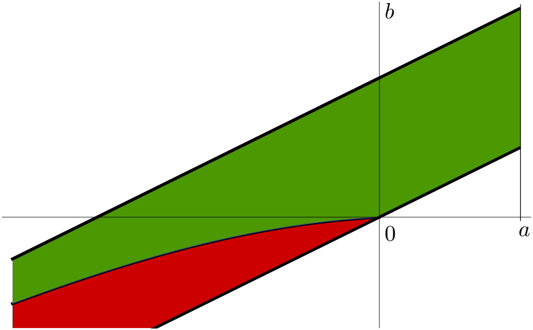
- Further numerical results [JD, Esteban, 2012] (coarse / refined / self-adaptive grids)



- Formal commutation of the non-symmetric branch near the bifurcation point [JD, Esteban, 2013]
- Asymptotic energy estimates [JD, Esteban, 2013]

Symmetry *versus* symmetry breaking: the sharp result in the critical case

A result based on entropies and nonlinear flows



[JD, Esteban, Loss (Inventiones 2016)]

The symmetry result in the critical case

The Felli & Schneider curve is defined by

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$

Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

Interpolation and subcritical Caffarelli-Kohn-Nirenberg inequalities

Caffarelli-Kohn-Nirenberg inequalities (with two weights)

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^q(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p} \, \|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)} \, \|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,, \quad \gamma-2 < \beta < \tfrac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with } p_\star := \tfrac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

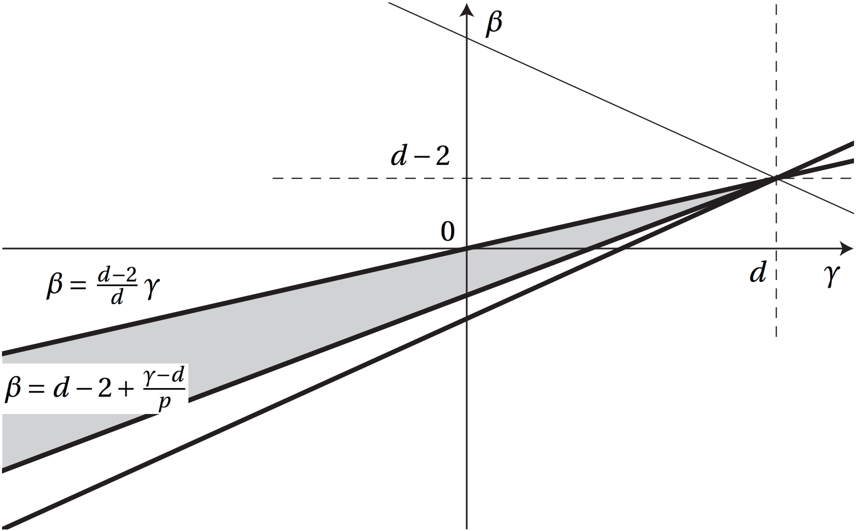
$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

- [JD, Muratori, Nazaret] $\beta = 0$, $\gamma > 0$ small...
- General case: is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters



With two weights: a symmetry breaking result

Let us define

$$\beta_{\rm FS}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad \text{and} \quad \beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$, $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal.

Symmetry and symmetry breaking

[JD, Esteban, Loss, Muratori, 2016]

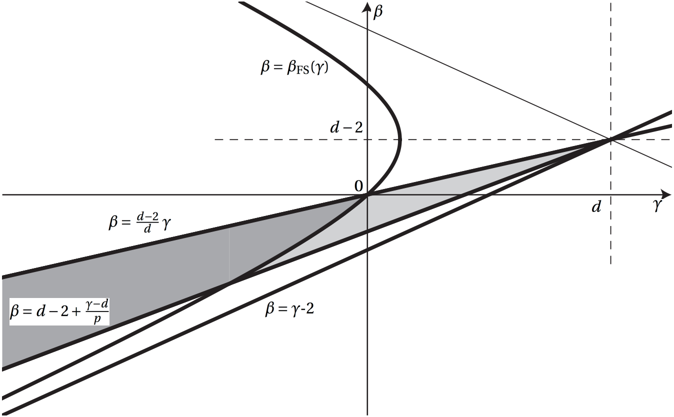
Let us define $\beta_{\rm FS}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$

Theorem

Symmetry breaking holds in (CKN) if and only if

$$\gamma < 0 \quad \textit{and} \quad \beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$, $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal.



The grey area corresponds to the admissible cone. The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha < \alpha_{\mathrm{FS}} \quad \text{with} \quad \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta \log\left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta)\log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{J}\left[v_\star + \varepsilon\,\mu_{\delta/2}\,f\right] = \tfrac{1}{2}\,\varepsilon^2\,\vartheta\,\mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \geq 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \geq n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta - n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta + n - 2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta - n)\,(\delta - 1)}$.

$\mathcal{Q} \geq 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \leq \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \iff \eta < 1$$

$$\iff \frac{d-1}{\alpha^2} = \eta\,(\eta + n - 2) < n - 1 \iff \alpha > \alpha_{\rm FS}$$

Fast diffusion equations with weights: a symmetry result

- Rényi entropy powers
- The symmetry result
- The strategy of the proof

Joint work with M.J. Esteban, M. Loss in the critical case $\beta = d - 2 + \frac{\gamma - d}{p}$

Joint work with M.J. Esteban, M. Loss and M. Muratori in the subcritical case $d - 2 + \frac{\gamma - d}{p} < \beta < \frac{d-2}{d}\,\gamma$

Rényi entropy powers (no weights)

We consider the flow $\frac{\partial u}{\partial t} = \Delta u^m$ and the Gagliardo-Nirenberg inequalities (GN)

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\rm GN}_{p,d}\,\|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)}\,\|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

where $u = w^{2p}$, that is, $w = u^{m-1/2}$ with $p = \frac{1}{2\,m-1}$. Straightforward computations show that (GN) can be brought into the form

$$\left(\int_{\mathbb{R}^d} u\,dx\right)^{(\sigma+1)\,m-1} \le C\,\mathcal{I}\,\mathcal{E}^{\sigma-1} \quad \text{where} \quad \sigma = \frac{2}{d\,(1-m)} - 1$$

where $\mathcal{E} := \int_{\mathbb{R}^d} u^m\,dx$ and $\mathcal{I} := \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$, $\mathsf{P} = \frac{m}{1-m}\,u^{m-1}$ is the *pressure variable*. If $\mathcal{F} = \mathcal{E}^\sigma$ is the *Rényi entropy power* and $\sigma = \frac{2}{d}\,\frac{1}{1-m} - 1$, then $\mathcal{F}''$ is proportional to

$$-2\,(1-m)\,\left\langle \mathrm{Tr}\left(\left(\mathrm{Hess}\,\mathsf{P} - \tfrac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right)^2\right)\right\rangle + (1-m)^2\,(1-\sigma)\,\left\langle\left(\Delta\mathsf{P} - \langle\Delta\mathsf{P}\rangle\right)^2\right\rangle$$

where we have used the notation $\langle A\rangle := \int_{\mathbb{R}^d} u^m\,A\,dx \big/ \int_{\mathbb{R}^d} u^m\,dx$

A rigidity result

We actually prove a rigidity (uniqueness) result

▷ critical case: [JD, Esteban, Loss; Inventiones]
▷ subcritical case: [JD, Esteban, Loss, Muratori]

Theorem

Assume that $\beta \le \beta_{\rm FS}(\gamma)$. Then all positive solutions in $\mathrm H^p_{\beta,\gamma}(\mathbb R^d)$ of

$$-\,div\left(|x|^{-\beta}\,\nabla w\right) = |x|^{-\gamma}\left(w^{2p-1} - \; w^p\right) \quad in \quad \mathbb R^d\setminus\{0\}$$

are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$

The strategy of the proof (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,, \quad \alpha = 1 + \frac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta} \quad \forall\,v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1+|x|^2\right)^{-1/(p-1)} \quad \forall\,x \in \mathbb{R}^d$$

The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathfrak{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

With $\mathcal{L}_\alpha = -\mathfrak{D}_\alpha^* \mathfrak{D}_\alpha = \alpha^2 \left(u'' + \frac{n-1}{s} u'\right) + \frac{1}{s^2} \Delta_\omega u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}_\alpha u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned}
&- \tfrac{d}{dt} \mathcal{G}[u(t,\cdot)] \left(\textstyle\int_{\mathbb{R}^d} u^m \, d\mu\right)^{1-\sigma} \\
&\geq (1-m)(\sigma-1) \int_{\mathbb{R}^d} u^m \left| \mathcal{L}_\alpha \mathsf{P} - \frac{\int_{\mathbb{R}^d} u \, |\mathfrak{D}_\alpha \mathsf{P}|^2 \, d\mu}{\int_{\mathbb{R}^d} u^m \, d\mu} \right|^2 d\mu \\
&+ 2 \int_{\mathbb{R}^d} \left(\alpha^4 \left(1 - \tfrac{1}{n}\right) \left| \mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega \mathsf{P}}{\alpha^2 (n-1) s^2} \right|^2 + \frac{2\alpha^2}{s^2} \left| \nabla_\omega \mathsf{P}' - \frac{\nabla_\omega \mathsf{P}}{s} \right|^2 \right) u^m \, d\mu \\
&+ 2 \int_{\mathbb{R}^d} \left((n-2) \left(\alpha_{\rm FS}^2 - \alpha^2\right) |\nabla_\omega \mathsf{P}|^2 + c(n,m,d) \frac{|\nabla_\omega \mathsf{P}|^4}{\mathsf{P}^2} \right) u^m \, d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \leq \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result.

(3/3: elliptic regularity, boundary terms)

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathcal{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathcal{L}_\alpha u^m\,d\mu\ge\mathcal{H}[u]\ge 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Comments and open questions

▷ Flows on the sphere, improvements in the subcritical range

▷ Can one prove Sobolev's inequalities with a heat flow ? Rényi entropy powers *vs.* the Bakry-Emery method

▷ The *bifurcation* point of view

▷ Some open problems: constraints, improved inequalities, nodal solutions

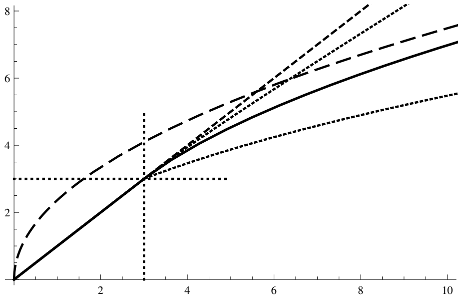
▷ A parabolic theory in the Euclidean case (so far: formal)

[Bakry, Emery, 1984]
[Bidault-Véron, Véron, 1991], [Bakry, Ledoux, 1996]
[Demange, 2008], [JD, Esteban, Loss, 2014 & 2015]

The bifurcation point of view (on the sphere)

$\mu(\lambda)$ is the optimal constant in the functional inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$



Here $d = 3$ *and* $p = 4$

Antipodal symmetry (on the sphere)

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

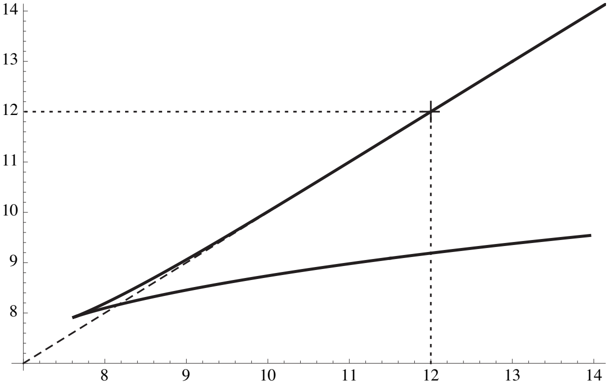
If $p \in (1,2) \cup (2,2^*)$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{p-2}\left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1}\right]\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{2}\frac{(d+3)^2}{(d+1)^2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

Branches of antipodal solutions



Case $d = 5$, $p = 3$: values of the shooting parameter a as a function of λ

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$\partial_t v + |x|^\gamma \, \nabla \cdot \Big[|x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$$\zeta := 1 - \left(1 - \frac{2-m}{(1-m)\,q}\right)\left(1 - \frac{2-m}{1-m}\,\theta\right)$$
$$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma} \text{ is in the range } 0<\theta<\frac{1-m}{2-m}<1$$

Theorem

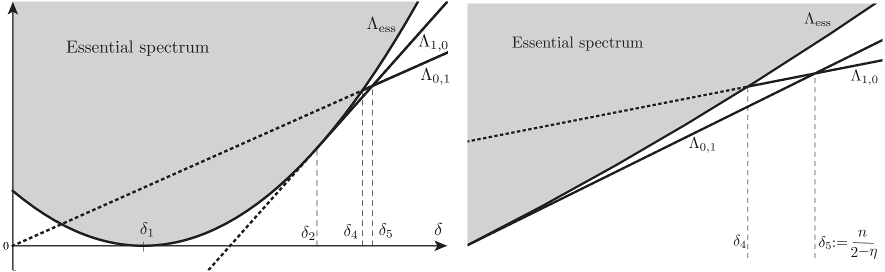
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q\in\left[\frac{2-m}{1-m},\infty\right]$, the function $w=v/\mathfrak{B}$ satisfies

$$\|w(t)-1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)}\le\mathcal{K}\,e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)}\quad\forall\,t\ge t_0$$

in the case $\gamma\in(0,d)$, and

$$\|w(t)-1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)}\le\mathcal{K}\,e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)}\quad\forall\,t\ge t_0$$

in the case $\gamma\le 0$



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum.

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\,\partial_t w = -\,\frac{1}{\mathfrak{B}}\,\nabla\cdot\Big(|x|^{-\beta}\,\mathfrak{B}\,w\,\nabla\left(\left(w^{m-1}-1\right)\mathfrak{B}^{m-1}\right)\Big) & \text{in } \mathbb{R}^+\times\mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*, relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

(1/2) Harnack inequality and Hölder regularity

We change variables: $x \mapsto |x|^{\alpha-1}\,x$ and adapt the ideas of F. Chiarenza and R. Serapioni to

$$\partial_t u + \mathsf{D}_\alpha^* \Big[\mathsf{a}\,(\mathsf{D} u + \mathsf{B}\,u) \Big] = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d$$

Proposition (A parabolic Harnack inequality)

Let $d \ge 2$, $\alpha > 0$ and $n > d$. If u is a bounded positive solution, then for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $r > 0$ such that $Q_r(t_0, x_0) \subset \mathbb{R}^+ \times B_1$, we have

$$\sup_{Q_r^-(t_0,x_0)} u \le H \inf_{Q_r^+(t_0,x_0)} u$$

The constant $H > 1$ depends only on the local bounds on the coefficients a, B *and on d, α, and n*

By adapting the classical method *à la De Giorgi* to our weighted framework: Hölder regularity at the origin

Regularity (2/2): from local to global estimates

Lemma

If w is a solution of the the Ornstein-Uhlenbeck equation with initial datum w_0 bounded from above and from below by a Barenblatt profile (+ relative mass condition) = "good solutions", then there exist $\nu \in (0,1)$ and a positive constant $\mathcal{K} > 0$, depending on d, m, β, γ, C, C_1, C_2 such that:

$$\|\nabla v(t)\|_{\mathrm{L}^\infty(B_{2\lambda}\setminus B_\lambda)} \le \frac{Q_1}{\lambda^{\frac{2+\beta-\gamma}{1-m}+1}} \quad \forall\, t \ge 1\,, \quad \forall\, \lambda > 1$$

$$\sup_{t\ge1} \|w\|_{C^k((t,t+1)\times B_\varepsilon^c)} < \infty \quad \forall\, k \in \mathbb{N}\,,\ \forall\, \varepsilon > 0$$

$$\sup_{t\ge1} \|w(t)\|_{C^\nu(\mathbb{R}^d)} < \infty$$

$$\sup_{\tau\ge t} |w(\tau) - 1|_{C^\nu(\mathbb{R}^d)} \le \mathcal{K} \sup_{\tau\ge t} \|w(\tau) - 1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \quad \forall\, t \ge 1$$

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, with $m \neq m_*$ with $m_* :=$. Under the relative mass condition, for any "good solution" v there exists a positive constant $\mathcal{C}$ such that

$$\mathcal{F}[v(t)] \leq \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\,t} \quad \forall\, t \geq 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates cane be obtained using "best matching techniques"

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t}\quad\forall\,t\ge0\quad\text{with}\quad\Lambda_\star:=\tfrac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

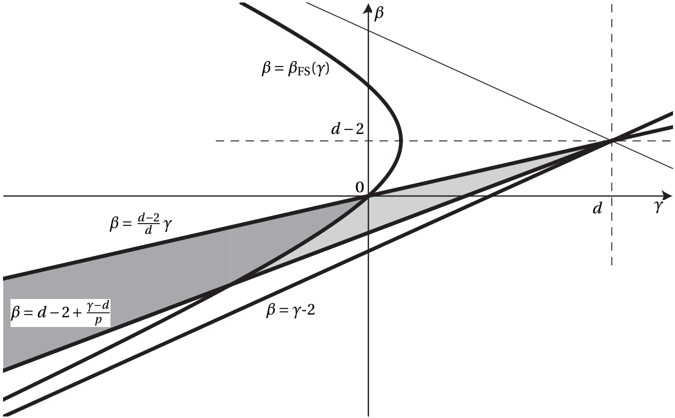
Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{F}[v]\le\mathcal{I}[v]\quad\forall\,v\in\mathcal{L}^{1,\gamma}(\mathbb{R}^d)\quad\text{such that}\quad\|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}=M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M):=\frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t}\quad\forall\,t\ge0$$

The symmetry breaking region (again)



... consequences for the entropy – entropy production inequalities (a measurement in relative entropy of the global rates of convergence)

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any* $M>0$, *we have*
$$0<\mathcal{K}(M)\le\frac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) and* $\beta\neq\beta_{\rm FS}(\gamma)$*,then*
$$\mathcal{K}(M)>\frac{1-m}{m}\,(2+\beta-\gamma)^2$$

Corollary

Assume that $m\in[m_1,1)$

(i) For any $M>0$*, if* $\Lambda(M)=\Lambda_\star$ *then* $\beta=\beta_{\rm FS}(\gamma)$

(ii) If $\beta>\beta_{\rm FS}(\gamma)$ *then* $\Lambda_{0,1}<\Lambda_\star$ *and* $\Lambda(M)\in(0,\Lambda_{0,1}]$ *for any* $M>0$

(iii) For any $M>0$*, if* $\beta<\beta_{\rm FS}(\gamma)$ *and if symmetry holds in (CKN), then*
$$\Lambda(M)>\Lambda_\star$$

Summary and comments

▷ Linearization / linear instability determines the symmetry breaking range

▷ *Ad hoc* flows can be build to obtain strict monotonicity results (parabolic point of view) and prove the uniqueness of critical points (elliptic point of view). These flows allow us to extend local results (bifurcations) to global results (uniqueness)

▷ Studying large time asymptotics amounts to linearize around asymptotic profiles

▷ Large time asymptotics *do not* determine the global rates as measured by *entropy methods*

▷ ... but large time asymptotics *do* determine the *symmetry range*: why ?

Entropy methods, gradient flows and rates of convergence

▷ The Bakry-Emery method (Fokker-Planck and Ornstein-Uhlenbeck)

▷ Gradient flow interpretation

▷ Linearization: why is the method optimal ?

The Fokker-Planck equation

The linear Fokker-Planck (FP) equation in the Euclidean case, *in presence of a potential / drift term*

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \phi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \phi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

Unique stationary solution (mass normalized to 1): $\gamma = \dfrac{e^{-\phi}}{\int_\Omega e^{-\phi}\,dx}$

With $v = u/\gamma$, (FP) is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \phi \cdot \nabla v =: \mathcal{L}\, v$$

The Bakry-Emery method

With v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$, the q-*entropy* is defined by

$$\mathcal{E}[v] := \frac{1}{q-1}\int_\Omega \left(v^q - 1 - q\,(v-1)\right) d\gamma$$

Under the action of (OU), with $w = v^{q/2}$, $\mathcal{I}[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$,

$$\frac{d}{dt}\mathcal{E}[v(t,\cdot)] = -\mathcal{I}[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\left(\mathcal{I}[v] - 2\,\lambda\,\mathcal{E}[v]\right) \le 0$$

with $\lambda := \inf_{w \in H^1(\Omega,d\gamma)\setminus\{0\}} \dfrac{\int_\Omega \left(2\,\frac{q-1}{q}\,\|\mathrm{Hess}\,w\|^2 + \mathrm{Hess}\,\phi : \nabla w \otimes \nabla w\right) d\gamma}{\int_\Omega |w|^2\,d\gamma}$

Proposition

[Bakry, Emery, 1984] [JD, Nazaret, Savaré, 2008] *Let Ω be convex. If $\lambda > 0$ and v is a solution of (OU), then $\mathcal{I}[v(t,\cdot)] \le \mathcal{I}[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ and $\mathcal{E}[v(t,\cdot)] \le \mathcal{E}[v(0,\cdot)]\,e^{-2\,\lambda\,t}$ for any $t \ge 0$ and, as a consequence,*

$$\mathcal{I}[v] \ge 2\,\lambda\,\mathcal{E}[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma)$$

Some remarks

- Grisvard's lemma: by convexity of Ω, boundary terms have the right sign

- *On a manifold, the curvature tensor replaces* $\mathrm{Hess}\,\phi$

- What is still to be covered:
  - The gradient flow interpretation (Euclidean space, linear flow)
  - The explanation of why the method for proving symmetry is optimal

Gradient flow interpretation (Ornstein-Uhlenbeck)

$\mathbb{R}^d$ *case:* a question by F. Poupaud. Let ϕ s.t. $\mathrm{Hess}\,\phi \geq \lambda\,\mathrm{I}$, $\mu := e^{-\phi}\,\mathcal{L}^d$

- *Entropy* : $\mathcal{E}(\rho) := \int_{\mathbb{R}^d} \psi(\rho)\,d\mu$
- *Action density* : $\phi(\rho, \mathbf{w}) := \frac{|\mathbf{w}|^2}{h(\rho)}$, $1/h = \psi''$
- *Action functional* : $\Phi(\rho, \mathbf{w}) := \int_{\mathbb{R}^d} \phi(\rho, \mathbf{w})\,d\gamma$

• $\Gamma(\mu_0, \mu_1)$: $(\mu_s, \boldsymbol{\nu}_s)_{s\in[0,1]}$ is an *admissible path* connecting μ_0 to μ_1 if there is a solution $(\mu_s, \boldsymbol{\nu}_s)_{s\in[0,1]}$ to the *continuity equation*

$$\partial_s \mu_s + \nabla \cdot \boldsymbol{\nu}_s = 0\,, \quad s \in [0,1]$$

• *h-Wasserstein distance*

$$W_h^2(\mu_0, \mu_1) := \inf\left\{ \int_0^1 \Phi(\mu_s, \boldsymbol{\nu}_s)\,ds \;:\; (\mu, \boldsymbol{\nu}) \in \Gamma(\mu_0, \mu_1)\right\}$$

[JD, Nazaret, Savaré]: (OU) is the gradient flow of $\mathcal{E}$ w.r.t. W_h

Optimality of the method in GN inequalities: linearization

Without weights, but with a nonlinear flow: let u be a solution to

$$\frac{\partial u}{\partial t}+\nabla\cdot\big(u\,(\nabla u^{m-1}-2\,x)\big)=0$$

Gagliardo-Nirenberg inequalities

Barenblatt functions $\mathcal{B}(x)=(1+|x|^2)^{1/(m-1)}$ are stationary solutions
Scalar products

$$\langle f_1,f_2\rangle_B=\int_{\mathbb{R}^d}f_1\,f_2\,B^{2-m}\,dx\quad\text{and}\quad\langle\!\langle f_1,f_2\rangle\!\rangle_B=\int_{\mathbb{R}^d}\nabla f_1\cdot\nabla f_2\,B\,dx$$

Linearization

$$u_\varepsilon=B\,\left(1+\varepsilon\,\tfrac{m}{m-1}\,B^{1-m}\,f\right)$$

Linearized evolution equation

$$\frac{\partial f}{\partial t}=\mathcal{L}_u\,f\quad\text{where}\quad\mathcal{L}_u\,f:=m\,u^{m-2}\,\nabla\cdot(u\,\nabla f)$$

Optimality of the method in GN inequalities: spectral gap

$$\frac{\partial f}{\partial t} = \mathcal{L}_u f \quad \text{where} \quad \mathcal{L}_u f := m\, u^{m-2}\, \nabla \cdot (u\, \nabla f)$$

At formal level, it is straightforward to check that

$$\frac{1}{2}\,\frac{d}{dt}\langle f, f\rangle_B = \int_{\mathbb{R}^d} f\, \mathcal{L}_u f\, B^{2-m}\, dx = -\int_{\mathbb{R}^d} |\nabla f|^2\, B\, dx = -\,\langle\!\langle f, f\rangle\!\rangle_B$$

With the spectral gap

$$\langle\!\langle f, f\rangle\!\rangle_B \ge \Lambda_\star \langle f, f\rangle_B$$

as $t \to +\infty$ one gets the *asymptotic rates of convergence*

$$\mathcal{F}[u] := \frac{1}{m-1}\int_{\mathbb{R}^d} \left(u^m - B^m - m\, B^{m-1}\,(u-B)\right)\, dx$$

$$\sim \frac{\varepsilon^2}{2}\,\langle f, f\rangle_B \le \frac{\varepsilon^2}{2\,\Lambda_\star}\,\langle\!\langle f, f\rangle\!\rangle_B \sim \frac{1}{2\,\Lambda_\star}\,\mathcal{I}[u]$$

$$\text{where} \quad \mathcal{I}[u] := \frac{m^2}{(m-1)^2}\int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - \nabla B^{m-1}\right|^2\, dx = -\frac{d}{dt}\mathcal{F}[u]$$

Optimality of the method in GN inequalities: global rates

Global rates of convergence: from $\frac{d}{dt}\mathcal{F}[u] = -\mathcal{I}[u]$ and the *entropy – entropy production* inequality

$$\mathcal{I}[u] \geq 2\,\Lambda_\star\,\mathcal{F}[u]$$

we get that

$$\mathcal{F}[u(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-\,2\,\Lambda_\star\,t}$$

Optimality in the variational problem

$$\inf_u \frac{\mathcal{I}[u]}{\mathcal{F}[u]} = \frac{1}{2\,\Lambda_\star}$$

is achieved in the limit corresponding to $u_\varepsilon = B\,\left(1+\varepsilon\,\frac{m}{m-1}\,B^{1-m}\,f\right)$ with $\varepsilon \to 0$ because the optimality in the variational problem

$$\inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \quad \text{where} \quad \mathcal{K}[u] := -\,\frac{d}{dt}\mathcal{I}[u(t,\cdot)]$$

is also achieved in the limit as $\varepsilon \to 0$

Optimality of the method in GN inequalities: conclusion

Optimality is achieved in the asymptotic regime

$$\frac{1}{2\,\varepsilon^2}\,\frac{d}{dt}\mathcal{I}[u(t,\cdot)]$$

$$\sim \frac{1}{2}\,\frac{d}{dt}\langle\langle f,f\rangle\rangle_B = \int_{\mathbb{R}^d} \nabla f\cdot\nabla\mathcal{L}_u\,f\,u\,dx$$

$$= -\,\langle\langle f,\mathcal{L}_u\,f\rangle\rangle_B \sim \frac{1}{2\,\varepsilon^2}\,\mathcal{K}[u(t,\cdot)]$$

Consider the eigenfunction of $\mathcal{L}$ associated with λ_1

$$-\,\mathcal{L}_u\,f_1 = \lambda_1\,f_1$$

In the asymptotic regime, the optimal contant in the entropy-entropy production inequality has to be given by the same eigenvalue problem

$$\lambda_1 = \frac{\langle f_1,\mathcal{L}\,f_1\rangle_B}{\langle f_1,f_1\rangle_B} = \frac{\langle\langle f_1,f_1\rangle\rangle_B}{\langle f_1,f_1\rangle_B} \ge \inf_u \frac{\mathcal{I}[u]}{\mathcal{F}[u]} \ge \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} = \frac{\langle\langle f_1,\mathcal{L}\,f_1\rangle\rangle_B}{\langle\langle f_1,f_1\rangle\rangle_B} = \lambda_1$$

... $\lambda_1 = \Lambda_\star$ is the best constant

Optimality of the method in CKN inequalities: weights !

With weights

▷ The infimum $\inf_u \dfrac{\mathcal{K}[u]}{\mathcal{I}[u]}$ is achieved in the asymptotic regime, and given by $\frac{1}{2\,\lambda_1}$, that is, the linearized problem (spectral gap)

▷ This gives a *non-optimal* estimate of

$$\inf_u \frac{\mathcal{I}[u]}{\mathcal{F}[u]} =: \frac{1}{2\,\Lambda_\star} \le \frac{1}{2\,\lambda_1}$$

but equality holds if and only if $\beta = \beta_{\mathrm{FS}}(\gamma)$. Otherwise, as far as rates of convergence are concerned, this estimate *is not* optimal

▷ The inequality $\mathcal{K}[u] \ge 2\,\lambda_1\,\mathcal{I}[u]$ proves the monotonicity that provides the symmetry result as long as $\lambda_1 \ge 0$. The condition $\lambda_1 < 0$ determines the symmetry breaking region. As far as symmetry issues are concerned, this estimate *is* optimal

Computing the second derivative of the entropy is known as the Γ_2 method, or *carré du champ* method, or *Bakry-Emery* method, but

- the so-called $\mathrm{CD}(\rho, N)$ condition encodes integrations by parts which are not easy to justify if the potential has singularities or if the manifold is not compact

Figure: © N. Ghoussoub, dec. 2014

- Non-linear flows (as shown by the Rényi entropy powers) are essential to cover ranges of exponents up to the critical exponent, and have not so much to do with the $\mathrm{CD}(\rho, N)$ condition (*a priori* !)

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Thank you for your attention !