Symmetry by flows

Extremal functions in subcritical Caffarelli-Kohn-Nirenberg inequalities: symmetry breaking, symmetry, rigidity and nonlinear flows

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Outline

- Caffarelli-Kohn-Nirenberg inequalities
  - ▷ the symmetry issue
  - ▷ the result
  - ▷ the critical and the subcritical regime
- The proof
  - ▷ Rényi entropy powers
  - ▷ A rigidity result
  - ▷ The strategy of the proof
- Inequalities and flows on compact manifolds: the sphere
  - ▷ Flows on the sphere, improvements in the subcritical range
  - ▷ Can one prove Sobolev's inequalities with a heat flow ?
  - ▷ The *bifurcation* point of view
  - ▷ Some open problems: constraints and improved inequalities
- Fast diffusion equations with weights: large time asymptotics
  - ▷ Relative uniform convergence
  - ▷ Asymptotic rates of convergence
  - ▷ From asymptotic to global estimates

Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)

and also: S. Filippas, A. Tertikas, G. Tarantello

- Caffarelli-Kohn-Nirenberg inequalities (...)

Background references

- Rigidity methods, uniqueness in nonlinear elliptic PDE's: [B. Gidas, J. Spruck, 1981], [M.-F. Bidaut-Véron, L. Véron, 1991]
- Probabilistic methods (Markov processes), semi-group theory and *carré du champ* methods (Γ_2 theory): [D. Bakry, M. Emery, 1984], [Bakry, Ledoux, 1996], [Demange, 2008], [JD, Esteban, Loss, 2014 & 2015] → *D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs

  ▷ Entropy-entropy production inequalities: Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Unterreiter, Villani..., [del Pino, JD, 2001], [Blanchet, Bonforte, JD, Grillo, Vázquez] → *A. Jüngel, Entropy Methods for Diffusive Partial Differential Equations (2016)*

  ▷ Mass transportation: [Otto] → *C. Villani, Optimal transport. Old and new (2009)*

  ▷ Rényi entropy powers (information theory) [Savaré, Toscani, 2014], [Dolbeault, Toscani]

Caffarelli-Kohn-Nirenberg inequalities and symmetry issues

Critical Caffarelli-Kohn-Nirenberg inequalities

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right)\,:\,|x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right)\right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b}\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

▷ *An optimal function among radial functions:*

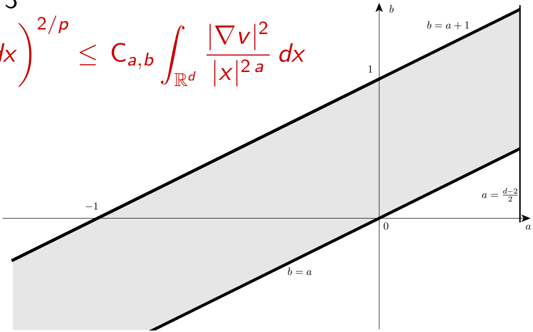
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le\mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a\le b\le a+1$ if $d\ge3$

$a<b\le a+1$ if $d=2$, $a+1/2<b\le a+1$ if $d=1$

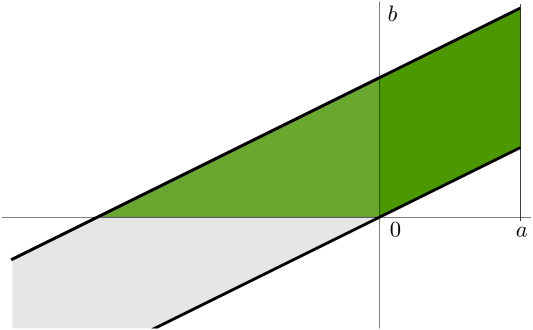
and $a<a_c:=(d-2)/2$

$$p=\frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]

[F. Catrina, Z.-Q. Wang (2001)]

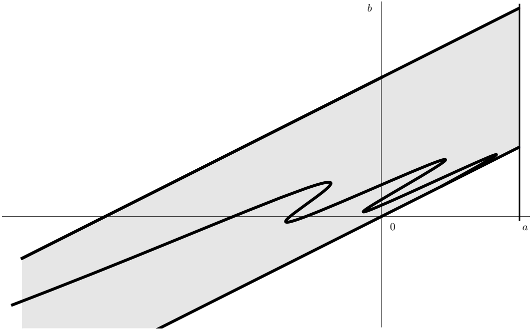
Symmetry 1: moving planes and symmetrization



[Chou, Chu], [Horiuchi]
[Betta, Brock, Mercaldo, Posteraro]
+ Perturbation results: [CS Lin, ZQ Wang], [Smets, Willem], [JD, Esteban, Tarantello 2007], [JD, Esteban, Loss, Tarantello, 2009]

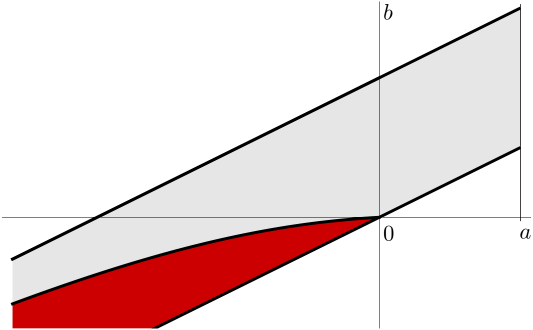
The threshold between symmetry and symmetry breaking

[F. Catrina, Z.-Q. Wang]



[JD, Esteban, Loss, Tarantello, 2009] There is a curve which separates the symmetry region from the symmetry breaking region, which is parametrized by a function $p \mapsto a + b$

Linear instability of radial minimizers: the Felli-Schneider curve

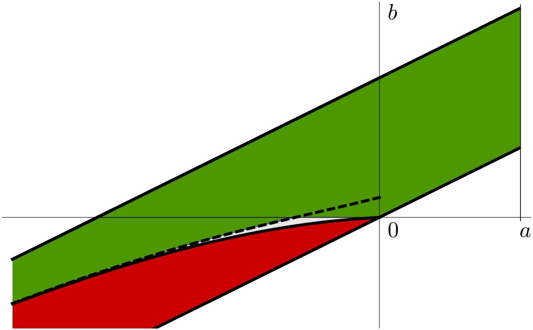


[Catrina, Wang], [Felli, Schneider] The functional

$$C^\star_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx-\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

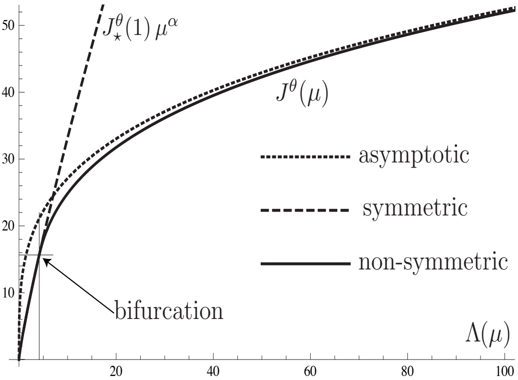
is linearly instable at $v=v_\star$

Direct spectral estimates



[JD, Esteban, Loss, 2011]: sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

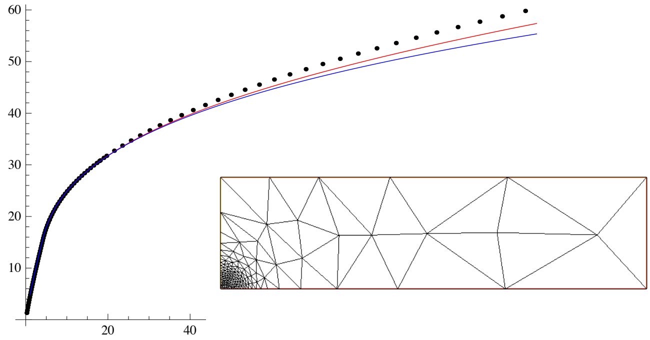
Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

Other evidences

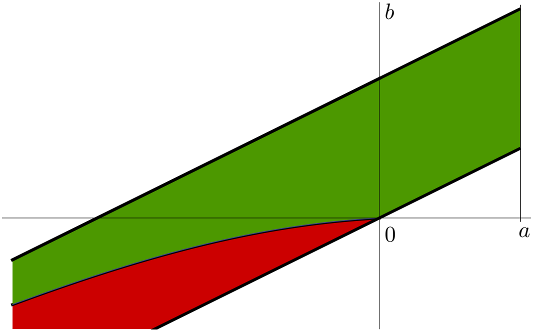
- Further numerical results [JD, Esteban, 2012] (coarse / refined / self-adaptive grids)



- Formal commutation of the non-symmetric branch near the bifurcation point [JD, Esteban, 2013]
- Asymptotic energy estimates [JD, Esteban, 2013]

Symmetry *versus* symmetry breaking: the sharp result in the critical case

A result based on entropies and nonlinear flows



[JD, Esteban, Loss (Inventiones 2016)]

The symmetry result in the critical case

The Felli & Schneider curve is defined by

$$b_{\rm FS}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c-a)^2+d-1}} + a - a_c$$

Theorem

Let $d \ge 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \ge b_{\rm FS}(a)$, then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, Caffarelli-Kohn-Nirenberg inequalities on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad \omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}\quad\forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda:=(a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}}\quad\text{with}\quad a=a_c\pm\sqrt{\Lambda}\quad\text{and}\quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

Interpolation and subcritical Caffarelli-Kohn-Nirenberg inequalities

With one weight: a perturbation result

On the space of smooth functions on $\mathbb{R}^d$ with compact support

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_\gamma \, \|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\vartheta} \, \|w\|_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta}$$

where $\vartheta := \frac{2^*_\gamma\,(p-1)}{2\,p\,(2^*_\gamma-p-1)} = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+2-2\,\gamma-p\,(d-2)\big)}$ and

$$\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q} \quad \text{and} \quad \|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$$

and $d \ge 3$, $\gamma \in (0,2)$, $p \in \big(1, 2^*_\gamma/2\big)$ with $2^*_\gamma := 2\,\frac{d-\gamma}{d-2}$

Theorem

[JD, Muratori, Nazaret] *Let $d \ge 3$. For any $p \in (1, d/(d-2))$, there exists a positive γ^* such that equality holds for all $\gamma \in (0,\gamma^*)$ with*

$$w_\star(x) := \big(1+|x|^{2-\gamma}\big)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

Caffarelli-Kohn-Nirenberg inequalities (with two weights)

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^{q}(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}^{\vartheta}\,\|w\|_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}^{1-\vartheta} \tag{CKN}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,,\quad \gamma-2<\beta<\frac{d-2}{d}\,\gamma\,,\quad \gamma\in(-\infty,d)\,,\quad p\in(1,p_\star]\quad\text{with } p_\star := \frac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

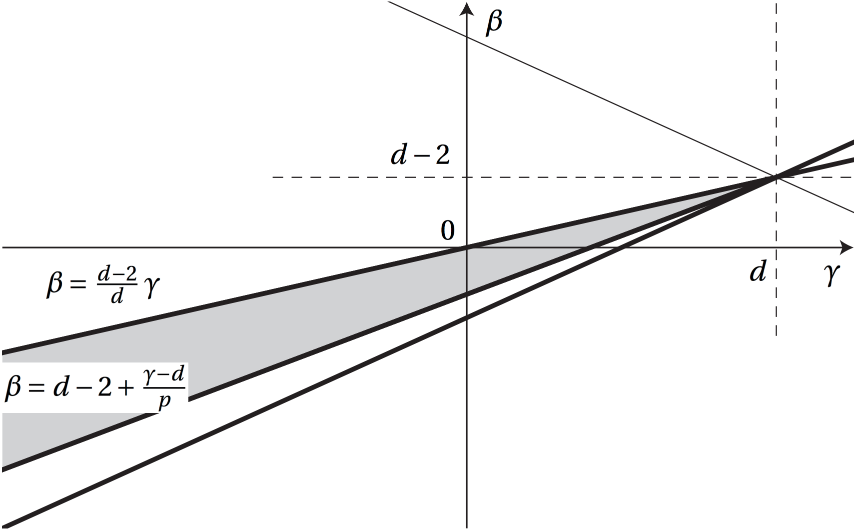
$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}\quad \forall\,x\in\mathbb{R}^d\quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions?

Range of the parameters



CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\tfrac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}$. Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{F}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}\,.$$

If v solves the *Fokker-Planck type equation*

$$\partial_t v+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0 \qquad \text{(WFDE-FP)}$$

then

$$\frac{d}{dt}\mathcal{F}[v(t,\cdot)]=-\,\frac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$$

Proposition

Let $m = \frac{p+1}{2\,p}$ and consider a solution to (WFDE-FP) with nonnegative initial datum $u_0 \in \mathcal{L}^{1,\gamma}(\mathbb{R}^d)$ such that $\|u_0^m\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d} u_0\,|x|^{2+\beta-2\gamma}\,dx$ are finite. Then

$$\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[u_0]\,e^{-(2+\beta-\gamma)^2\,t} \quad \forall\, t \ge 0$$

if one of the following two conditions is satisfied:
(i) either u_0 is a.e. radially symmetric
(ii) or symmetry holds in (CKN)

With two weights: a symmetry breaking result

Let us define

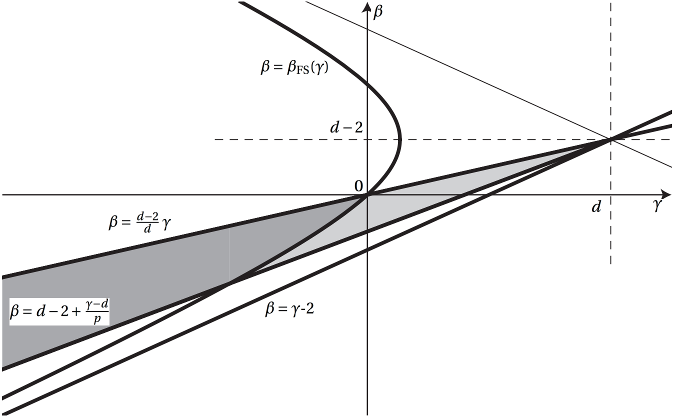
$$\beta_{\mathrm{FS}}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad \textit{and} \quad \beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

In the range $\beta_{\mathrm{FS}}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$, $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal.



The grey area corresponds to the admissible cone. The light grey area is the region of symmetry, while the dark grey area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

A useful change of variables

With

$$\alpha = 1 + \frac{\beta - \gamma}{2} \quad \text{and} \quad n = 2\,\frac{d - \gamma}{\beta + 2 - \gamma}\,,$$

(CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$. Parameters are in the range

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,, \quad p_\star := \frac{n}{n-2}$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1 + |x|^2\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d$$

The symmetry breaking condition (Felli-Schneider) now reads

$$\alpha < \alpha_{\rm FS} \quad \text{with} \quad \alpha_{\rm FS} := \sqrt{\frac{d-1}{n-1}}$$

The second variation

$$\mathcal{J}[v] := \vartheta\,\log\left(\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\right) + (1-\vartheta)\,\log\left(\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\right) + \log \mathsf{K}_{\alpha,n,p} - \log\left(\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\right)$$

Let us define $d\mu_\delta := \mu_\delta(x)\,dx$, where $\mu_\delta(x) := (1+|x|^2)^{-\delta}$. Since $v_\star$ is a critical point of $\mathcal{J}$, a Taylor expansion at order ε^2 shows that

$$\|\mathfrak{D}_\alpha v_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\mathcal{J}\left[v_\star + \varepsilon\,\mu_{\delta/2}\,f\right] = \tfrac12\,\varepsilon^2\,\vartheta\,\mathcal{Q}[f] + o(\varepsilon^2)$$

with $\delta = \frac{2\,p}{p-1}$ and

$$\mathcal{Q}[f] = \int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2\,|x|^{n-d}\,d\mu_\delta - \frac{4\,p\,\alpha^2}{p-1}\int_{\mathbb{R}^d} |f|^2\,|x|^{n-d}\,d\mu_{\delta+1}$$

We assume that $\int_{\mathbb{R}^d} f\,|x|^{n-d}\,d\mu_{\delta+1} = 0$ (mass conservation)

Symmetry breaking: the proof

Proposition (Hardy-Poincaré inequality)

Let $d \geq 2$, $\alpha \in (0,+\infty)$, $n > d$ and $\delta \geq n$. If f has 0 average, then

$$\int_{\mathbb{R}^d} |\mathfrak{D}_\alpha f|^2 \, |x|^{n-d} \, d\mu_\delta \geq \Lambda \int_{\mathbb{R}^d} |f|^2 \, |x|^{n-d} \, d\mu_{\delta+1}$$

with optimal constant $\Lambda = \min\{2\,\alpha^2\,(2\,\delta-n), 2\,\alpha^2\,\delta\,\eta\}$ where η is the unique positive solution to $\eta\,(\eta+n-2) = (d-1)/\alpha^2$. The corresponding eigenfunction is not radially symmetric if $\alpha^2 > \frac{(d-1)\,\delta^2}{n\,(2\,\delta-n)\,(\delta-1)}$.

$\mathcal{Q} \geq 0$ iff $\frac{4\,p\,\alpha^2}{p-1} \leq \Lambda$ and symmetry breaking occurs in (CKN) if

$$2\,\alpha^2\,\delta\,\eta < \frac{4\,p\,\alpha^2}{p-1} \iff \eta < 1$$

$$\iff \frac{d-1}{\alpha^2} = \eta\,(\eta+n-2) < n-1 \iff \alpha > \alpha_{\mathrm{FS}}$$

Fast diffusion equations with weights: a symmetry result

- Rényi entropy powers
- The symmetry result
- The strategy of the proof

Joint work with M.J. Esteban, M. Loss in the critical case $\beta = d - 2 + \frac{\gamma - d}{p}$

Joint work with M.J. Esteban, M. Loss and M. Muratori in the subcritical case $d - 2 + \frac{\gamma - d}{p} < \beta < \frac{d-2}{d}\,\gamma$

Rényi entropy powers (no weights)

We consider the flow $\frac{\partial u}{\partial t} = \Delta u^m$ and the Gagliardo-Nirenberg inequalities (GN)

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}_{p,d}^{\mathrm{GN}} \, \|\nabla w\|_{L^2(\mathbb{R}^d)}^{\theta} \, \|w\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

where $u = w^{2p}$, that is, $w = u^{m-1/2}$ with $p = \frac{1}{2\,m-1}$. Straightforward computations show that (GN) can be brought into the form

$$\left(\int_{\mathbb{R}^d} u \, dx\right)^{(\sigma+1)\,m-1} \le C \, \mathcal{I} \, \mathcal{E}^{\sigma-1} \quad \text{where} \quad \sigma = \frac{2}{d\,(1-m)} - 1$$

where $\mathcal{E} := \int_{\mathbb{R}^d} u^m \, dx$ and $\mathcal{I} := \int_{\mathbb{R}^d} u \, |\nabla \mathsf{P}|^2 \, dx$, $\mathsf{P} = \frac{m}{1-m} \, u^{m-1}$ is the *pressure variable*. If $\mathcal{F} = \mathcal{E}^{\sigma}$ is the *Rényi entropy power* and $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$, then $\mathcal{F}''$ is proportional to

$$-2\,(1-m)\,\left\langle \mathrm{Tr}\left(\left(\mathrm{Hess}\,\mathsf{P} - \tfrac{1}{d}\,\Delta \mathsf{P}\,\mathrm{Id}\right)^2\right)\right\rangle + (1-m)^2\,(1-\sigma)\,\left\langle \left(\Delta \mathsf{P} - \langle \Delta \mathsf{P} \rangle\right)^2 \right\rangle$$

where we have used the notation $\langle A \rangle := \int_{\mathbb{R}^d} u^m \, A \, dx \big/ \int_{\mathbb{R}^d} u^m \, dx$

A rigidity result

We actually prove a rigidity (uniqueness) result

▷ critical case: [JD, Esteban, Loss; Inventiones]
▷ subcritical case: [JD, Esteban, Loss, Muratori]

Theorem

Assume that $\beta \le \beta_{\mathrm{FS}}(\gamma)$. Then all positive solutions in $\mathrm{H}^p_{\beta,\gamma}(\mathbb{R}^d)$ of

$$-\,div\left(|x|^{-\beta}\,\nabla w\right)=|x|^{-\gamma}\left(w^{2p-1}-\,w^p\right)\quad in\quad \mathbb{R}^d\setminus\{0\}$$

are radially symmetric and, up to a scaling and a multiplication by a constant, equal to $w_\star(x)=\left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$

The strategy of the proof (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,,\quad \alpha = 1 + \frac{\beta-\gamma}{2}\quad\text{and}\quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathfrak{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}\quad \forall\,v\in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathfrak{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$ and

$$d\ge 2\,,\quad \alpha>0\,,\quad n>d\quad\text{and}\quad p\in(1,p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1+|x|^2\right)^{-1/(p-1)}\quad \forall\,x\in\mathbb{R}^d$$

The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathfrak{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

With $\mathcal{L}_\alpha=-\,\mathfrak{D}_\alpha^*\,\mathfrak{D}_\alpha=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t}=\mathcal{L}_\alpha u^m$$

in the subcritical range $1-1/n<m<1$. The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\,\left(\textstyle\int_{\mathbb{R}^d}u^m\,d\mu\right)^{1-\sigma}\\
&\ge(1-m)\,(\sigma-1)\textstyle\int_{\mathbb{R}^d}u^m\left|\mathcal{L}_\alpha\mathsf{P}-\frac{\int_{\mathbb{R}^d}u\,|\mathfrak{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d}u^m\,d\mu}\right|^2d\mu\\
&+2\textstyle\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\frac{1}{n}\right)\left|\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,d\mu\\
&+2\textstyle\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2+c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,d\mu=:\mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d)>0$. Hence if $\alpha\le\alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result.

(3/3: elliptic regularity, boundary terms)

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathcal{L}_\alpha u^m$

$$0 = \int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u] \cdot \mathcal{L}_\alpha u^m \, d\mu \geq \mathcal{H}[u] \geq 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma \operatorname{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p} \operatorname{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right) + |\nabla w^{1-p}|^2 + |x|^{-\gamma}\left(c_1\,w^{1-p} - c_2\right) = 0$$

Inequalities and flows on compact manifolds

▷ Flows on the sphere, improvements in the subcritical range

▷ Can one prove Sobolev's inequalities with a heat flow ?

▷ The *bifurcation* point of view

▷ Some open problems: constraints and improved inequalities

[Bakry, Emery, 1984]
[Bidault-Véron, Véron, 1991], [Bakry, Ledoux, 1996]
[Demange, 2008], [JD, Esteban, Loss, 2014 & 2015]

The interpolation inequalities

On the d-dimensional sphere, let us consider the interpolation inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

where the measure $d\mu$ is the uniform probability measure on $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ corresponding to the measure induced by the Lebesgue measure on $\mathbb{R}^{d+1}$, and the exposant $p \ge 1$, $p \ne 2$, is such that

$$p \le 2^* := \frac{2\,d}{d-2}$$

if $d \ge 3$. We adopt the convention that $2^* = \infty$ if $d = 1$ or $d = 2$. The case $p = 2$ corresponds to the logarithmic Sobolev inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \,\log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \setminus \{0\}$$

The Bakry-Emery method

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \le -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \le 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \ge d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \le 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

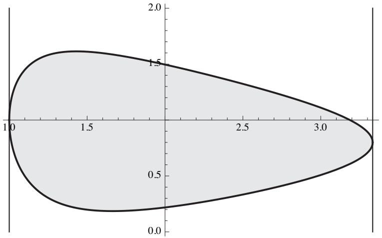
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m \,. \tag{1}$$

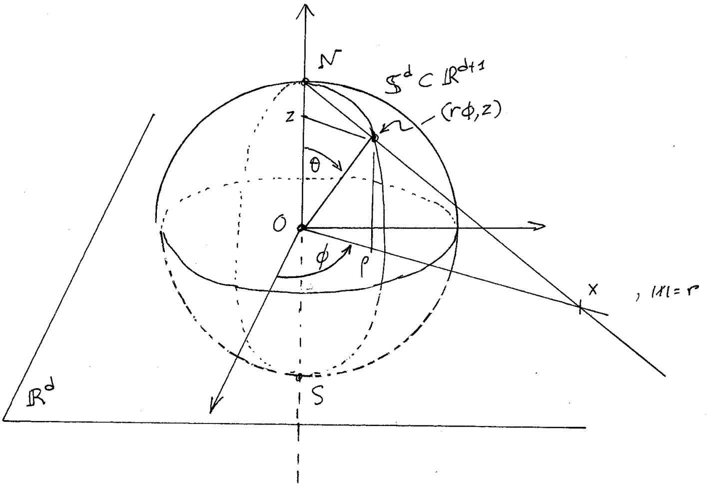
[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0$$



(p, m) *admissible region*, $d = 5$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z=\cos\theta$, $v(\theta)=f(z)$, $d\nu_d:=\nu^{\frac{d}{2}-1}\,dz/Z_d$, $\nu(z):=1-z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\,f:=(1-z^2)\,f''-d\,z\,f'=\nu\,f''+\frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1,\mathcal{L}\,f_2\rangle=-\int_{-1}^1 f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge1$. For any $f\in\mathrm{H}^1([-1,1],d\nu_d)$,

$$-\langle f,\mathcal{L}\,f\rangle=\int_{-1}^1|f'|^2\,\nu\,d\nu_d\ge d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathcal{L}\, g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\, \frac{|f'|^2}{f}\, \nu$$

$$-\frac{1}{2}\, \frac{d}{dt} \int_{-1}^{1} |f'|^2\, \nu\, d\nu_d = \frac{1}{2}\, \frac{d}{dt}\, \langle f, \mathcal{L}\, f \rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f \rangle + (p-1)\, \langle \frac{|f'|^2}{f}\, \nu, \mathcal{L}\, f \rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\, d\, \mathcal{I}[g(t,\cdot)] = \frac{d}{dt} \int_{-1}^{1} |f'|^2\, \nu\, d\nu_d + 2\, d \int_{-1}^{1} |f'|^2\, \nu\, d\nu_d$$

$$= -\, 2 \int_{-1}^{1} \left(|f''|^2 + (p-1)\, \frac{d}{d+2}\, \frac{|f'|^4}{f^2} - 2\, (p-1)\, \frac{d-1}{d+2}\, \frac{|f'|^2\, f''}{f} \right) \nu^2\, d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\, \frac{d}{d+2}\, \frac{|f'|^4}{f^2} - 2\, (p-1)\, \frac{d-1}{d+2}\, \frac{|f'|^2\, f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\, \frac{d-1}{d+2} \right]^2 \le (p-1)\, \frac{d}{d+2} \iff p \le \frac{2\, d^2 + 1}{(d-1)^2} = 2^\# < \frac{2\, d}{d-2} = 2^*$$

The rigidity point of view (nonlinear flow)

$u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$... Which computation do we have to do ?

$$-\,\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with

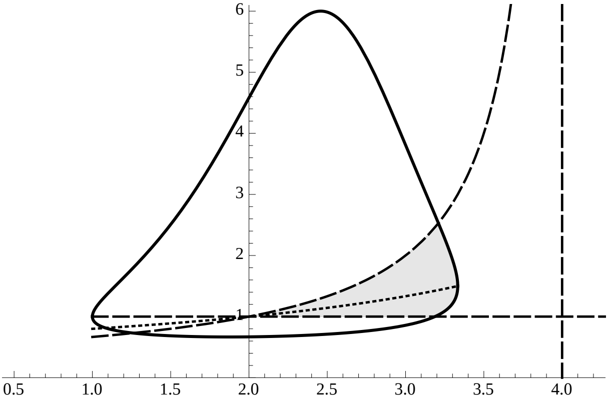
$$\int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d = 0 \quad \text{if } p=2^* \text{ and } \beta = \frac{4}{6-p}$$

Improved functional inequalities

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

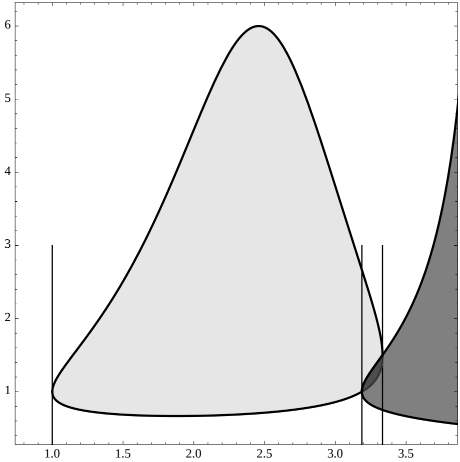
$$\rho = |u|^{\beta p}$$

$$m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$$



(p, β) representation of the admissible range of parameters when $d = 5$ [JD, Esteban, Kowalczyk, Loss]

Can one prove Sobolev's inequalities with a heat flow ?

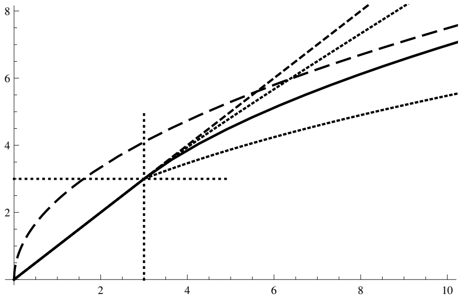


(p, β) *representation when* $d = 5$. *In the dark grey area, the functional is not monotone under the action of the heat flow [JD, Esteban, Loss]*

The bifurcation point of view

$\mu(\lambda)$ is the optimal constant in the functional inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$



Here $d = 3$ *and* $p = 4$

- A critical point of $u \mapsto \mathcal{Q}_\lambda[u] := \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$ solves

$$-\,\Delta u + \lambda\, u = |u|^{p-2}\, u \tag{EL}$$

up to a multiplication by a constant (and a conformal transformation if $p = 2^*$)

- The best constant $\mu(\lambda) = \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)\setminus\{0\}} \mathcal{Q}_\lambda[u]$ is such that $\mu(\lambda) < \lambda$ if $\lambda > \frac{d}{p-2}$, and $\mu(\lambda) = \lambda$ if $\lambda \le \frac{d}{p-2}$ so that

$$\frac{d}{p-2} = \min\{\lambda > 0 \,:\, \mu(\lambda) < \lambda\}$$

- *Rigidity* : the unique positive solution of (EL) is $u = \lambda^{1/(p-2)}$ if $\lambda \le \frac{d}{p-2}$

Constraints and improvements

- Taylor expansion:

$$d=\inf_{u\in\mathrm{H}^1(\mathbb{S}^d,d\mu)\setminus\{0\}}\frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

is achieved in the limit as $\varepsilon\to 0$ with $u=1+\varepsilon\,\varphi_1$ such that

$$-\Delta\varphi_1=d\,\varphi_1$$

▷ This suggest that improved inequalities can be obtained under appropriate orthogonality constraints...

Integral constraints

With the heat flow...

Proposition

For any $p \in (2, 2^\#)$, the inequality

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{\lambda}{p-2} \, \|f\|_2^2 \geq \frac{\lambda}{p-2} \, \|f\|_p^2$$

$$\forall\, f \in \mathrm{H}^1((-1,1), d\nu_d) \text{ s.t. } \int_{-1}^{1} z \, |f|^p \, d\nu_d = 0$$

holds with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

... and with a nonlinear diffusion flow ?

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

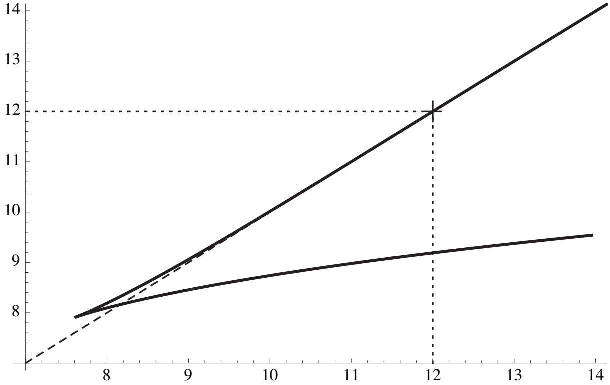
If $p \in (1,2) \cup (2, 2^*)$, *we have*

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{p-2} \left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1} \right] \left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ *with antipodal symmetry. The limit case* $p = 2$ *corresponds to the improved logarithmic Sobolev inequality*

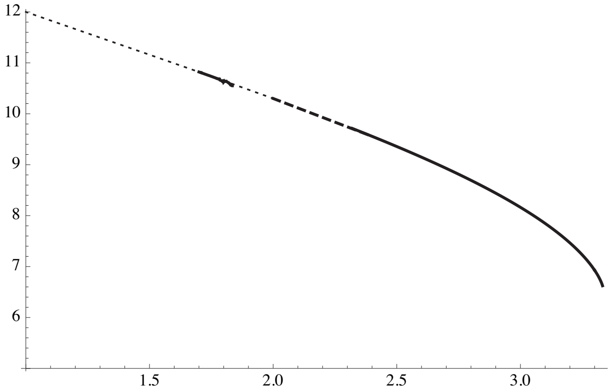
$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \geq \frac{d}{2} \frac{(d+3)^2}{(d+1)^2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu$$

The larger picture: branches of antipodal solutions



Case $d = 5$, $p = 3$: values of the shooting parameter a as a function of λ

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d = 5$ *and* $1 \le p \le 10/3 \approx 3.33$*. The limiting value of the constant is numerically found to be equal to* $\lambda_\star = 2^{1-2/p}\, d \approx 6.59754$ *with* $d = 5$ *and* $p = 10/3$

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$\partial_t v + |x|^\gamma \, \nabla \cdot \Big[\, |x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret **[...]**

Relative uniform convergence

$$\zeta := 1 - \left(1 - \tfrac{2-m}{(1-m)\,q}\right)\left(1 - \tfrac{2-m}{1-m}\,\theta\right)$$

$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma}$ is in the range $0 < \theta < \frac{1-m}{2-m} < 1$

Theorem

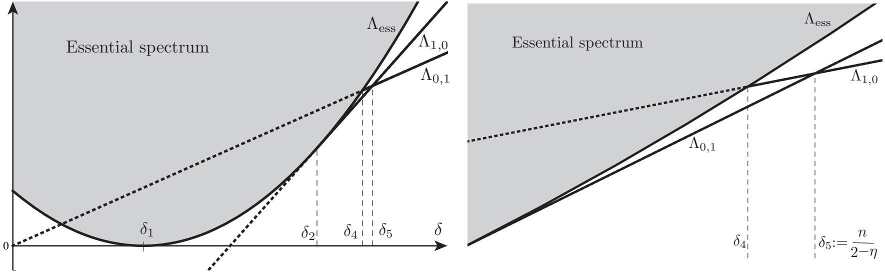
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0, d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum.

Main steps of the proof:

- Existence of weak solutions, $\mathrm{L}^{1,\gamma}$ contraction, Comparison Principle, conservation of relative mass
- Self-similar variables and the Ornstein-Uhlenbeck equation in relative variables: the ratio $w(t,x) := v(t,x)/\mathfrak{B}(x)$ solves

$$\begin{cases} |x|^{-\gamma}\,\partial_t w = -\,\frac{1}{\mathfrak{B}}\,\nabla\cdot\left(|x|^{-\beta}\,\mathfrak{B}\,w\,\nabla\left(\left(w^{m-1}-1\right)\mathfrak{B}^{m-1}\right)\right) & \text{in } \mathbb{R}^+\times\mathbb{R}^d \\ w(0,\cdot) = w_0 := v_0/\mathfrak{B} & \text{in } \mathbb{R}^d \end{cases}$$

- *Regularity*, relative uniform convergence (without rates) and asymptotic rates (linearization)
- The relative free energy and the relative Fisher information: linearized free energy and linearized Fisher information
- A Duhamel formula and a bootstrap

(1/2) Harnack inequality and Hölder regularity

We change variables: $x \mapsto |x|^{\alpha-1}\,x$ and adapt the ideas of F. Chiarenza and R. Serapioni to

$$\partial_t u + \mathsf{D}_\alpha^*\Big[\mathsf{a}\,(\mathsf{D}u + \mathsf{B}\,u)\Big] = 0 \quad \text{in} \quad \mathbb{R}^+ \times \mathbb{R}^d$$

Proposition (A parabolic Harnack inequality)

Let $d \geq 2$, $\alpha > 0$ and $n > d$. If u is a bounded positive solution, then for all $(t_0, x_0) \in \mathbb{R}^+ \times \mathbb{R}^d$ and $r > 0$ such that $Q_r(t_0, x_0) \subset \mathbb{R}^+ \times B_1$, we have

$$\sup_{Q_r^-(t_0,x_0)} u \leq H \inf_{Q_r^+(t_0,x_0)} u$$

The constant $H > 1$ depends only on the local bounds on the coefficients a, B *and on d, α, and n*

By adapting the classical method *à la De Giorgi* to our weighted framework: Hölder regularity at the origin

Regularity (2/2): from local to global estimates

Lemma

If w is a solution of the the Ornstein-Uhlenbeck equation with initial datum w_0 bounded from above and from below by a Barenblatt profile (+ relative mass condition) = "good solutions", then there exist $\nu \in (0,1)$ and a positive constant $\mathcal{K} > 0$, depending on d, m, β, γ, C, C_1, C_2 such that:

$$\|\nabla v(t)\|_{\mathrm{L}^\infty(B_{2\lambda}\setminus B_\lambda)} \le \frac{Q_1}{\lambda^{\frac{2+\beta-\gamma}{1-m}+1}} \quad \forall\, t \ge 1\,, \quad \forall\, \lambda > 1$$

$$\sup_{t\ge1} \|w\|_{C^k((t,t+1)\times B_\varepsilon^c)} < \infty \quad \forall\, k \in \mathbb{N}\,,\ \forall\, \varepsilon > 0$$

$$\sup_{t\ge1} \|w(t)\|_{C^\nu(\mathbb{R}^d)} < \infty$$

$$\sup_{\tau\ge t} |w(\tau)-1|_{C^\nu(\mathbb{R}^d)} \le \mathcal{K} \sup_{\tau\ge t} \|w(\tau)-1\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \quad \forall\, t \ge 1$$

Asymptotic rates of convergence

Corollary

Assume that $m \in (0,1)$, with $m \neq m_*$ with $m_* :=$. Under the relative mass condition, for any "good solution" v there exists a positive constant $\mathcal{C}$ such that

$$\mathcal{F}[v(t)] \leq \mathcal{C}\, e^{-\,2\,(1-m)\,\Lambda\,t} \quad \forall\, t \geq 0\,.$$

- With Csiszár-Kullback-Pinsker inequalities, these estimates provide a rate of convergence in $\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$
- Improved estimates cane be obtained using "best matching techniques"

From asymptotic to global estimates

When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{F}[v]\le\frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t}\quad\forall\,t\ge0\quad\text{with}\quad\Lambda_\star:=\tfrac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

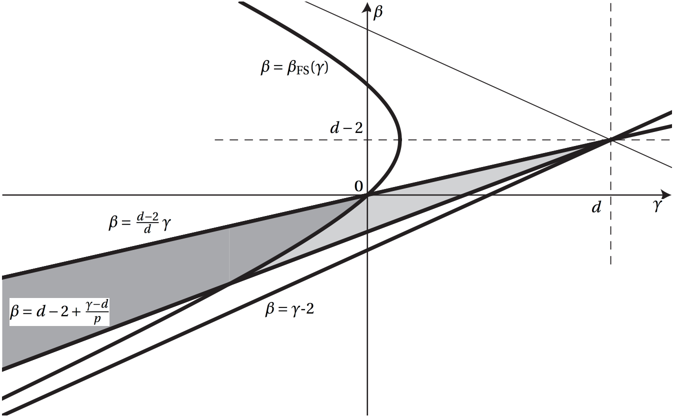
Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{F}[v]\le\mathcal{I}[v]\quad\forall\,v\in\mathcal{L}^{1,\gamma}(\mathbb{R}^d)\quad\text{such that}\quad\|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}=M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M):=\frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{F}[v(t)]\le\mathcal{F}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t}\quad\forall\,t\ge0$$

The symmetry breaking region (again)



... consequences for the entrapy – entropy production inequalities (a measurement in relative entropy of the global rates of convergence)

Symmetry breaking and global entropy – entropy production inequalities

Proposition

- *In the symmetry breaking range of (CKN), for any $M > 0$, we have*
$$0 < \mathcal{K}(M) \le \tfrac{2}{m}\,(1-m)^2\,\Lambda_{0,1}$$
- *If symmetry holds in (CKN) then*
$$\mathcal{K}(M) > \tfrac{1-m}{m}\,(2+\beta-\gamma)^2$$

Corollary

Assume that $m \in [m_1, 1)$

(i) For any $M > 0$*, if* $\Lambda(M) = \Lambda_\star$ *then* $\beta = \beta_{\mathrm{FS}}(\gamma)$

(ii) If $\beta > \beta_{\mathrm{FS}}(\gamma)$ *then* $\Lambda_{0,1} < \Lambda_\star$ *and* $\Lambda(M) \in (0, \Lambda_{0,1}]$ *for any* $M > 0$

(iii) For any $M > 0$*, if* $\beta < \beta_{\mathrm{FS}}(\gamma)$ *and if symmetry holds in (CKN), then*
$$\Lambda(M) > \Lambda_\star$$

Conclusion

Some concluding remarks

- The fast diffusion equation (without weights) has a rich structure: a lot has been done (for instance, with parabolic methods or gradient flow techniques) and this is a fundamental equation to explore qualitative behaviors, sharp rates, *entropy methods in PDEs*, etc.

- With *weights*, self-similar Barenblatt solutions attract all solutions (in good spaces) on the long time range, the linearization of the entropy determines the sharp asymptotic rates... but when *symmetry breaking* occurs, there are other critical points and Barenblatt solutions are not optimal for entropy – entropy production ineq.

- Entropy methods can be used *as a tool* to produce symmetry / uniqueness / rigidity results which go well beyond results of elliptic PDEs (rearrangement, moving planes), energy / calculus of variations methods (concentration-compactness methods) and methods of spectral theory (so far)

- Stability results can also be produced in the subcritical case and in the critical case (work in progress)

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Thank you for your attention !