Lieb-Thirring type inequalities and Gagliardo-Nirenberg inequalities for systems

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First eigenvalues and Gagliardo-Nirenberg inequalities (1)

Let $H = -\Delta + V$ in $\mathbb{R}^d$, $d > 1$ and consider $\lambda_1(V)$, its lowest eigenvalue

$$\lambda_1(V) = \inf_{\substack{u \in H^1(\mathbb{R}^d) \\ u \not\equiv 0 \ a.e.}} \frac{\int_{\mathbb{R}^d} |\nabla u|^2 \, dx + \int_{\mathbb{R}^d} V \, |u|^2 \, dx}{\int_{\mathbb{R}^d} |u|^2 \, dx}$$

Consider the variational problem

$$C_1 = \sup_{\substack{V \in \mathcal{D}(\mathbb{R}^d) \\ V \leq 0}} \frac{|\lambda_1(V)|^\gamma}{\int_{\mathbb{R}^d} |V|^{\gamma + \frac{d}{2}} \, dx}$$

By density of $\mathcal{D}(\mathbb{R}^d)$ in $L^{\gamma + \frac{d}{2}}(\mathbb{R}^d)$, it is equivalent to

$$C_1 = \sup_{\substack{V \in X_\gamma \\ V \leq 0, \ V \not\equiv 0 \ a.e.}} \frac{|\lambda_1(V)|^\gamma}{\int_{\mathbb{R}^d} |V|^{\gamma + \frac{d}{2}} \, dx}$$

where $X_\gamma := \left\{ V \in L^{\gamma + \frac{d}{2}}(\mathbb{R}^d) \ : \ V \leq 0 \, , \ V \not\equiv 0 \ a.e. \right\}$

$$R(u,V) := \frac{\int_{\mathbb{R}^d} |V|\,|u|^2\,dx - \int_{\mathbb{R}^d} |\nabla u|^2\,dx}{\int_{\mathbb{R}^d} |u|^2\,dx \quad \|V\|_{L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)}^{1+\frac{d}{2\gamma}}}$$

The variational problem amounts to

$$C_1 = \sup_{\substack{V \in X_\gamma \\ V \not\equiv 0 \ a.e.}} \ \sup_{\substack{u \in H^1(\mathbb{R}^d) \\ u \not\equiv 0 \ a.e.}} R(u,V)$$

Invariance under scalings: $\forall\,\lambda > 0$, if $u_\lambda = u(\lambda\,\cdot)$, $V_\lambda = \lambda^2 V(\lambda\,\cdot)$

$$R(u_\lambda, V_\lambda) = R(u,V)$$

Hint: optimize first on V. With $q := \frac{2\gamma+d}{2\gamma+d-2}$,

$$|V|^{\gamma+\frac{d}{2}-2}\,V = -|u|^2 \quad \Longleftrightarrow \quad V = V_u = -|u|^{\frac{4}{2\gamma+d-2}} = -|u|^{2(q-1)}$$

$$R(u,V) \le R(u,V_u) = \frac{\int_{\mathbb{R}^d} |u|^{2q}\,dx - \int_{\mathbb{R}^d} |\nabla u|^2\,dx}{\int_{\mathbb{R}^d} |u|^2\,dx \left(\int_{\mathbb{R}^d} |u|^{2q}\,dx\right)^{\frac{1}{\gamma}}}$$

$$C_{\mathrm{GN}}(\gamma) = \inf_{\substack{u \in H^1(\mathbb{R}^d) \\ u \not\equiv 0 \ a.e.}} \frac{\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2\gamma+d}} \|u\|_{L^2(\mathbb{R}^d)}^{\frac{2\gamma}{2\gamma+d}}}{\|u\|_{L^{2q}(\mathbb{R}^d)}}$$

Theorem 1 *Let* $d \in \mathbb{N}^*$*. For any* $\gamma > \max(0, 1 - \frac{d}{2})$*,*

$$C_1 = \kappa_1(\gamma) \left[C_{\mathrm{GN}}(\gamma)\right]^{-\kappa_2(\gamma)}$$

$$\kappa_1(\gamma) = \frac{2\gamma}{d} \left(\frac{d}{2\gamma + d}\right)^{1+\frac{d}{2\gamma}} \quad \textit{and} \quad \kappa_2(\gamma) = 2 + \frac{d}{\gamma}$$

Range: $q := \frac{2\gamma+d}{2\gamma+d-2}$. For $\gamma > \max(0, 1 - d/2)$, $q > 1$ and $2\,q < \frac{2d}{d-2}$

Optimality:

$$\Delta u + |u|^{2(q-1)} u - u = 0 \quad \text{in} \quad \mathbb{R}^d$$

Summary (1): $\gamma > \max(0, 1 - \frac{d}{2})$

$$[\lambda_1(V)]^\gamma \leq \; C_1 \int_{\mathbb{R}^d} V^{\frac{d}{2}+\gamma} \, dx$$

$C_{\mathsf{GN}}(\gamma)$ is the best constant of the Gagliardo-Nirenberg inequality

$$\|u\|_{L^{2q}(\mathbb{R}^d)} \leq C_{\mathsf{GN}}(\gamma)^{-1} \|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2\gamma+d}} \|u\|_{L^2(\mathbb{R}^d)}^{\frac{2\gamma}{2\gamma+d}}$$

$q := \frac{2\gamma+d}{2\gamma+d-2}$

[Benguria,Loss], [Veling], [Weinstein]

First eigenvalues and Gagliardo-Nirenberg inequalities (2)

Consider now a nonnegative smooth potential $V \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\lim_{|x| \to +\infty} V(x) = +\infty$$

and denote by $\lambda_1(V)$, $\lambda_2(V)$, ... the positive eigenvalues of $H := -\Delta + V$

$$Y_\gamma := \left\{ V^{\frac{d}{2}-\gamma} \in L^1(\mathbb{R}^d) \ : \ V \geq 0 \, , \ V \not\equiv +\infty \ a.e. \right\}$$

Let

$$q := \frac{2\gamma - d}{2(\gamma+1) - d} \in (0,1)$$

Second type Gagliardo-Nirenberg inequality:

$$C^*_{\mathsf{GN}}(\gamma) = \inf_{\substack{u \in H^1(\mathbb{R}^d),\ u \not\equiv 0 \ a.e. \\ \int_{\mathbb{R}^d} |u|^{2q} \, dx < \infty}} \frac{\|\nabla u\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2\gamma}} \left(\int_{\mathbb{R}^d} |u|^{2q} \, dx \right)^{\frac{1}{2q}\left(1 - \frac{d}{2\gamma}\right)}}{\|u\|_{L^2(\mathbb{R}^d)}}$$

With

$$q := \frac{2\gamma - d}{2(\gamma + 1) - d} \in (0, 1)$$

Theorem 2 *Let $d \in \mathbb{N}^*$. For any $\gamma > d/2$, for any $V \in Y_\gamma$,*

$$[\lambda_1(V)]^{-\gamma} \leq C_2 \int_{\mathbb{R}^d} V^{\frac{d}{2}-\gamma}\,dx$$

$$C_2 = \kappa_1(\gamma)\,[C^*_{\mathrm{GN}}(\gamma)]^{-\kappa_2(\gamma)}$$

where $\kappa_1(\gamma) = \frac{(2q)^{\gamma-\frac{d}{2}}(d(1-q))^{\frac{d}{2}}}{(d(1-q)+2q)^\gamma}$ *and* $\kappa_2(\gamma) = 2\gamma$

Notice that $q < 1$, and $2q > 1$ if and only if $\gamma > 1 + d/2$.

$$R(u, V) := \frac{\int_{\mathbb{R}^d} |u|^2\,dx \quad \left(\int_{\mathbb{R}^d} V^{\frac{d}{2}-\gamma}\,dx\right)^{\frac{1}{\gamma}}}{\int_{\mathbb{R}^d} |\nabla u|^2\,dx + \int_{\mathbb{R}^d} V\,|u|^2\,dx}$$

is invariant under the transformation

$$(u, V) \mapsto (u_\lambda = u(\lambda\,\cdot),\ V_\lambda = \lambda^2 V(\lambda\,\cdot))$$

Summary (2): $\gamma > d/2$

$$[\lambda_1(V)]^{-\gamma} \leq \ C_2 \int_{\mathbb{R}^d} V^{\frac{d}{2}-\gamma}\,dx$$

$C^*_{\mathsf{GN}}(\gamma)$ is the best constant of the Gagliardo-Nirenberg inequality

$$\|u\|_{L^2(\mathbb{R}^d)} \leq C^*_{\mathsf{GN}}(\gamma)^{-1} \|\nabla u\|^{\frac{d}{2\gamma}}_{L^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} |u|^{2q}\,dx\right)^{\frac{1}{2q}(1-\frac{d}{2\gamma})}$$

$q := \frac{2\gamma-d}{2(\gamma+1)-d} \in (0,1)$

Optimal functions for the GN inequality

$$-\Delta u + u^{2q-1} - u = 0\,, \quad u \geq 0$$

[DelPino and coll.] u has compact support, V_u is finite only on a ball

$$\mathcal{C}_F^{(1)} = \sup_V \frac{F(\lambda_1(V))}{\int_{\mathbb{R}^d} G(V(x))\,dx} \le 1$$

$$\mathcal{C}_F^{(1)} = \sup_{\substack{V,\,\phi \\ \int_{\mathbb{R}^d} |\phi|^2\,dx = 1}} \frac{F\left(\int_{\mathbb{R}^d} (|\nabla\phi|^2 + V\,|\phi|^2)\,dx\right)}{\int_{\mathbb{R}^d} G(V(x))\,dx}$$

Duality condition to relate F and G... Optimization with respect to V

$$\kappa\,|\phi|^2 - G'(V) = 0$$

$$\mathcal{C}_F^{(1)} = \sup_{\substack{\phi \in H^1(\mathbb{R}^d) \\ \int_{\mathbb{R}^d} |\phi|^2\,dx = 1}} \frac{F\left(\int_{\mathbb{R}^d} (|\nabla\phi|^2 + |\phi|^2\,(G')^{-1}(\kappa\,|\phi|^2))\,dx\right)}{\int_{\mathbb{R}^d} \left(G \circ (G')^{-1}\right)(\kappa\,|\phi|^2)\,dx}$$

$$\kappa = \left(\mathcal{C}_F^{(1)}\right)^{-1} F'\left(\int_{\mathbb{R}^d} (|\nabla\phi|^2 + |\phi|^2\,(G')^{-1}(\kappa\,|\phi|^2))\,dx\right)$$

A simple case: $q=1$ or $F(s)=e^{-s}$, $G(s)=(4\pi)^{-d/2}\,e^{-s}$

$$\mathcal{C}_F^{(1)} = \sup_{\substack{V,\,\phi \\ \int_{\mathbb{R}^d}|\phi|^2\,dx=1}} \frac{e^{-\int_{\mathbb{R}^d}(|\nabla\phi|^2+V\,|\phi|^2)\,dx}}{(4\pi)^{-d/2}\int_{\mathbb{R}^d}e^{-V}\,dx}$$

The optimization with respect to V gives

$$V=-\log(|\phi|^2)$$

The Lagrange multiplier $\kappa=1$ is such that $\int_{\mathbb{R}^d}e^{-V}\,dx=\int_{\mathbb{R}^d}|\phi|^2\,dx=1$
This is equivalent to the usual logarithmic Sobolev inequality: for any $\phi\in H^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}|\phi|^2\,dx=1$,

$$\int_{\mathbb{R}^d}|\phi|^2\log(|\phi|^2)\,dx+\log\left(\frac{(4\pi)^{d/2}}{\mathcal{C}_F^{(1)}}\right)\le\int_{\mathbb{R}^d}|\nabla\phi|^2\,dx$$

Optimal functions ϕ are gaussian

$$\mathcal{C}_F^{(1)}=\left(\frac{2}{e}\right)^d$$

Summary (3):

$$e^{-\lambda_1(V)} \le (\pi\, e^2)^{-d/2} \int_{\mathbb{R}^d} e^{-V}\, dx$$

is equivalent to the logarithmic Sobolev inequality:
for any $\phi \in H^1(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} |\phi|^2\, dx = 1$,

$$\int_{\mathbb{R}^d} |\phi|^2 \log(|\phi|^2)\, dx + \frac{d}{2} \log\left(\pi\, e^2\right) \le \int_{\mathbb{R}^d} |\nabla\phi|^2\, dx$$

[Gross], [Carlen,Loss], [Bobkov,Götze]

Lieb-Thirring inequalities

Given a smooth bounded nonpositive potential V on $\mathbb{R}^d$, if

$$\lambda_1(V) < \lambda_2(V) \leq \lambda_3(V) \leq \ldots \lambda_N(V) < 0$$

is the finite sequence of all negative eigenvalues of

$$H = -\Delta + V$$

then we have the Lieb-Thirring inequality

$$\sum_{i=1}^{N} |\lambda_i(V)|^\gamma \leq C_{\mathrm{LT}}(\gamma) \int_{\mathbb{R}^d} |V|^{\gamma+\frac{d}{2}}\, dx \tag{1}$$

For $\gamma = 1$, $\sum_{i=1}^{N} |\lambda_i(V)|$ is the *complete ionization energy*

[...], [Laptev-Weidl] for $\gamma \geq 3/2$ the sharp constant is semiclassical
Lieb-Thirring conjecture: $d = 1$, $1/2 < \gamma < 3/2$, $C_{\mathrm{LT}}(\gamma) = C_1$

A new inequality of Lieb-Thirring type

Let V be a nonnegative unbounded smooth potential on $\mathbb{R}^d$: the eigen-values of H_V are

$$0 < \lambda_1(V) < \lambda_2(V) \leq \lambda_3(V) \leq \ldots \lambda_N(V) \ldots$$

Theorem 3 *For any $\gamma > d/2$, for any nonnegative $V \in C^\infty(\mathbb{R}^d)$ such that $V^{d/2-\gamma} \in L^1(\mathbb{R}^d)$,*

$$\sum_{i=1}^{N} \lambda_i(V)^{-\gamma} \leq \mathcal{C}(\gamma) \int_{\mathbb{R}^d} V^{\frac{d}{2}-\gamma}\, dx$$

$$\mathcal{C}(\gamma) = (2\pi)^{-d/2} \frac{\Gamma(\gamma - d/2)}{\Gamma(\gamma)}$$

Theorem 4 *An inequality by Golden, Thompson and Symanzik*
Let V be in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and bounded from below. Assume moreover that e^{-tV} is in $L^1(\mathbb{R}^d)$ for any $t > 0$. Then

$$\mathrm{Tr}\left(e^{-t\,(-\Delta+V)}\right) \le (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-t\,V(x)}\,dx \qquad (2)$$

Proof. The usual proof is based on the Feynman-Kac formula. Here: an elementary approach

$$G(x,t) = (4\pi t)^{-\frac{d}{2}}\, e^{-\frac{|x|^2}{4t}}\,, \quad e^{t\Delta} f = G(\cdot,t) * f$$

Trotter's formula:

$$e^{-t(-\Delta+V)} = \lim_{n\to\infty} \left(e^{\frac{t}{n}\Delta}\, e^{-\frac{t}{n}V}\right)^n$$

Compute the trace...

$$\ldots = \int_{(\mathbb{R}^d)^n} dx\, dx_1\, dx_2 \ldots\, dx_n\, G\left(\frac{t}{n}, x - x_1\right)\, e^{-\frac{t}{n} V(x_1)}\, G\left(\frac{t}{n}, x_1 - x_2\right)$$
$$\cdot e^{-\frac{t}{n} V(x_2)} \ldots\ldots\; G\left(\frac{t}{n}, x_n - x\right)\, e^{-\frac{t}{n} V(x)}$$

Notation $x = x_0 = x_{n+1}$:

$$\int_{(\mathbb{R}^d)^n} dx_0\, dx_1\, dx_2 \ldots\, dx_n\, \prod_{j=0}^{n} G\left(\frac{t}{n}, x_j - x_{j+1}\right)\, e^{-\frac{t}{n} \sum_{k=0}^{n-1} V(x_k)}$$

Convexity of $x \mapsto e^{-x}$

$$e^{-\frac{t}{n} \sum_{k=0}^{n-1} V(x_k)} \leq \frac{1}{n} \sum_{k=0}^{n-1} e^{-t\, V(x_k)}$$

Main ingredient:

$$\int_{(\mathbb{R}^d)^{n-1}} dx_0\, dx_1\, dx_2 \ldots dx_{k-1}\, dx_{k+1} \ldots dx_n \prod_{j=0}^{n} G\left(\frac{t}{n}, x_j - x_{j+1}\right)$$

$$\text{and} \quad G(t, x_k - x_k) = (4\pi t)^{-\frac{d}{2}}$$

$$\begin{aligned} \mathrm{Tr}\left(e^{\frac{t}{n}\Delta} e^{-\frac{t}{n}V}\right)^n \leq \; & \frac{1}{n} \sum_{k=0}^{n-1} \int_{(\mathbb{R}^d)^n} dx_0\, dx_1\, dx_2 \ldots dx_n \\ & \cdot \prod_{j=0}^{n} G\left(\frac{t}{n}, x_j - x_{j+1}\right) e^{-t\,V(x_k)} \\ = \; & (4\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} e^{-t\,V(x)}\, dx \end{aligned}$$

$\square$

Proof of the Main Theorem. By definition of the Γ function, for any $\gamma > 0$ and $\lambda > 0$,

$$\lambda^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} e^{-t\lambda}\, t^{\gamma-1} dt$$

The operator $-\Delta + V$ is essentially self-adjoint on $L^2(\mathbb{R}^d)$, and positive:

$$\mathrm{Tr}\left((-\Delta + V)^{-\gamma}\right) = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} \mathrm{Tr}\left(e^{-t(-\Delta+V)}\right)\, t^{\gamma-1} dt$$

Since $V^{\frac{d}{2}-\gamma} \in L^1(\mathbb{R}^d)$, we get

$$\begin{aligned} \mathrm{Tr}\left((-\Delta + V)^{-\gamma}\right) &\leq \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} \int_{\mathbb{R}^d} (4\pi t)^{-\frac{d}{2}} e^{-tV(x)}\, t^{\gamma-1} dx\, dt \\ &\leq \frac{\Gamma(\gamma - \frac{d}{2})}{(4\pi)^{\frac{d}{2}} \Gamma(\gamma)} \int_{\mathbb{R}^d} V(x)^{\frac{d}{2}-\gamma} dx \end{aligned}$$

□

Generalization: Let f be a nonnegative function on $\mathbb{R}_+$ such that

$$\int_0^\infty f(t)\,\left(1+t^{-d/2}\right)\,\frac{dt}{t}<\infty$$

$$F(s):=\int_0^\infty e^{-t\,s}\,f(t)\,\frac{dt}{t}\quad\text{and}\quad G(s):=\int_0^\infty e^{-t\,s}\,(4\pi\,t)^{-d/2}\,f(t)\,\frac{dt}{t}$$

Theorem 5 *Let V be in $L^1_{\rm loc}(\mathbb{R}^d)$ and bounded from below. If $G(V)\in L^1(\mathbb{R}^d)$, then*

$$\sum_{i\in\mathbb{N}^*}F(\lambda_i(V))=\mathrm{Tr}\,[F\,(-\Delta+V)]\le\int_{\mathbb{R}^d}G(V(x))\,dx$$

If $F(s)=s^{-\gamma}$, then $G(s)=\frac{\Gamma(\gamma-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}\Gamma(\gamma)}\,s^{\frac{d}{2}-\gamma}$

If $F(s)=e^{-s}$, then $f(s)=\delta(s-1)$ and $G(s)=(4\pi)^{-d/2}\,e^{-s}$

$$\sum_{i\in\mathbb{N}^*}e^{-\lambda_i(V)}\le\frac{1}{(4\pi)^{d/2}}\int_{\mathbb{R}^d}e^{-V(x)}\,dx$$

If $V(x) = A^2\,|x|^2 + B$, eigenvalues are:

$$B + \sum_{j=1}^{d} \Big(2\,n_j + 1\Big)\ A\,, \qquad n_1,\ n_2\ \ldots\ n_d \in \mathbb{N}$$

$$\mathrm{Tr}\left(e^{-t\,(-\Delta+V)}\right) = \frac{e^{-B\,t}}{[2\,\sinh(At)]^d}$$

$$\frac{1}{(4\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-V(x)}\,dx = \frac{e^{-B}}{(2\,A)^d}$$

and the quotient

$$\frac{\sum_{i\in\mathbb{N}^*} e^{-\lambda_i(V)}}{\int_{\mathbb{R}^d} e^{-V(x)}\,dx} = \frac{1}{(4\pi)^{d/2}}\left(\frac{A}{\sinh A}\right)^d \le \frac{1}{(4\pi)^{d/2}}$$

achieves the optimal value in the limit $A \to 0_+$. If $\gamma > d/2$

$$\frac{\mathrm{Tr}\Big((-\Delta + V)^{-\gamma}\Big)}{\mathcal{C}(\gamma)\ \int_{\mathbb{R}^d} V^{\frac{d}{2}-\gamma}\,dx} = \frac{s^d}{\Gamma(\gamma - d)}\int_0^\infty \frac{t^{\gamma-1}\,e^{-t}}{(\sinh(s\,t))^d}\,dt$$

with $s := B/A$ converges to 1 in the limit $s \to 0_+$

Optimality in the power law case? $V_\varepsilon \equiv 1$ in $(0, \varepsilon^{-1}\,\pi)^d = \Omega_\varepsilon$, $V_\varepsilon \equiv +\infty$ in Ω_ε^c. Eigenvalues are:

$$1 + \varepsilon^2 \sum_{j=1}^{d} n_j^2 \,, \quad n_1,\; n_2 \;\ldots\; n_d \in \mathbb{N}^*$$

$$\mathrm{Tr}\left((-\Delta + V_\varepsilon)^{-\gamma}\right) = \sum_{n_1,\, n_2 \ldots n_d \in \mathbb{N}^*} \left(1 + \varepsilon^2 \sum_{j=1}^{d} n_j^2\right)^{-\gamma}$$

which behaves asymptotically as ε tends to zero as

$$\sum_{n_1,\, n_2 \ldots n_d \in \mathbb{N}^*} \; \underset{\substack{n_j - 1 \le x_j \le n_j \\ j=1,2,\ldots d}}{\int\!\!\int \cdots \int} \frac{dx}{\left(1 + \varepsilon^2\, |x|^2\right)^\gamma} = \frac{1}{(2\,\varepsilon)^d} \int_{\mathbb{R}^d} \frac{dx}{\left(1 + |x|^2\right)^\gamma}$$

$$= \frac{|S^{d-1}|}{(2\,\varepsilon)^d} \int_0^\infty \frac{r^{d-1}}{(1 + r^2)^\gamma}\, dr$$

The conclusion holds using: $(\pi/\varepsilon)^d = \int_{\mathbb{R}^d} V_\varepsilon^{d/2-\gamma}\, dx$

Stability for the linear Schrödinger equation

$E[\psi] := \int_{\mathbb{R}^d}(|\nabla\psi|^2 + V\,|\psi|^2)\,dx$, $H := -\Delta + V$ has an infinite nondecreasing sequence of eigenvalues $(\lambda_i(V))_{i\in\mathbb{N}^*}$:

$$\lambda_i(V) := \inf_{\substack{F \subset L^2(\mathbb{R}^d)\\ \dim(F)=i}} \sup_{\psi\in F} E[\psi]$$

The eigenfunction $\bar\psi_i$ form an orthonormal sequence:

$$(\bar\psi_i, \bar\psi_j)_{L^2(\mathbb{R}^d)} = \delta_{ij} \quad \forall\, i,\, j \in \mathbb{N}^*$$

Free energy of the *mixed state* $(\boldsymbol{\nu}, \boldsymbol{\psi}) = ((\nu_i)_{i\in\mathbb{N}^*}, (\psi_i)_{i\in\mathbb{N}^*})$:

$$\mathcal{F}[\boldsymbol{\nu}, \boldsymbol{\psi}] := \sum_{i\in\mathbb{N}^*} \beta(\nu_i) + \sum_{i\in\mathbb{N}^*} \nu_i\, E[\psi_i]$$

Assumption (H) holds if β is a strictly convex function, $\beta(0) = 0$,

$$\Big|\sum_{i\in\mathbb{N}^*} \beta(\bar\nu_i)\Big| < \infty \quad \text{and} \quad \Big|\sum_{i\in\mathbb{N}^*} \bar\nu_i\, \lambda_i(V)\Big| < \infty$$

where $\bar\nu_i := (\beta')^{-1}(-\lambda_i(V))$ for any $i \in \mathbb{N}^*$

Theorem 5 for $F(\lambda) = -\beta(\nu) - \lambda\,\nu$, $\nu = (\beta')^{-1}(-\lambda) \Rightarrow$ (H)

Lemma 6 *Under Assumption* (H), *if* $\boldsymbol{\psi} = (\psi_i)_{i\in\mathbb{N}^*}$ *is an orthonormal sequence,*

$$\mathcal{F}_n[\boldsymbol{\nu}, \boldsymbol{\psi}] - \mathcal{F}_n[\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{\psi}}]$$

$$= \sum_{i=1}^n \left(\beta(\nu_i) - \beta(\bar{\nu}_i) - \beta'(\bar{\nu}_i)(\nu_i - \bar{\nu}_i)\right) + \sum_{i=1}^n \nu_i \left(E[\psi_i] - E[\bar{\psi}_i]\right)$$

Lemma 7 *Assume that* $\inf_{s>0} \beta''(s)\, s^{2-p} =: \alpha > 0$, $p \in [1,2]$. *If* $\sum_{i\in\mathbb{N}^*} \beta(\nu_i)$ *and* $\sum_{i\in\mathbb{N}^*} \nu_i\, \beta'(\bar{\nu}_i)$ *are absolutely convergent, then* $(\nu_i - \bar{\nu}_i)_{i\in\mathbb{N}^*} \in \ell^p$ *and*

$$\sum_{i\in\mathbb{N}^*} \left(\beta(\nu_i) - \beta(\bar{\nu}_i) - \beta'(\bar{\nu}_i)(\nu_i - \bar{\nu}_i)\right) \geq \frac{\alpha}{2^{2/p}} \|\boldsymbol{\nu} - \bar{\boldsymbol{\nu}}\|^2_{\ell^p} \cdot \min\left\{\|\boldsymbol{\nu}\|^{p-2}_{\ell^p}, \|\bar{\boldsymbol{\nu}}\|^{p-2}_{\ell^p}\right\}$$

For any n functions $\psi_1, \ldots, \psi_n$ that are orthonormal in $L^2(\mathbb{R}^d)$,

$$\sum_{i=1}^n E[\psi_i] \geq \sum_{i=1}^n \lambda_i(V)$$

Lemma 8 *For any orthogonal family $(\phi_i)_{1\le i\le n}$ in $L^2(\mathbb{R}^d)$, with $\|\phi_i\|^2 = \nu_i$ and $\nu_1 \ge \ldots \ge \nu_n$,*

$$\sum_{i=1}^{n} E[\phi_i] \ge \sum_{i=1}^{n} \nu_i\,\lambda_i(V)$$

Consequences:

1) There exists a minimizer $(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{\psi}})$ of $\mathcal{F}$ under the constraint

$$(\bar\psi_i, \bar\psi_j)_{L^2(\mathbb{R}^d)} = \delta_{ij} \quad \forall\, i\,,\ j \in \mathbb{N}^*$$

where the sequence $\bar{\boldsymbol{\psi}} = (\bar\psi_i)_{i\in\mathbb{N}^*}$ is made of eigenfunctions and

$$\bar\nu_i = (\beta')^{-1}(\lambda_i(V))$$

2) If $(\boldsymbol{\nu}, \boldsymbol{\psi}(t))$ obeys to

$$i\,\partial_t\psi_j = -\Delta\psi_j + V\psi_j\,, \quad x\in\mathbb{R}^d,\ t>0$$

then

$$\mathcal{F}[\boldsymbol{\nu}, \boldsymbol{\psi}(t)] = \mathcal{F}[\boldsymbol{\nu}, \boldsymbol{\psi}^0] \quad \forall\, t>0\,.$$

$\Rightarrow$ Stability

Examples

To various functions β with $-F(s) = (\beta \circ (\beta')^{-1})(-s) + s\,(\beta')^{-1}(-s)$ correspond various generalized Lieb-Thirring inequalities

Example 1. Let $m > 1$ and consider $\beta(\nu) := (m-1)^{m-1} m^{-m}\, \nu^m$. With $\beta'(\nu) = (m-1)^{m-1} m^{1-m}\, \nu^{m-1} = -\lambda$ and $m = \frac{\gamma}{\gamma-1}$, we get:

$$-(\beta(\nu) + \lambda\,\nu) = F(\lambda) = (-\lambda)^\gamma$$

The case $\gamma \in (0,1)$ is formally covered by

$$\beta(\nu) := -(1-m)^{m-1} |m|^{-m}\, \nu^m$$

with $m \in (-\infty, 0)$, $m = \frac{\gamma}{\gamma-1}$ again and $F(s) = (-s)^\gamma$, but in this case, β is not convex and the free energy $\mathcal{F}$ cannot be defined as above.

Example 2. For $m < 1$ and $\beta(\nu) := -(1-m)^{m-1} m^{-m}\, \nu^m$, with $\beta'(\nu) = -(1-m)^{m-1} m^{1-m}\, \nu^{m-1} = -\lambda$ and $m = \frac{\gamma}{\gamma+1}$, we get:

$$F(\lambda) = \lambda^{-\gamma}$$

Example 3. If $\beta(\nu) := \nu \log \nu - \nu$, then $\beta'(\nu) = \log \nu = -\lambda$

$$\sum_{i \in \mathbb{N}^*} e^{-\lambda_i(V)} \leq \frac{1}{(4\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-V(x)} \, dx$$

This case can formally be seen as the limit case $m \to 1$ in Examples 1 and 2. Here $F(s) = e^{-s}$, $G(s) = (4\pi)^{-d/2} \, e^{-s}$

Example 4. If $\beta(\nu) := \nu \log \nu + (1-\nu) \log(1-\nu)$, then $\beta'(\nu) = \log\left(\frac{\nu}{1-\nu}\right) = -\lambda$ and $F(s) = \log(1 + e^{-s})$

$$\sum_{i \in \mathbb{N}^*} \log\left(1 + e^{-\lambda_i(V)}\right) \leq \int_{\mathbb{R}^d} G(V(x)) \, dx$$

Interpolation: Gagliardo-Nirenberg ineq. for systems

Assume that $H = -\Delta + V$ has an infinite sequence $(\lambda_i(V))_{i\in\mathbb{N}^*}$ of eigenvalues. Let F and G be such that

$$\sum_{i\in\mathbb{N}^*} F(\lambda_i(V)) = \mathrm{Tr}\,[F\,(-\Delta + V)] \le \int_{\mathbb{R}^d} G(V(x))\,dx$$

Let $\bar\lambda := \lim_{i\to\infty} \lambda_i(V) \le \infty$ and assume that

$$\mathrm{Spectrum}(-\Delta + V) \cap (-\infty, \bar\lambda) = \{\lambda_i(V)\ :\ i \in \mathbb{N}^*\}$$

Define $\sigma(s) := -F'(s)$ and $\beta(s) := -\int_0^s \sigma^{-1}(t)\,dt$. Notice that

$$F(s) = \int_s^{\bar\lambda} \sigma(t)\ dt = \int_s^{\bar\lambda} (\beta')^{-1}(-t)\ dt = -\min_{\nu>0}\,[\beta(\nu) + \nu\, s]$$

$$\sum_{i\in\mathbb{N}^*} \nu_i \int_{\mathbb{R}^d} \left(|\nabla\psi_i|^2 + V\,|\psi_i|^2\right)\,dx + \sum_{i\in\mathbb{N}^*} \beta(\nu_i) + \int_{\mathbb{R}^d} G(V(x))\,dx \geq 0$$

for any sequence of nonnegative occupation numbers $(\nu_i)_{i\in\mathbb{N}^*}$ and any sequence $(\psi_i)_{i\in\mathbb{N}^*}$ of orthonormal $L^2(\mathbb{R}^d)$ functions

Method: For fixed $\boldsymbol{\nu} = (\nu_i)_{i\in\mathbb{N}^*}$, $\boldsymbol{\psi} = (\psi_i)_{i\in\mathbb{N}^*}$

$$K[\boldsymbol{\nu},\boldsymbol{\psi}] := \int_{\mathbb{R}^d} \sum_{i\in\mathbb{N}^*} \nu_i\,|\nabla\psi_i|^2\,dx \quad \text{and} \quad \rho := \sum_{i\in\mathbb{N}^*} \nu_i\,|\psi_i|^2$$

$$H(s) := -\left[G\circ(G')^{-1}(-s) + s\,(G')^{-1}(-s)\right]$$

Assume that G' is invertible and optimize on V: The optimal potential V has to satisfy

$$G'(V) + \rho = 0$$

$$\int_{\mathbb{R}^d} V\,\rho\,dx + \int_{\mathbb{R}^d} G(V(x))\,dx = -\int_{\mathbb{R}^d} H(\rho(x))\,dx$$

Theorem 9

$$K[\boldsymbol{\nu}, \boldsymbol{\psi}] + \sum_{i \in \mathbb{N}^*} \beta(\nu_i) \geq \int_{\mathbb{R}^d} H(\rho)\, dx$$

with $\rho = \sum_{i \in \mathbb{N}^*} \nu_i \, |\psi_i|^2$.

Here $(\nu_i)_{i \in \mathbb{N}^*}$ *is any nonnegative sequence of occupation numbers and* $(\psi_i)_{i \in \mathbb{N}^*}$ *is any sequence of orthonormal* $L^2(\mathbb{R}^d)$ *functions.*

Example 1. Let $m>1$ (standard Lieb-Thirring inequality) and consider $\beta(\nu) := c_m\,\nu^m$, $c_m := (m-1)^{m-1}m^{-m}$, $m=\frac{\gamma}{\gamma-1}$,
$F(s)=(-s)^\gamma$ and $G(s)=C_{\rm LT}(\gamma)(-s)^{\gamma+d/2}$

$$q := \frac{2\gamma+d}{2\gamma+d-2} \quad \text{and} \quad \mathcal{K}^{-1} := q\,[C_{\rm LT}(\gamma)\,(\gamma+d/2)]^{q-1}$$

Corollary 10 *For any* $m\in(1,+\infty)$,

$$K[\boldsymbol{\nu},\boldsymbol{\psi}] + c_m \sum_{i\in\mathbb{N}^*} \nu_i^m \ge \mathcal{K}\int_{\mathbb{R}^d}\rho^q\,dx$$

$$\Big(K[\boldsymbol{\nu},\boldsymbol{\psi}]\Big)^\theta\Big(\sum_{i\in\mathbb{N}^*}\nu_i^m\Big)^{(1-\theta)} \ge \mathcal{L}\int_{\mathbb{R}^d}\rho^q\,dx\;, \qquad \theta=\frac{d}{2(\gamma-1)+d}$$

The case $m=\frac{\gamma}{\gamma-1}\in(-\infty,0)$, which corresponds to $\gamma\in(0,1)$, $q\in(1+d/2,d/(d-2))$, $\beta(\nu):=c_m\,\nu^m$, $c_m:=-(1-m)^{m-1}|m|^{-m}$ is not covered

Notice that $q\in(1,1+d/2)\;\Leftrightarrow\; m>1$

The case $\gamma=1$, $q=1+d/2$ is not covered

Example 2. $m \in (0,1)$, $\beta(\nu) := -c_m\,\nu^m$, $c_m := (1-m)^{m-1} m^{-m}$, $m = \frac{\gamma}{\gamma+1}$, $F(\lambda) = \lambda^{-\gamma}$ and $G(s) = \mathcal{C}(\gamma)\, s^{d/2-\gamma}$

$$q := \frac{2\gamma - d}{2(\gamma+1) - d} \in (0,1) \quad \text{and} \quad \mathcal{K}^{-1} := q\,[\mathcal{C}(\gamma)\,(\gamma - d/2)]^{q-1}$$

Notice that $\gamma > d/2 \;\Rightarrow\; m \in (d/(d+2), 1)$

Corollary 11 *For any* $m \in (\frac{d}{d+2}, 1)$*,*

$$K[\boldsymbol{\nu}, \boldsymbol{\psi}] + \mathcal{K} \int_{\mathbb{R}^d} \rho^q\, dx \geq c_m \sum_{i \in \mathbb{N}^*} \nu_i^m$$

Scale invariant version:

$$\left(K[\boldsymbol{\nu}, \boldsymbol{\psi}]\right)^\theta \left(\int_{\mathbb{R}^d} \rho^q\, dx\right)^{(1-\theta)} \geq \mathcal{L} \sum_{i \in \mathbb{N}^*} \nu_i^m$$

with $\theta = \frac{d}{2(\gamma+1)}$

Remark: $\nu_1 = 1$, $\nu_i = 0$ for any $i \geq 2 \Rightarrow$ standard Gagliardo-Nirenberg inequalities

Example 3. If $\beta(\nu) := \nu \log \nu - \nu$, then $\beta'(\nu) = \log \nu = -\lambda$, $F(s) = e^{-s}$ and $G(s) = (4\pi)^{-d/2}\, e^{-s}$.

Corollary 12

$$K[\boldsymbol{\nu}, \boldsymbol{\psi}] + \sum_{i \in \mathbb{N}^*} \nu_i \log \nu_i \geq \int_{\mathbb{R}^d} \rho \log \rho \, dx + d/2 \log(4\pi) \int_{\mathbb{R}^d} \rho \, dx$$

$$\int_{\mathbb{R}^d} \rho \log \rho \, dx \leq \sum_{i \in \mathbb{N}^*} \nu_i \log \nu_i + \frac{d}{2} \log \left(\frac{e}{2\pi \, d} \frac{K[\boldsymbol{\nu}, \boldsymbol{\psi}]}{\int_{\mathbb{R}^d} \rho \, dx} \right) \int_{\mathbb{R}^d} \rho \, dx$$

Conclusion: Motivations

The free energy is a measurement of the distance to the minimizer of a variational problem corresponding to mixed states problems in quantum mechanics (systems with temperature) which is expected to be robust with respect to, e.g. semi-classical limits, or when one takes mean field nonlinearities into account (Poisson coupling / time-dependent Hartree-Fock with temperature, etc). ...[J.D., Felmer, Paturel, Rein]

Confinement potentials arise in a natural way in dispersion problems when one goes to self-similar variables. In the standard Gagliardo-Nirenberg inequalities, the two types of solutions correspond to the two types of self-similar Barenblatt special solutions for the porous media equation. The two "dual" classes of inequalities should therefore not be a surprise. Confinement also arises in a natural way in the modeling, for instance of semi-conductor devices. [...]

Drift-Diffusion limits are the main open question. [Chavanis, Laurençot, Lemou,...], [Degond, Ringhofer], [J.D, Mayorga, Méhats]