L^1 and L^∞ intermediate asymptotics for scalar conservation laws

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P. Lax (1957): if U is the unique entropy solution to

$$U_\tau + f(U)_\xi = 0\ , \quad U(0,\xi) = U_0(\xi)$$

with $f \in C^2$ near the origin, $f(0) = f'(0) = 0$ and $f'' > 0$, and if $U_0 \geq 0$ is of compact support in the bounded interval (s_-, s_+), then the following estimate holds:

$$\|U(\tau,\cdot) - W_\infty(\tau, \cdot - s_-)\|_1 = O(\tau^{-1/2}) \quad \text{as} \quad \tau \to \infty$$

where $W_\infty(\tau,\xi) = \frac{\xi}{f''(0)}\,\tau^{-1}$ if $0 < \xi < -s_- + s_+ + \sqrt{2\,\|u_0\|_1\, f''(0)}\,\tau^{-1/2}$ and 0 elsewhere.

Let $q > 1$ and consider a nonnegative entropy solution of

$$\begin{cases} U_\tau + (U^q)_\xi = 0\ , \quad \xi \in \mathbb{R}\ , \quad \tau > 0 \\ U(\tau = 0, \cdot) = U_0 \end{cases} \tag{1}$$

T.-P. Liu & M. Pierre: $\lim_{\tau\to\infty} \tau^{\frac{1}{q}(1-\frac{1}{p})} \|U(\tau) - U_\infty(\tau)\|_p = 0$

where U_∞ is the self-similar solution $U_\infty(\tau,\xi) = \left(\frac{|\xi|}{q\,\tau}\right)^{1/(q-1)} \chi_{\xi\le c(\tau)}$

Y.-J. Kim (2001)

Theorem 1 *Let U be a global, piecewise C^1 entropy solution of (1) corresponding to a nonnegative initial data U_0 in $L^1 \cap L^\infty(\mathbb{R})$ which is compactly supported in $(\xi_0, +\infty)$ for some $\xi_0 \in \mathbb{R}$ and such that*

$$\liminf_{\substack{\xi\to\xi_0\\ \xi>\xi_0}} \frac{U_0(\xi)}{|\xi-\xi_0|^{1/(q-1)}} > 0$$

Then, for any $\alpha \in (0, \frac{q}{q-1})$ and $\epsilon > 0$,

$$\limsup_{\tau\to+\infty} \tau^{\alpha-\epsilon} \int_{\mathbb{R}} |U(\tau,\xi) - U_\infty(\tau,\xi-\xi_0)| \,\frac{d\xi}{|\xi-\xi_0|^\alpha} = 0$$

$\frac{1}{2} < \alpha(1 - 1/q) \Longleftrightarrow q/(2(q-1)) < \alpha$

Corollary 1 *For any $\beta < 1$, there exists a constant C_β such that*

$$\|U(\tau,\cdot) - U_\infty(\tau,\xi-\xi_0)\|_1 \leq C_\beta \, \tau^{-\beta}$$

$$M := \int_{\mathbf{R}} U_0 \, d\xi \quad \text{and} \quad c_M := \left(\frac{q\int_{\mathbf{R}} U_0 \, d\xi}{q-1}\right)^{(q-1)/q}$$

Theorem 2 *Under the same assumptions as in Theorem 1,*

$$\lim_{\tau\to+\infty} \sup_{\xi\in \mathrm{supp}(U(\tau,\cdot))} \tau^{1/q} \, |U(\tau,\xi) - U_\infty(\tau,\cdot-\xi_0)| = 0$$

and $\rho(\tau) := \max\left[\mathrm{supp}(U(\tau,\cdot))\right]$ *satisfies as* $\tau \to +\infty$

$$\lim_{\tau\to+\infty} (1+q\,\tau)^{-1/q} \rho(\tau) = c_M \,, \quad \rho(\tau) \geq (1+q\,\tau)^{1/q} c_M \,(1+O(\tau^{-1}))$$

Notions of solution, time-dependent rescaling

Proposition 2 *Let U be a nonnegative piecewise C^1 entropy solution of (1), whose points of discontinuity are given by the curves $\xi_1(\tau) < \xi_2(\tau) < \cdots < \xi_n(\tau)$. Then the rescaled function*

$$u(t,x) = e^t U\left((e^{qt}-1)/q, e^t x\right)$$

is a piecewise C^1 function, whose points of discontinuity are given by the curves $s_i(t) \equiv e^{-t}\xi_i((e^{qt}-1)/q)$, which satisfy

$$s_i'(t) = \frac{(u_i^+)^q - s_i(t)\,u_i^+ - (u_i^-)^q + s_i(t)\,u_i^-}{u_i^+ - u_i^-}$$

for any $i = 1, 2, \ldots\ n$. Out of the curves $x = s_i(t)$ the function u is a classical solution of

$$u_t = (x\,u - u^q)_x \tag{2}$$

and across these curves it satisfies

$$u_i^- := \lim_{\substack{x \to s_i(t) \\ x < s_i(t)}} u(t,x) > \lim_{\substack{x \to s_i(t) \\ x > s_i(t)}} u(t,x) := u_i^+$$

Moreover u *and* U *have the same initial data* $U_0 := U(0,\cdot) = u(0,\cdot) =: u_0$. *Finally, if* $U_0 \in L^1(\mathbb{R})$, *then, for all* $t > 0$, *we have:* $||u(t)||_1 = ||U_0||_1$.

Rankine-Hugoniot condition

$$\xi_i'(\tau) = \frac{(U_i^+)^q - (U_i^-)^q}{U_i^+ - U_i^-}$$

$$U_i^- := \lim_{\substack{\xi \to \xi_i(\tau) \\ \xi < \xi_i(\tau)}} U(\tau,\xi) > \lim_{\substack{\xi \to \xi_i(\tau) \\ \xi > \xi_i(\tau)}} U(\tau,\xi) := U_i^+$$

For every $c > 0$, let u_∞^c be the stationary solution of (2) :

$$u_\infty^c(x) = \begin{cases} x^{1/(q-1)} & 0 \leq x \leq c \\ \\ 0 & \text{if } x < 0 \text{ or } x > c \end{cases}$$

If $c = c_M := (q\,M/(q-1))^{(q-1)/q}$, $u_\infty := u_\infty^{c_M}$. $\|u_\infty\|_1 = M$.

Comparison results

Lemma 3 *Consider two solutions U and V of (1)*

$$U_\tau = -(U^q)_\xi \quad \textit{and} \quad V_\tau = -(V^q)_\xi$$

with nonnegative initial data U_0 and V_0 such that $U_0 \leq A_0\, V_0$ a.e. for some positive constant A_0. Then

$$U(\tau, \cdot) \leq A_0\, V(A_0^{q-1}\tau, \cdot) \text{ a.e.} \quad \forall\, \tau \in \mathbb{R}^+$$

Corollary 4 *Let u be a solution of (2) with a nonnegative initial data u_0 satisfying*

$$u_0 \leq A_0\, u_\infty^c \; a.e.$$

for some positive constants A_0 and c. Then

$$u(t,x) \leq A(t)\, u_\infty^{c(t)}(x) \; a.e. \quad \forall\, t \in \mathbb{R}^+$$

with $A(t) = \frac{A_0\, e^{qt/(q-1)}}{\left[1+A_0^{q-1}(e^{qt}-1)\right]^{1/(q-1)}}$ *and* $c(t) = c\left(\frac{A_0}{A(t)}\right)^{(q-1)/q}$.

$u(t,\cdot)$ *is supported in* $[0, c(t)] \subset [0, c\,(\max(A_0, 1))^{(q-1)/q}]$

$$\|\,(u - x^{1/q})_+\,\|_\infty \leq (A(t)-1)_+ \to 0 \quad \textit{as}\; t \to +\infty\,.$$

L^1 INTERMEDIATE ASYMPTOTICS

Relative entropy Σ of the solution u with respect to the stationary solution u^c_∞: For any positive constants c and c', let

$$\Sigma(t) = \int_0^{c'} \mu(x)\,|u(t,x) - u^c_\infty(x)|\,dx = \int_0^{c} \mu\,|u - u^c_\infty|\,dx + \int_c^{c'} \mu\,u\,dx$$

Define $f(v) = v - v^q$ for $v > 0$.

Proposition 5 *Consider a nonnegative solution u with initial data u_0, with compact support in $[0, +\infty)$, such that*

$$u_0(x) \leq A_0\,x^{1/(q-1)} \quad \forall\, x \in \mathbb{R}^+$$

for some $A_0 > 0$. Assume that $\lim_{x\to 0,\, x>0} \mu(x)\,u^q_\infty(x) = 0$. *Let $c' > 0$ and suppose that the functions $\mu' u^q_\infty$ and $\mu\,u_\infty$ are integrable on $(0, c')$. Then for every fixed $c \in (0, c')$, for $t \geq 0$,*

$$\frac{d\Sigma}{dt} \leq \int_0^c \mu'(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right) \right| \, dx - \int_c^{c'} \mu'(u_\infty^{c'})^q \, f\left(\frac{u}{u_\infty^{c'}}\right) dx$$

$$- \, \mu(c)\, c^{\frac{q}{q-1}} \left\{ f\left(\frac{u^+(c)}{c^{\frac{1}{q-1}}}\right) + \left| f\left(\frac{u^-(c)}{c^{\frac{1}{q-1}}}\right) \right| \right\}$$

$$+ \, \mu(c')\,(c')^{\frac{q}{q-1}} \, f\left(\frac{u^-(c')}{(c')^{\frac{1}{q-1}}}\right)$$

where $u^{\pm}(c) := \lim_{\substack{x\to c \\ \pm(x-c)>0}} u(x)$. *If* $c = c'$, *then*

$$\frac{d\Sigma}{dt} \leq \int_0^c \mu'(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right) \right| \, dx \leq 0 \, .$$

Proof. $\Sigma(t) = \int_0^c \mu \, [u - u_\infty^c] \left[1\!\!\mathrm{l}_{u > u_\infty^c} - 1\!\!\mathrm{l}_{u < u_\infty^c} \right] \, dx + \int_c^{c'} \mu \, u(t) \, dx$.
We assume for simplicity that $u(t, .)$ has exactly one shock at $x = s(t)$. Let $u^\pm = u^\pm(t)$ and $v^\pm = u^\pm(t)/u_\infty^{c'}$, where $u_\infty^{c'}$ stands for $u_\infty^{c'}(s(t))$: $v^- > v^+$ and

$$s'(t) = -(u_\infty^{c'})^{q-1} \, \frac{f(v^+) - f(v^-)}{v^+ - v^-}$$

Case $0 < s = s(t) < c$.

$$\frac{d\Sigma}{dt} = \int_0^c \mu \, u_t \left[1\!\!\mathrm{l}_{u > u_\infty^c} - 1\!\!\mathrm{l}_{u < u_\infty^c} \right] \, dx + \int_c^{c'} \mu \, u_t \, dx$$
$$+ \left[\mu(s) |u - u_\infty^c(s)| \cdot s'(t) \, \right]_{u=u^+}^{u=u^-}$$

$$\frac{d\Sigma}{dt} \leq \int_0^c \mu'\,(u_\infty^c)^q \left| f\left(\frac{u}{u_\infty^c}\right)\right| \,dx + \; \mu(s)\,(u_\infty^c(s))^q \Psi(v^-, v^+)$$
$$- \,\mu(c)\, c^{\frac{q}{q-1}} \;\left[\; \left| f\left(c^{-\frac{1}{q-1}}\, u^-(t,c)\right)\right| + f\left(c^{-\frac{1}{q-1}}\, u^+(t,c)\right)\right]$$
$$- \int_c^{c'} \mu'\,(u_\infty^{c'})^q\, f\left(\frac{u}{u_\infty^{c'}}\right) dx + \; \mu(c')\,(c')^{q/(q-1)}\, f\left(\frac{u^-(c')}{(c')^{\frac{1}{q-1}}}\right)$$

$$\Psi(v^-, v^+) := \left[f(v^+) - f(v^-)\right] \cdot \tfrac{|v^+-1|-|v^--1|}{v^+-v^-} + |f(v^+)| - |f(v^-)|$$

(i) $1 \leq v^+ \leq v^-$: $f(v^-) \leq f(v^+) \leq 0$ and $\Psi(v^-, v^+) = 0$.

(ii) $v^+ < 1 \leq v^-$: $f(v^-) \leq 0 < f(v^+)$

$$\begin{aligned} \tfrac{1}{2}\,\Psi(v^-, v^+) = \;& \tfrac{v^--1}{v^--v^+}\, f(v^+) + \tfrac{1-v^+}{v^--v^+}\, f(v^-) \\ & \leq f\left(\tfrac{v^--1}{v^--v^+}\, v^+ + \tfrac{1-v^+}{v^--v^+}\, v^-\right) = f(1) = 0 \end{aligned}$$

(iii) $v^+ < v^- \leq 1$: $f(v^-) \geq 0$ and $f(v^+) \geq 0$, $\Psi(v^-, v^+) = 0$

Rates of decay To emphasize the dependence in α, we denote by Σ_α the quantity Σ in case $\mu(x) = |x|^{-\alpha}$.

Proposition 6 *Assume that $c \le c_M$, $c \le c'$ and $c = c_M$ if $c' > c$. Then $\lim_{t\to+\infty} \Sigma_\alpha(t) = 0$ and*

$$\frac{d\Sigma_\alpha}{dt} + (q-1)\,\alpha\,\Sigma_\alpha(t) - \alpha \int_c^{c'} x^{-\alpha}\,u\,dx - r(c') = o(\Sigma_\alpha(t)) \quad \text{as } t \to +\infty$$

with $r(c') = \mu(c')\,(c')^{q/(q-1)}\,f\left((c')^{-1/(q-1)}\,u^-(c')\right)$

Proof.

$$\frac{d\Sigma_\alpha}{dt} \le -\alpha \int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right| dx + \alpha \int_c^{c'} x^{-\alpha-1+\frac{q}{q-1}}\, f\left(\frac{u}{u_\infty^{c'}}\right) dx + r(c')$$

$$\int_c^{c'} x^{-\alpha-1+\frac{q}{q-1}}\, f\left(\frac{u}{u_\infty^{c'}}\right) dx = \int_c^{c'} x^{-\alpha}\,u\,dx - \int_c^{c'} x^{-\alpha-1}\,u^q\,dx \le \int_c^{c'} x^{-\alpha}\,u\,dx$$

$$f\left(\frac{u}{u_\infty^c}\right) = (1-q)\left(\frac{u}{u_\infty^c}-1\right) + q(1-q)\left(\frac{u}{u_\infty^c}-1\right)^2 \int_0^1 (1-\theta)\left(\theta\frac{u}{u_\infty^c}+1-\theta\right)^{q-2} d\theta$$

$$\int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right| \, dx \geq (q-1)\,\Sigma_\alpha - C_q \int_0^c x^{-\alpha+\frac{1}{q-1}} \left(\frac{u}{u_\infty^c} - 1\right)^2 dx$$

with $C_q = \frac{1}{2}\, q\, (q-1)$ if $s(t) > c(t)$, and

$$\begin{aligned} \int_0^c x^{-\alpha-1+\frac{q}{q-1}} \left| f\left(\frac{u}{u_\infty^c}\right)\right| \, dx \;\; &\geq \;\; (q-1) \left(\Sigma_\alpha - \int_{s(t)}^c \mu\, u_\infty^c \, dx\right) \\ &\qquad - C_q \int_0^{s(t)} x^{-\alpha+\frac{1}{q-1}} \left(\frac{u}{u_\infty^c} - 1\right)^2 dx \end{aligned}$$

if $s(t) \leq c$. Thus, with $c(t) := \min(s(t), c)$, $\chi(t) \equiv 0$ if $s(t) > c(t)$ and $\chi(t) := \int_{s(t)}^c \mu\, u_\infty^c \, dx$ if $s(t) \leq c(t)$

$$\begin{aligned} \frac{d\Sigma_\alpha}{dt} + (q-1)\,\alpha\,\Sigma_\alpha(t) - r(c') \leq C_q \int_0^{c(t)} x^{-\alpha+\frac{1}{q-1}} \left(\frac{u}{u_\infty^c} - 1\right)^2 \, dx \\ + q\,\alpha \int_c^{c'} x^{-\alpha}\, u \, dx + \chi(t) \end{aligned}$$

As we shall see

$$\left\| \frac{u}{u_\infty^c} - 1 \right\|_{L^\infty(0,c(t))} \to 0 \quad \text{as} \quad t \to +\infty \tag{3}$$

so that

$$\int_0^c x^{-\alpha+\frac{1}{q-1}} \left(\frac{u}{u_\infty^c} - 1 \right)^2 dx = \left\| \frac{u}{u_\infty^c} - 1 \right\|_{L^\infty(0,c(t))} \int_0^c \mu \, |u - u_\infty^c| \, dx$$

is neglectible compared to $\Sigma_\alpha(t)$.

Corollary 7 *Under the assumptions of Proposition 6, if* $c = c_M$ *and if* $\operatorname{supp} u(t,\cdot) \subset (0, c') \quad \forall\, t \geq 0$*, then for any* $\epsilon > 0$*, there exists a positive constant* $C_\alpha(\epsilon)$ *such that*

$$\Sigma_\alpha(t) \leq C_\alpha(\epsilon)\, e^{-[(q-1)\,\alpha-\epsilon]\,t} \quad \forall\, t \geq 0$$

Uniform estimates: Refined *graph convergence*

Proposition 8 *Let $\epsilon > 0$, $M = \int u_0 \, dx$ and consider a piecewise C^1 nonnegative initial data u_0 with compact support contained in $[0, +\infty)$, such that $\liminf_{\substack{x \to 0 \\ x>0}} x^{1/(1-q)} u_0(x) > 0$. Then there exists a positive T such that, for any $t > T$,*
(i) the support of $u(\cdot, t)$ is an interval $[0, s(t)]$
(ii) $\inf_{x \in [0, s(t))} x^{1/(1-q)} u(x,t) > 0$
(iii) there exists a constant $A_0 > 0$ such that

$$u \leq A(t)\, u_\infty^{s(t)} \quad \text{with } A(t) = \frac{A_0 \, e^{q(t-1)/(q-1)}}{\left[1 + A_0^{q-1}(e^{q(t-1)} - 1)\right]^{1/(q-1)}}$$

(iv) for any $\epsilon > 0$, there exists a constant κ such that

$$u \geq (1 - \kappa \, e^{-qt})\, u_\infty^{c_M - \epsilon}$$

Theorem 3 *Let u be an entropy solution of*

$$u_t + (u^q - xu)_x = 0 \tag{2}$$

corresponding to a piecewise C^1 nonnegative initial data u_0 with compact support contained in $[0, +\infty)$ and assume that $\liminf_{\substack{x\to 0\\ x>0}} x^{1/(1-q)} u_0(x) > 0$. Then

$$\lim_{t\to\infty} \sup_{x\in(0,s(t))} |u(t,x) - u_\infty^{s(t)}| = 0$$

where $[0, s(t)]$ is the support of $u(\cdot, t)$ for $t > 0$ large enough. Moreover,

$$\lim_{t\to+\infty} s(t) = c_M \quad \textit{and} \quad s(t) \geq c_M - O(e^{-qt}) \quad \textit{as} \quad t \to +\infty$$

Lemma 9 *Consider a solution u of (2) as in Theorem 3. Let $s(t)$ be the upper extremity of the support of u for t karge enough and consider $h(t) := \lim_{\substack{x \to s(t) \\ x < s(t)}} u(x,t)$. Then*

$$\begin{cases} \dfrac{ds}{dt} = h^{q-1} - s \\ \\ \dfrac{dh}{dt} = h\left(1 - (u^{q-1})_x\right) \end{cases} \tag{4}$$

where by $(u^{q-1})_x$ we denote the quantity $\lim_{\substack{x \to s(t) \\ x < s(t)}} (u^{q-1})_x(x,t)$.

Lemma 10 *Consider a solution u of (2) as in Theorem 3. Then*

$$(u^{q-1})_x \leq (1 - e^{-qt})^{-1}$$

in the distribution sense.

Lemma 11 *(i) For any $\epsilon > 0$, there exists $t_1 > 0$ such that*

$$s(t) \geq c_M - \epsilon \quad \forall\, t > t_1$$

For any $\epsilon > 0$, $\delta \in (0,1)$, $t_0 > 0$, there exists $t_1 > t_0$ such that

$$h(t_1) \geq (1-\delta)\, u_\infty(s(t_1))$$

(ii) Assume that $h(t_1) = h_1 > 0$ for some $t_1 > 0$. Then

$$h(t) \geq h_1\, (1 - e^{-qt_1})^{1/q} \quad \forall\, t > t_1$$

(iii) As $t \to +\infty$, $s(t)$ converges to c_M and for any $\eta \in (0,1)$, there exists a $t_1 \geq 0$ such that

$$u(\cdot, t) \geq (1-\eta)\, u_\infty^{s(t)} \quad \forall\, t \geq t_1$$

Proof of (iv). Let us prove that $s(t)$ converges to c_M. Integrating $(u^{q-1})_x \leq (1-e^{-qt})^{-1}$ with respect to x, we get

$$u(x,t) \geq \left((h(t))^{q-1} - \frac{s(t)-x}{1-e^{-qt}}\right)_+^{1/(q-1)} =: v(x,t) \quad \forall\, x \in [c_M - \epsilon, s(t))$$

Integrating u on $(0, s(t))$, we obtain a lower estimate for the mass:

$$M \geq (1-\eta) \int_0^{c_M-\epsilon} u_\infty \, dx + \int_{c_M-\epsilon}^{s(t)} v \, dx$$

as soon as t is large enough so that $\kappa\, e^{-q\,t} < \eta$. Take $h_0 = h(t_1)$

$$(h(t))^{q-1} \geq (1-\delta)^{q-1}\, s(t_1) \left(1 - e^{-qt_1}\right)^{(q-1)/q} =: h_1 \quad \forall\, t > t_1$$

Conclusion by contradiction on $(c_M - \epsilon, s(t))$