Free energies, nonlinear flows and functional inequalities

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A – Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall \, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \; dx \quad \forall \, v \in \mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2\,d}{d-2}$. Can we recover this using a nonlinear flow approach ? Can we improve it ?

Keller-Segel model: another motivation [Carrillo, Carlen and Loss] and [Blanchet, Carlen and Carrillo]

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\;dx + \mathsf{S}_d\left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\;dx\right)^{\frac{2}{d}}\int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\;dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2}$

A first statement

Proposition

[J.D.] *Assume that* $d \geq 3$ *and* $m = \frac{d-2}{d+2}$*. If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$*, then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

Theorem

[J.D.] *Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^{\alpha}\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. delPino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]
For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Improved inequality: proof (1/2)

$\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa\,T = \frac{2\,d}{d+2}\,\frac{T}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2}$$

Improved inequality: proof (2/2)

By the **Cauchy-Schwarz inequality**, we have

$$\frac{\mathsf{J}'^2}{(m+1)^2} = \|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2$$
$$\le \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx = Cst\;\mathsf{J}''\,\mathsf{J}$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q} - 1)\,, \quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \le \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \le 0$$

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1 - e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, the proof is completed □

Fast diffusion equations

1. entropy methods
2. linearization of the entropy
3. improved Gagliardo-Nirenberg inequalities

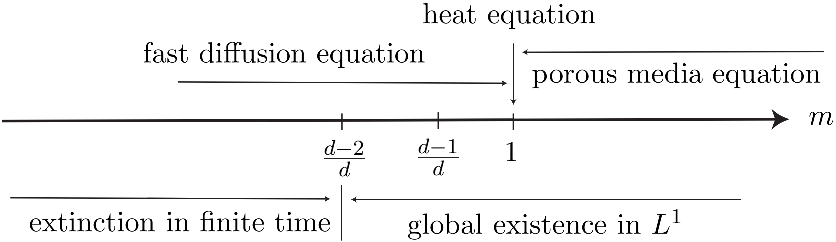
B1 – Fast diffusion equations: entropy methods

Existence, classical results

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$ as $t \to +\infty$

[Friedmann, Kamin, 1980] $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$



Existence theory, critical values of the parameter m

Time-dependent rescaling, Free energy

- **Time-dependent rescaling**: Take $u(\tau, y) = R^{-d}(\tau)\, v\,(t, y/R(\tau))$ where

$$\frac{dR}{d\tau} = R^{d(1-m)-1}\,, \quad R(0) = 1\,, \quad t = \log R$$

- The function v solves a Fokker-Planck type equation

$$\frac{\partial v}{\partial t} = \Delta v^m + \nabla \cdot (x\, v)\,, \quad v_{|\tau=0} = u_0$$

- [Ralston, Newman, 1984] Lyapunov functional: **Generalized entropy** or **Free energy**

$$\mathcal{F}[v] := \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}\,|x|^2 v \right) dx - \mathcal{F}_0$$

Entropy production is measured by the **Generalized Fisher information**

$$\frac{d}{dt}\mathcal{F}[v] = -\mathcal{I}[v]\,, \quad \mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$

Relative entropy and entropy production

- **Stationary solution:** choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) := \left(C + \frac{1-m}{2\,m}\,|x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{F}_0$ so that $\mathcal{F}[v_\infty] = 0$

- **Entropy – entropy production inequality**

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[v] \geq 2\,\mathcal{F}[v]$$

Corollary

A solution v with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies $\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[u_0]\,e^{-\,2\,t}$

An equivalent formulation: Gagliardo-Nirenberg inequalities

$$\mathcal{F}[v] = \int_{\mathbb{R}^d} \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \mathcal{F}_0 \leq \frac{1}{2} \int_{\mathbb{R}^d} v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} \mathcal{I}[v]$$

Rewrite it with $p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \geq 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} w^{2p}\, dx \right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With $1 < p \leq \frac{d}{d-2}$ (fast diffusion case) and $d \geq 3$*

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \leq A \, \|\nabla w\|^{\theta}_{L^2(\mathbb{R}^d)} \, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}} \,, \quad \theta = \frac{d(p-1)}{p(d+2-(d-2)p)} \,, \quad y = \frac{p+1}{p-1}$$

... a proof by the Bakry-Emery method

Consider the generalized Fisher information

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v\,|Z|^2\;dx \quad \text{with} \quad Z := \frac{\nabla v^m}{v} + x$$

and compute

$$\frac{d}{dt}\mathcal{I}[v(t,\cdot)] + 2\,\mathcal{I}[v(t,\cdot)] = -2\int_{\mathbb{R}^d} u^m \left[|\nabla Z|^2 - (1-m)(\nabla\cdot Z)^2\right]\,dx$$

- the Fisher information decays exponentially:

$$\mathcal{I}[v(t,\cdot)] \le \mathcal{I}[u_0]\,e^{-\,2\,t}$$

- $\lim_{t\to\infty}\mathcal{I}[v(t,\cdot)] = 0$ and $\lim_{t\to\infty}\mathcal{F}[v(t,\cdot)] = 0$
- $\frac{d}{dt}\Big(\mathcal{I}[v(t,\cdot)] - 2\,\mathcal{F}[v(t,\cdot)]\Big) \le 0$ means $\mathcal{I}[v] \ge 2\,\mathcal{F}[v]$

[Otto], [Carrillo, Toscani], [Jüngel, Markowich, Toscani], [Carrillo, Jüngel, Markowich, Toscani, Unterreiter], [Carrillo, Vázquez]

B2 – Fast diffusion equations: sharp asymptotic rates by linearization of the entropy

Entropy methods and linearization: sharp rates

Generalized Barenblatt profiles: $V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}}$

Assumptions on the initial datum v_0

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_*$, $(v_0 - V_D) \in L^1$ *for some* $D \in [D_1, D_0]$

Theorem

[Blanchet, Bonforte, J.D., Grillo, Vázquez] *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \ne m_* := \frac{d-4}{d-2}$, *the entropy decays according to*

$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

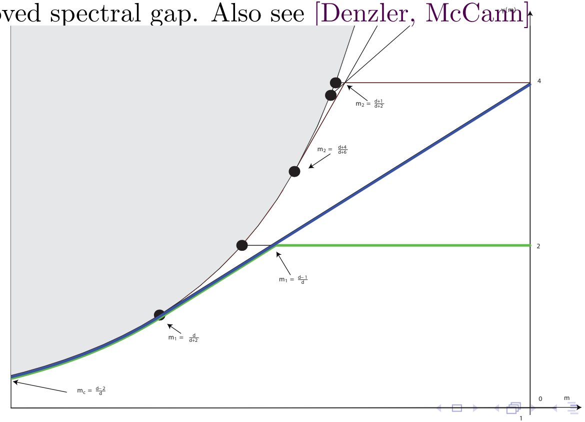
where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$

Improved asymptotic rates

[Bonforte, J.D., Grillo, Vázquez] Assume that $m \in (m_1, 1)$, $d \geq 3$. Under Assumption (H1), if v is a solution of the fast diffusion equation with initial datum v_0 such that $\int_{\mathbb{R}^d} x\, v_0\, dx = 0$, then the asymptotic convergence holds with an improved rate corresponding to the improved spectral gap. Also see [Denzler, McCann]



Higher order matching asymptotics

[J.D., G. Toscani] For some $m \in (m_c, 1)$ with $m_c := (d-2)/d$, we consider on $\mathbb{R}^d$ the fast diffusion equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left(u \, \nabla u^{m-1} \right) = 0$$

Without choosing R, we may define the function v such that

$$u(\tau, y + x_0) = R^{-d} \, v(t, x) \,, \quad R = R(\tau) \,, \quad t = \tfrac{1}{2} \log R \,, \quad x = \frac{y}{R}$$

Then v has to be a solution of

$$\frac{\partial v}{\partial t} + \nabla \cdot \left[v \left(\sigma^{\frac{d}{2}(m-m_c)} \, \nabla v^{m-1} - 2\, x \right) \right] = 0 \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

with (as long as we make no assumption on R)

$$2\, \sigma^{-\frac{d}{2}(m-m_c)} = R^{\,1-d\,(1-m)} \, \frac{dR}{d\tau}$$

Refined relative entropy

Consider the family of the Barenblatt profiles

$$B_\sigma(x) := \sigma^{-\frac{d}{2}} \left(C_M + \frac{1}{\sigma}\,|x|^2\right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d \tag{4}$$

Note that σ is a function of t: as long as $\frac{d\sigma}{dt} \neq 0$, the Barenblatt profile B_σ *is not* a solution (it plays the role of a **local Gibbs state**) but we may still consider the relative entropy

$$\mathcal{F}_\sigma[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[v^m - B_\sigma^m - m\,B_\sigma^{m-1}\,(v - B_\sigma)\right]\,dx$$

The time derivative of this relative entropy is

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = \underbrace{\frac{d\sigma}{dt}\left(\frac{d}{d\sigma}\,\mathcal{F}_\sigma[v]\right)_{|\sigma=\sigma(t)}}_{\text{choose it } =0} + \frac{m}{m-1}\int_{\mathbb{R}^d}\left(v^{m-1} - B_{\sigma(t)}^{m-1}\right)\frac{\partial v}{\partial t}\,dx$$

$\iff$ Minimize $\mathcal{F}_\sigma[v]$ w.r.t. σ $\iff$ $\int_{\mathbb{R}^d} |x|^2\,B_\sigma\,dx = \int_{\mathbb{R}^d} |x|^2\,v\,dx$

The entropy / entropy production estimate

Using the new change of variables, we know that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\frac{m\,\sigma(t)^{\frac{d}{2}(m-m_c)}}{1-m}\int_{\mathbb{R}^d} v\left|\nabla\left[v^{m-1}-B_{\sigma(t)}^{m-1}\right]\right|^2 dx$$

Let $w := v/B_\sigma$ and observe that the relative entropy can be written as

$$\mathcal{F}_\sigma[v] = \frac{m}{1-m}\int_{\mathbb{R}^d}\left[w-1-\frac{1}{m}\left(w^m-1\right)\right] B_\sigma^m\,dx$$

(Repeating) define the *relative Fisher information* by

$$\mathcal{I}_\sigma[v] := \int_{\mathbb{R}^d}\left|\frac{1}{m-1}\nabla\left[\left(w^{m-1}-1\right)B_\sigma^{m-1}\right]\right|^2 B_\sigma\,w\,dx$$

so that $\quad \displaystyle\frac{d}{dt}\mathcal{F}_{\sigma(t)}[v(t,\cdot)] = -\,m\,(1-m)\,\sigma(t)\,\mathcal{I}_{\sigma(t)}[v(t,\cdot)] \quad \forall\, t>0$

When linearizing, one more mode is killed and $\sigma(t)$ scales out

Improved rates of convergence

Theorem (J.D., G. Toscani)

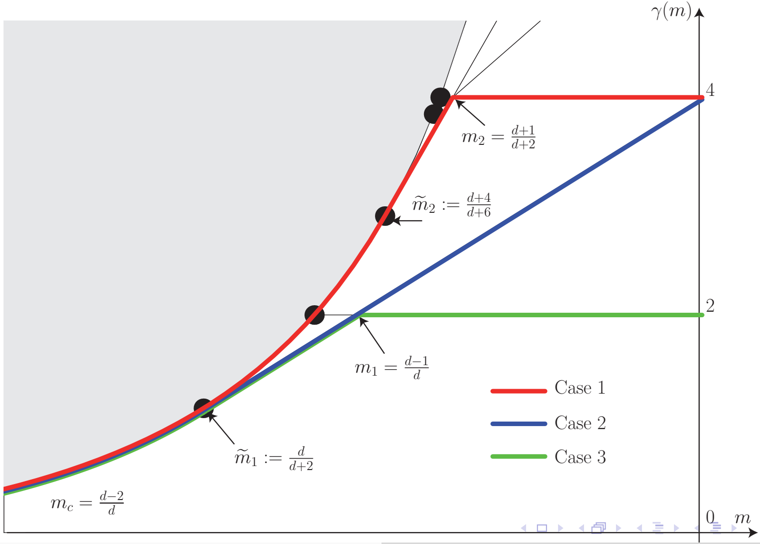
Let $m \in (\widetilde{m}_1, 1)$, $d \geq 2$, $v_0 \in L^1_+(\mathbb{R}^d)$ *such that* v_0^m, $|y|^2\, v_0 \in L^1(\mathbb{R}^d)$

$$\mathcal{E}[v(t,\cdot)] \leq C\, e^{-2\,\Lambda(m)\,t} \quad \forall\, t \geq 0$$

where

$$\Lambda(m) = \begin{cases} \frac{((d-2)\,m-(d-4))^2}{4\,(1-m)} & \text{if } m \in (\widetilde{m}_1, \widetilde{m}_2] \\ 4\,(d+2)\,m - 4\,d & \text{if } m \in [\widetilde{m}_2, m_2] \\ 4 & \text{if } m \in [m_2, 1) \end{cases}$$

Spectral gaps and best constants



B3 – Gagliardo-Nirenberg and Sobolev inequalities : improvements

[J.D., G. Toscani]

Best matching Barenblatt profiles

(Repeating) Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t} + \nabla \cdot \left[u \left(\sigma^{\frac{d}{2}(m-m_c)} \nabla u^{m-1} - 2\,x \right) \right] = 0 \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t) = \frac{1}{K_M} \int_{\mathbb{R}^d} |x|^2\, u(x,t)\, dx\,, \quad K_M := \int_{\mathbb{R}^d} |x|^2\, B_1(x)\, dx$$

where

$$B_\lambda(x) := \lambda^{-\frac{d}{2}} \left(C_M + \tfrac{1}{\lambda}\, |x|^2 \right)^{\frac{1}{m-1}} \quad \forall\, x \in \mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - B_\lambda^m - m\, B_\lambda^{m-1}\, (u - B_\lambda) \right]\, dx$$

Three ingredients for *global improvements*

1. $\inf_{\lambda>0} \mathcal{F}_\lambda[u(x,t)] = \mathcal{F}_{\sigma(t)}[u(x,t)]$ so that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(x,t)] = -\mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

where the relative Fisher information is

$$\mathcal{J}_\lambda[u] := \lambda^{\frac{d}{2}(m-m_c)}\,\frac{m}{1-m}\int_{\mathbb{R}^d} u\,\left|\nabla u^{m-1} - \nabla B_\lambda^{m-1}\right|^2 dx$$

2. In the *Bakry-Emery method*, there is **an additional (good) term**

$$4\left[1 + 2\,C_{m,d}\,\frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{M^\gamma\,\sigma_0^{\frac{d}{2}(1-m)}}\right]\frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right) \geq \frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

3. The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u] \geq \frac{m}{8\int_{\mathbb{R}^d} B_1^m\,dx}\,C_M^2\,\|u - B_\sigma\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$$

Improved decay for the relative entropy

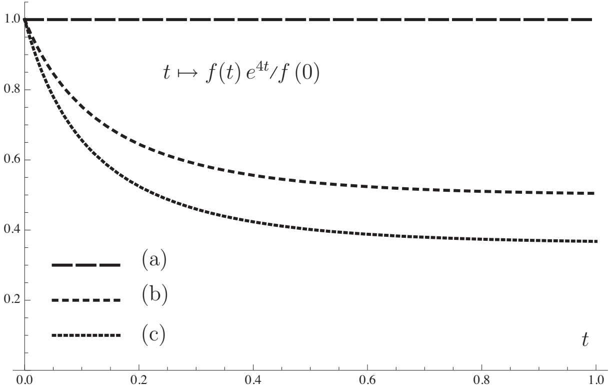


Figure: Upper bounds on the decay of the relative entropy: $t \mapsto f(t)\, e^{4t}/f(0)$.
(a): estimate given by the entropy-entropy production method
(b): exact solution of a simplified equation.
(c): numerical solution (found by a shooting method)

An improved Sobolev inequality: the setting

Sobolev's inequality on $\mathbb{R}^d$, $d \geq 3$ can be written as

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx - \mathsf{S}_d \left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{d}} \geq 0 \quad \forall \, f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

and optimal functions take the form

$$f_{M,y,\sigma}(x) = \frac{1}{\sigma^{\frac{d}{2}} \left(C_M + \frac{|x-y|^2}{\sigma} \right)^{d-2}} \quad \forall \, x \in \mathbb{R}^d$$

where C_M is uniquely determined in terms of M by the condition that $\int_{\mathbb{R}^d} f_{M,y,\sigma}^{\frac{2d}{d-2}} \, dx = M$ and $(M, y, \sigma) \in \mathcal{M}_d := (0, \infty) \times \mathbb{R}^d \times (0, \infty)$. Define the manifold of the optimal functions as

$$\mathfrak{M}_d := \left\{ f_{M,y,\sigma} \ : \ (M, y, \sigma) \in \mathcal{M}_d \right\}$$

and consider the *relative entropy* functional

$$\mathcal{R}[f] := \inf_{g \in \mathfrak{M}_d} \int_{\mathbb{R}^d} \left[g^{-\frac{2}{d-2}} \left(|f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}} \right) - \frac{d}{d-1} \left(|f|^{2\frac{d-1}{d-2}} - g^{2\frac{d-1}{d-2}} \right) \right] dx$$

An improved Sobolev inequality: the result (1/2)

Theorem

[J.D., G. Toscani] *Let $d \geq 3$. For any $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx - \mathsf{S}_d \left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}} \, dx \right)^{\frac{d-2}{d}} \geq \frac{\mathsf{C}_d \, \mathcal{R}[f]^2}{\left\| |x|^2 \, f^{\frac{2d}{d-2}} \right\|_{\mathrm{L}^1(\mathbb{R}^d)}}$$

The functional $\mathcal{R}[f]$ is a measure of the distance of f to $\mathfrak{M}_d$ and because of the **Csiszár-Kullback inequality**, we get

$$\frac{\mathcal{R}[f]}{\left\| |x|^2 \, f^{\frac{2d}{d-2}} \right\|_{\mathrm{L}^1(\mathbb{R}^d)}^{1/2}} \geq \frac{\mathsf{C}_{\mathrm{CK}}}{\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{3\,d+2}{d-2}}} \inf_{g \in \mathfrak{M}_d} \left\| |f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}} \right\|_{\mathrm{L}^1(\mathbb{R}^d)}^2$$

with explicit expressions for C_d and C_{CK}

An improved Sobolev inequality: the result (2/2)

Corollary

[J.D., G. Toscani] *Let $d \geq 3$. For any $f \in \mathcal{D}^{1,2}(\mathbb{R}^d)$, we have*

$$\int_{\mathbb{R}^d} |\nabla f|^2\, dx - \mathsf{S}_d \left(\int_{\mathbb{R}^d} |f|^{\frac{2d}{d-2}}\, dx \right)^{\frac{d-2}{d}} \geq \frac{\mathfrak{C}_d}{\|f\|^{2\,\frac{3\,d+2}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}} \inf_{g \in \mathfrak{M}_d} \left\| |f|^{\frac{2d}{d-2}} - g^{\frac{2d}{d-2}} \right\|^4_{\mathrm{L}^1(\mathbb{R}^d)}$$

- The expression of $\mathfrak{C}_d$ is also explicit
- This solves an old open question of [Brezis, Lieb (1985)] with (partial) answers given in [Bianchi-Egnell (1990)] and [Cianchi, Fusco, Maggi, Pratelli (2009)]
- A similar result holds for Gagliardo-Nirenberg inequalities with $p \in (1, \frac{d}{d-2})$

$d=2$: Onofri's and log HLS inequalities

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta\log\left(v/\mu\right) \quad t>0\,, \quad x\in\mathbb{R}^2$$

Proposition

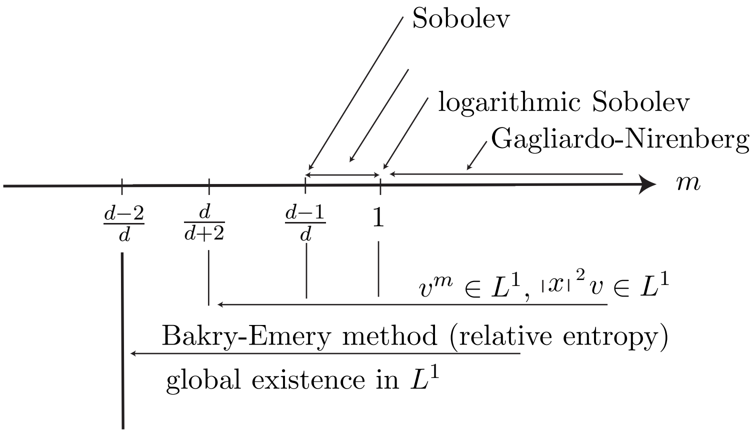
If $v=\mu\,e^{u/2}$ is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0\in\mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu\in\mathrm{L}^1(\mathbb{R}^2)$, then

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

$$\geq \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \int_{\mathbb{R}^2} u\,d\mu - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \geq 0$$

Fast diffusion: finite mass regime

Inequalities...



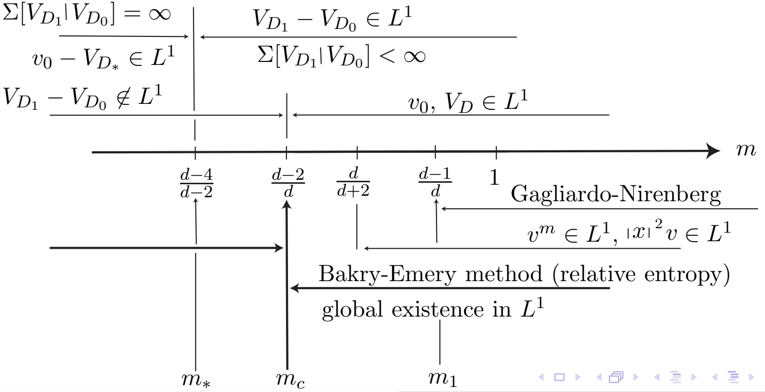
... existence of solutions of $u_t = \Delta u^m$

Removed

Extension to the infinite mass regime, finite time vanishing

- If $m > m_c := \frac{d-2}{d} \le m < m_1$, solutions globally exist in $L^1(\mathbb{R}^d)$ and the Barenblatt self-similar solution has finite mass
- For $m \le m_c$, the Barenblatt self-similar solution has infinite mass

Extension to $m \le m_c$? Work in relative variables !



Entropy methods and linearization: intermediate asymptotics, vanishing

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez]

$$\frac{\partial u}{\partial \tau} = -\nabla \cdot (u \nabla u^{m-1}) = \frac{1-m}{m} \Delta u^m \tag{5}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$

$$R(\tau) := (T + \tau)^{\frac{1}{d\,(m-m_c)}}$$

- $0 < m < m_c$, $T < +\infty$: vanishing in finite time $\lim_{\tau \nearrow T} u(\tau, y) = 0$

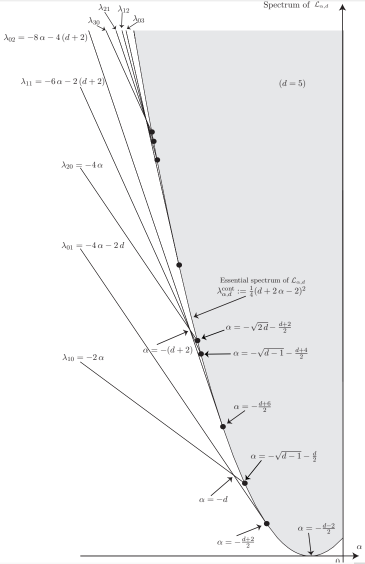
$$R(\tau) := (T - \tau)^{-\frac{1}{d\,(m_c-m)}}$$

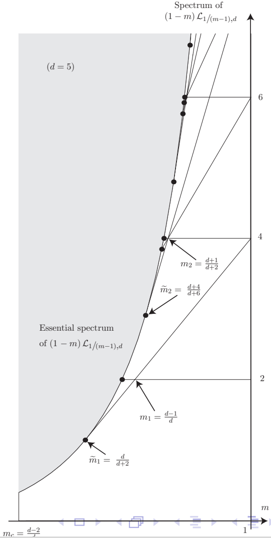
Self-similar *Barenblatt type solutions* exists for any m

$$t := \frac{1-m}{2} \log\left(\frac{R(\tau)}{R(0)}\right) \quad \text{and} \quad x := \sqrt{\frac{1}{2\,d\,|m-m_c|}}\,\frac{y}{R(\tau)}$$

Generalized Barenblatt profiles: $V_D(x) := \left(D + |x|^2\right)^{\frac{1}{m-1}}$

Plots ($d = 5$)


Spectrum of $\mathcal{L}_{\alpha,d}$
λ_{21}
λ_{12}
λ_{30}
λ_{03}
$\lambda_{02} = -8\,\alpha - 4\,(d+2)$
$\lambda_{11} = -6\,\alpha - 2\,(d+2)$
$(d = 5)$
$\lambda_{20} = -4\,\alpha$
$\lambda_{01} = -4\,\alpha - 2\,d$
Essential spectrum of $\mathcal{L}_{\alpha,d}$
$\lambda^{\rm cont}_{\alpha,d} := \frac{1}{4}(d+2\,\alpha-2)^2$
$\alpha = -\sqrt{2\,d} - \frac{d+2}{2}$
$\alpha = -(d+2)$
$\alpha = -\sqrt{d-1} - \frac{d+4}{2}$
$\lambda_{10} = -2\,\alpha$
$\alpha = -\frac{d+6}{2}$
$\alpha = -\sqrt{d-1} - \frac{d}{2}$
$\alpha = -d$
$\alpha = -\frac{d-2}{2}$
$\alpha = -\frac{d+2}{2}$
α
0


Spectrum of
$(1-m)\,\mathcal{L}_{1/(m-1),d}$
$(d = 5)$
6
4
$m_2 = \frac{d+1}{d+2}$
$\widetilde{m}_2 = \frac{d+4}{d+6}$
Essential spectrum
of $(1-m)\,\mathcal{L}_{1/(m-1),d}$
2
$m_1 = \frac{d-1}{d}$
$\widetilde{m}_1 = \frac{d}{d+2}$
m
1
$m_c = \frac{d-2}{d}$

C: Keller-Segel model

1. Small mass results
2. Spectral analysis
3. Collecting estimates: towards exponential convergence

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\[2ex] -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\[2ex] u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| \, u(t,y) \, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \, u(t,y) \, dy$$

Mass conservation: $\displaystyle \frac{d}{dt} \int_{\mathbb{R}^2} u(t,x) \, dx = 0$

Blow-up

$M=\int_{\mathbb{R}^2} n_0\,dx>8\pi$ and $\int_{\mathbb{R}^2}|x|^2\,n_0\,dx<\infty$: blow-up in finite time
a solution u of

$$\frac{\partial u}{\partial t}=\Delta u-\nabla\cdot(u\,\nabla v)$$

satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\,u(t,x)\;dx$$

$$=-\int_{\mathbb{R}^2}2x\,\Delta u\;dx+\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2}\underbrace{\frac{2x\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\;dx\,dy}_{\frac{(x-y)\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\;dx\,dy}$$

$$=4\,M-\frac{M^2}{2\pi}<0\quad\text{if}\quad M>8\pi$$

Existence and free energy

$M=\int_{\mathbb{R}^2} n_0\,dx\le 8\pi$: global existence [Jäger, Luckhaus], [JD, Perthame], [Blanchet, JD, Perthame], [Blanchet, Carrillo, Masmoudi]

If u solves

$$\frac{\partial u}{\partial t}=\nabla\cdot\left[u\,(\nabla\,(\log u)-\nabla v)\right]$$

the free energy

$$F[u]:=\int_{\mathbb{R}^2} u\log u\;dx-\frac{1}{2}\int_{\mathbb{R}^2} u\,v\;dx$$

satisfies

$$\frac{d}{dt}F[u(t,\cdot)]=-\int_{\mathbb{R}^2} u\left|\nabla\,(\log u)-\nabla v\right|^2\,dx$$

Log HLS inequality [Carlen, Loss]: F is bounded if $M\le 8\pi$
Existence: $n_0\in L^1_+(\mathbb{R}^2,(1+|x|^2)\,dx)$, $n_0\log n_0\in L^1(\mathbb{R}^2,dx)$

Time-dependent rescaling

$$u(x,t)=\frac{1}{R^2(t)}\,n\left(\frac{x}{R(t)},\tau(t)\right)\quad\text{and}\quad v(x,t)=c\left(\frac{x}{R(t)},\tau(t)\right)$$

with $R(t)=\sqrt{1+2t}$ and $\tau(t)=\log R(t)$

$$\begin{cases}\dfrac{\partial n}{\partial t}=\Delta n-\nabla\cdot(n\,(\nabla c-x)) & x\in\mathbb{R}^2\,,\ t>0\\ c=-\dfrac{1}{2\pi}\,\log|\cdot|*n & x\in\mathbb{R}^2\,,\ t>0\\ n(\cdot,t=0)=n_0\geq 0 & x\in\mathbb{R}^2\end{cases}$$

[Blanchet, JD, Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty}\|n(\cdot,\cdot+t)-n_\infty\|_{L^1(\mathbb{R}^2)}=0\quad\text{and}\quad\lim_{t\to\infty}\|\nabla c(\cdot,\cdot+t)-\nabla c_\infty\|_{L^2(\mathbb{R}^2)}=0$$

means "intermediate asymptotics" in original variables:

$$\|u(x,t)-\tfrac{1}{R^2(t)}\,n_\infty\left(\tfrac{x}{R(t)},\tau(t)\right)\|_{L^1(\mathbb{R}^2)}\searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M\,\frac{e^{\,c_\infty-|x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty-|x|^2/2}\,dx} = -\Delta c_\infty\;, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- A minimizer of the free energy in self-similar variables
- Radial symmetry [Naito]
- Uniqueness [Biler, Karch, Laurençot, Nadzieja]
- As $|x|\to+\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon\in(0,1)$ [Blanchet, JD, Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M\to 0_+}\|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[Joseph, Lundgreen] [JD, Stańczy]

Parabolic-elliptic case: large time asymptotics

Theorem

[Blanchet, JD, Escobedo, Fernández] Under the above conditions, if $M \le M^* < 8\pi$, there is a unique solution

Moreover, there are two positive constants, C and δ, such that

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \, \frac{dx}{n_\infty} \le C \, e^{-\,\delta\, t} \quad \forall \; t > 0$$

As a function of M, δ is such that $\lim_{M \to 0_+} \delta(M) = 1$

Smallness conditions in the proof:

- Uniform estimate: the *method of the trap*
- *Spectral gap* of a linearized operator $\mathcal{L}$
- Comparison of the (nonlinear) relative entropy with $\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \, \frac{dx}{n_\infty}$

A parametrization of the solutions and the linearized operator

[Campos, JD]

$$-\Delta c = M\,\frac{e^{-\frac{1}{2}\,|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,|x|^2+c}\,dx}$$

Solve

$$-\phi'' - \frac{1}{r}\,\phi' = e^{-\frac{1}{2}\,r^2+\phi}\,, \quad r>0$$

with initial conditions $\phi(0)=a$, $\phi'(0)=0$ and get

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}\,r^2+\phi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}\,r^2+\phi_a(r)}}{2\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}\,r^2+\phi_a}\,dx} = e^{-\frac{1}{2}\,r^2+\phi_a(r)}$$

With $-\Delta\,\varphi_f = n_a\,f$, consider the operator defined by

$$\mathcal{L}\,f := \frac{1}{n_a}\nabla\cdot\left(n_a(\nabla(f-\varphi_f))\right)\,, \quad x\in\mathbb{R}^2$$

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

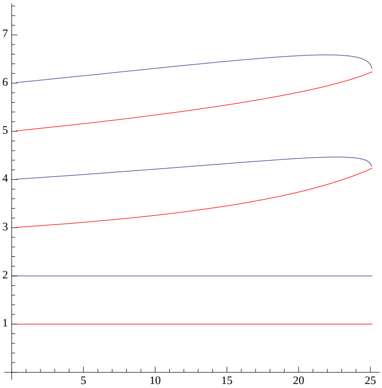


Figure: The lowest eigenvalues of $-\mathcal{L}$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_M$ be the solution of

$$-\Delta\, f_0 = n_M\, f_0$$

and observe that $g_0 = f_0/c_M$ is such that

$$\frac{1}{n_M}\nabla\cdot\big(n_M\nabla(f_0 - c_M\, g_0)\big) =: \mathcal{L}\, f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_M}\,\frac{\partial n_M}{\partial x_1}$ associated with $g_1 = \frac{1}{c_M}\,\frac{\partial c_M}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_1 = f_1$

With $D := x\cdot\nabla$, let$f_2 = 1 + \frac{1}{2}\, D\log n_M = 1 + \frac{1}{2\, n_M}\, D\, n_M$. Then

$$-\Delta\,(D\, c_M) + 2\,\Delta c_M = D\, n_M = 2\,(f_2 - 1)\, n_M$$

and so $g_2 := \frac{1}{c_M}\,(-\Delta)^{-1}(n_M\, f_2)$ is such that $-\mathcal{L}\, f_2 = 2\, f_2$

Functional setting...

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_M}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_M)\, c - c_M)\, dx$$

achieves its minimum for $n = n_M$ according to log HLS and

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2}\, F[n_M(1 + \varepsilon\, f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_M\, dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma

For any $f \in H^1(\mathbb{R}^2, n_M\, dx)$ such that $\int_{\mathbb{R}^2} f\, n_M\, dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(g\, c_M)|^2\, n_M\, dx \leq 2 \int_{\mathbb{R}^2} |f|^2\, n_M\, dx$$

… and eigenvalues

With g such that $-\Delta(g\,c_M) = f\,n_M$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\,f_2\,n_M\,dx - \int_{\mathbb{R}^2} f_1\,n_M\,(g_2\,c_M)\,dx$$

on the orthogonal to f_0 in $L^2(n_M\,dx)$ and with $G_2(x) := -\frac{1}{2\pi}\,\log|x|$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\,c_M)|^2\,n_M\,dx \quad \text{with} \quad g = \frac{1}{c_M}\,G_2 * (f\,n_M)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\,f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma

In this setting, $\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is positive

Concluding remarks

- The spectral gap inequality of $\mathcal{L}$ is a refined version of

Theorem (Onofri type inequality)

For any $M \in (0, 8\pi)$, if $n_M = M \frac{e^{c_M - \frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c_M - \frac{1}{2}|x|^2}\,dx}$ with $c_M = (-\Delta)^{-1} n_M$, $d\mu_M = \frac{1}{M} n_M\,dx$, we have the inequality

$$\log\left(\int_{\mathbb{R}^2} e^{\phi}\,d\mu_M\right) - \int_{\mathbb{R}^2} \phi\,d\mu_M \le \frac{1}{2\,M} \int_{\mathbb{R}^2} |\nabla\phi|^2\,dx \quad \forall\,\phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

- [Campos, JD] Uniform convergence of $n(t,\cdot)$ to n_M can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [Diaz, Nagai, Rakotoson]
- Exponential convergence of the relative entropy should follow [Campos, JD, work in progress]

Thank you for your attention !