Optimal functions and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

Jean Dolbeault

dolbeaul@ceremade.dauphine.fr

CEREMADE

CNRS & Université Paris-Dauphine

http://www.ceremade.dauphine.fr/∼dolbeaul

IN COLLABORATION WITH

J. CAMPOS, M. DEL PINO, M. ESTEBAN, P. FELMER, J. FERNÁNDEZ, S. FILIPPAS, M. LOSS, J. SALOMON, G. TARANTELLO, A. TERTIKAS

OBERWOLFACH

MFO, FEBRUARY 8, 2010

http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/

Outline

- Introduction
- Symmetry results (moving planes): some simple remarks
- Caffarelli-Kohn-Nirenberg inequalities
- Extension and limit cases
- Symmetry breaking in gravitational models

Introduction

Symmetry and symmetry breaking

The symmetry principle of Pierre Curie:

"Effects are at least as symmetric as their causes"

The target was electromagnetism more than a century ago, but soon it was realized that physics is not as simple: with ferromagnetism for instance, it can only be required that even if symmetry is broken after magnetization, all final states are equally likely to occur. These states are usually not radially symmetric. Symmetry breaking is a key concept in QFT.

Mathematically: symmetry in PDEs has been widely used to understand the uniqueness or multiplicity properties of the solutions. Standard scheme goes as follows:

- prove some symmetry properties by symmetrization or comparison techniques of the solutions (ground states) of an (Euler-Lagrange) equation
- prove uniqueness by ODE techniques

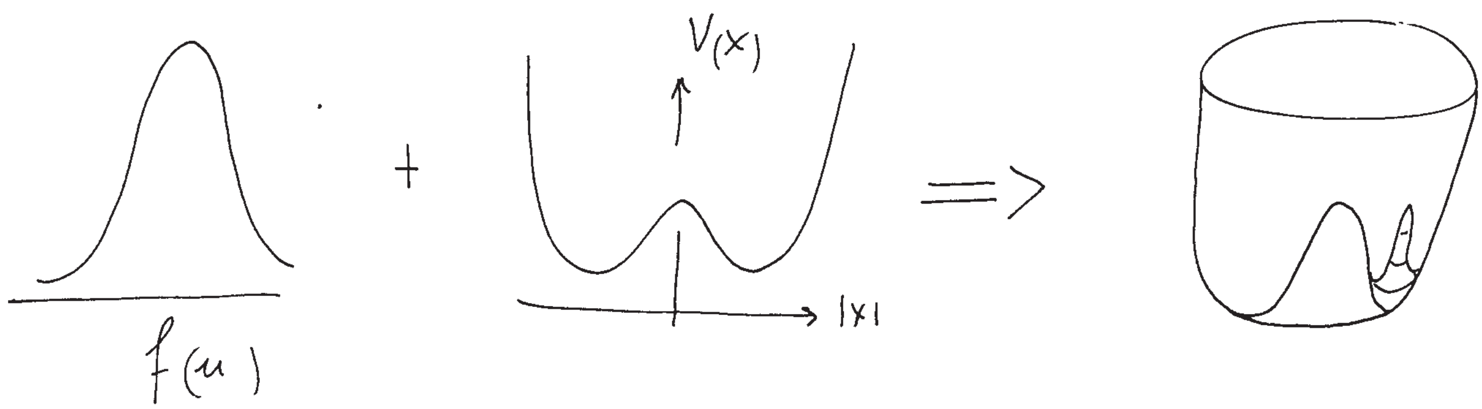
but also: bifurcation analysis, branches of solutions within certain classes of symmetry, direct analysis of the solution set,...

A symmetry breaking mechanism

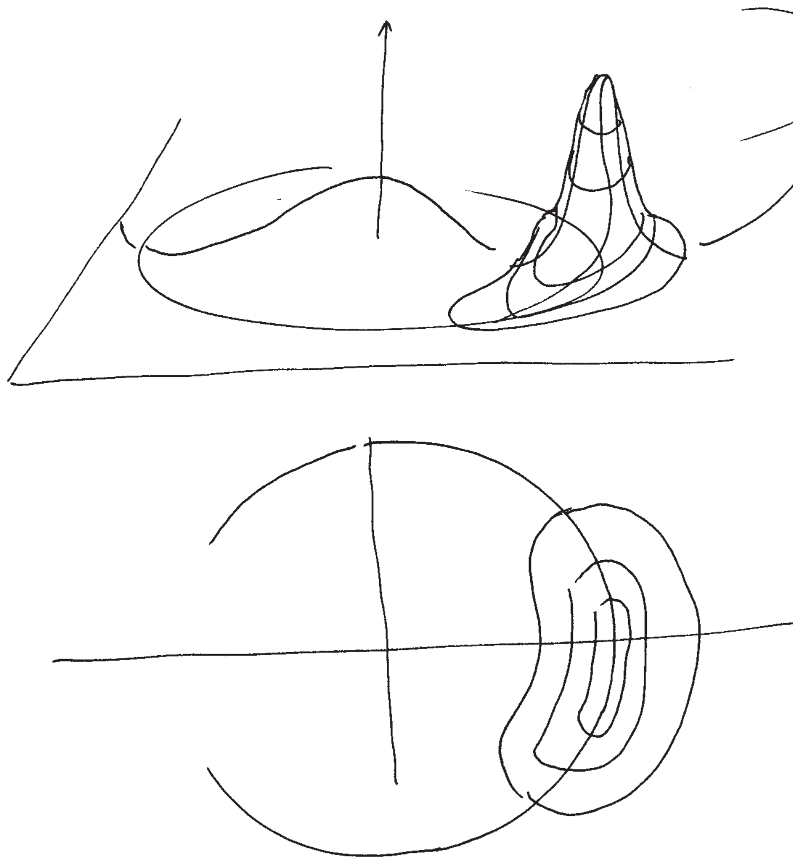
Much less is known concerning symmetry breaking. Known results are based on

- energy considerations + linear analysis
- characterization of some asymptotic regimes

What is the reason for symmetry breaking ? In this talk: the competition of a nonlinearity which tends to aggregate or concentrate the solution and of an (external) potential term which “prefers”

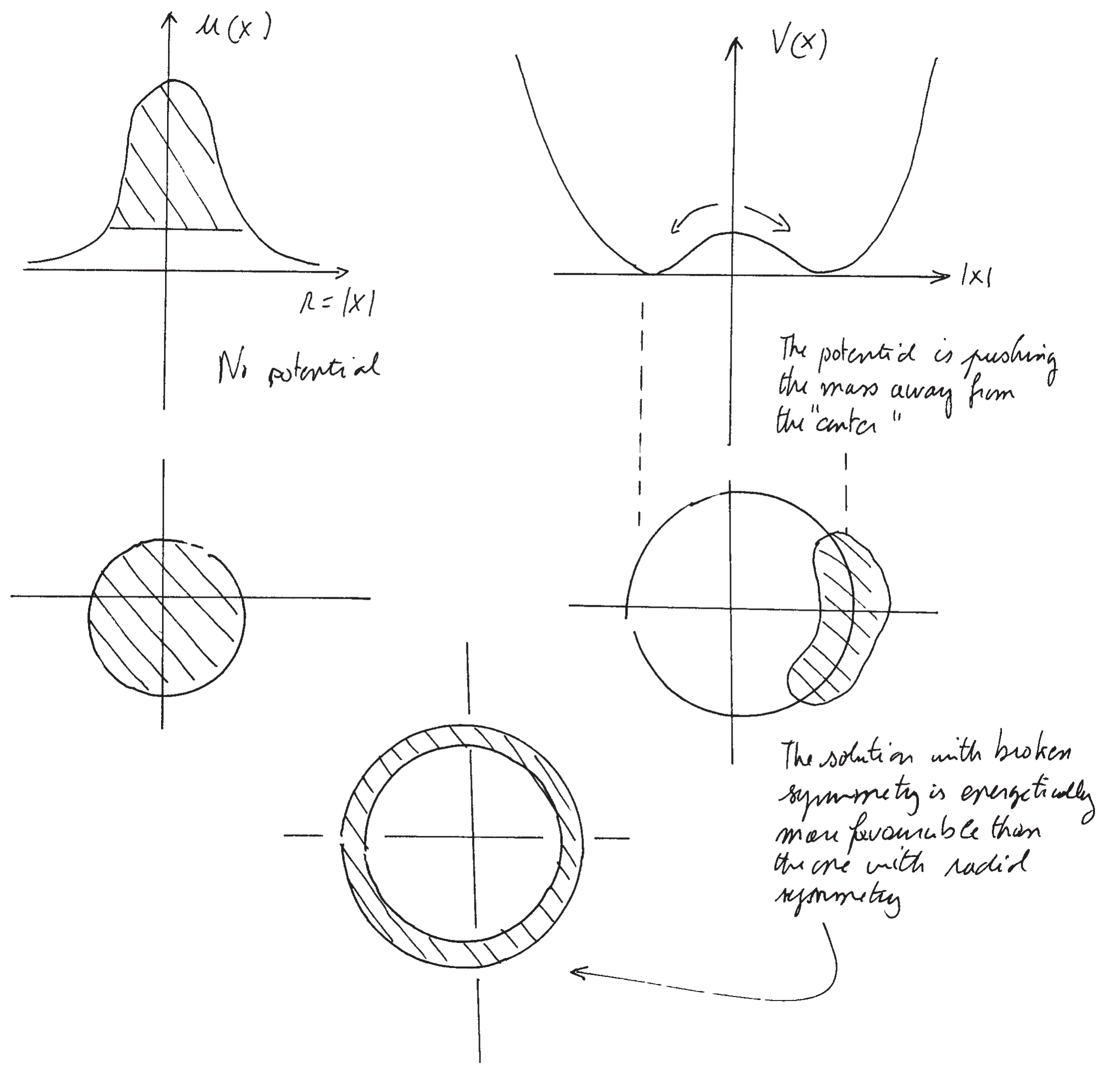


The solution with broken symmetry



Can we understand the transition from a regime of ground states with symmetry to a regime where symmetry is broken ? Can we quantify this phenomenon ?

The energy point of view (ground state)



Symmetry results (moving planes)

Some simple remarks

The theorem of Gidas, Ni and Nirenberg - extensions

Theorem 1. *[Gidas, Ni and Nirenberg, 1979 and 1980] Let $u \in C^2(B)$, $B = B(0,1) \subset \mathbb{R}^N$, be a solution of*

$$\Delta u + f(u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that f is Lipschitz. If u is positive, then it is radially symmetric and decreasing along any radius: $u'(r) < 0$ for any $r \in (0,1]$

Extension: $\Delta u + f(r,u) = 0$, $r = |x|$ if $\frac{\partial f}{\partial r} \leq 0$... a “cooperative" case

Theorem 2. *[JD, Felmer, 1999] Consider solutions of*

$$\Delta u + \lambda\, f(r,u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ (no assumption on the sign of $\frac{\partial f}{\partial r}$). There exists λ_1, λ_2 with $0 < \lambda_1 \leq \lambda_2$ such that

- *(i) if $\lambda \in (0,\lambda_1)$, then $\frac{d}{dr}(u - \lambda\, u_0) < 0$ where u_0 is the solution of $\Delta u_0 + \lambda f(r,0) = 0$.*
- *(ii) if $\lambda \in (0,\lambda_2)$, then u is radially symmetric*

Proof: spectral issues

The counter-example of Gidas, Ni and Nirenberg if $\frac{\partial f}{\partial r} \geq 0$ is based on eigenfunctions and eigenvalues

A sketch of the proof of (ii): $\bar{x} = (-x_1, x')$, $|x'| = |x|$, $\bar{u}(x) = u(\bar{x})$

$$\Delta \bar{u} + \lambda\, f(r, \bar{u}) = 0$$

$v = \bar{u} - u$, $c = (f(r, \bar{u}) - f(r, u))/(u - \bar{u})$

$$\Delta v + \lambda\, c\, v = 0$$

$\lambda_1 = \sup\{\lambda > 0 \, : \, \Delta v + \lambda\, c\, v = 0 \implies v = 0\}$
$\lambda_2 = \sup\{\lambda > 0 \, : \, \Delta v + \lambda\, c\, v = 0$ and v changes sign $\implies v = 0\}$

If $\lambda < \lambda_2$,

- either $\lambda = \lambda_1$ and v is nonnegative... but $v(\bar{x}) = u(x) - u(\bar{x}) = -v(x)$ and so $v \equiv 0$: $u = \bar{u}$
- or $\lambda \neq \lambda_1$: $v \equiv 0$, same conclusion $\square$

Caffarelli-Kohn-Nirenberg inequalities

Joint work(s) with M. Esteban, M. Loss and G. Tarantello

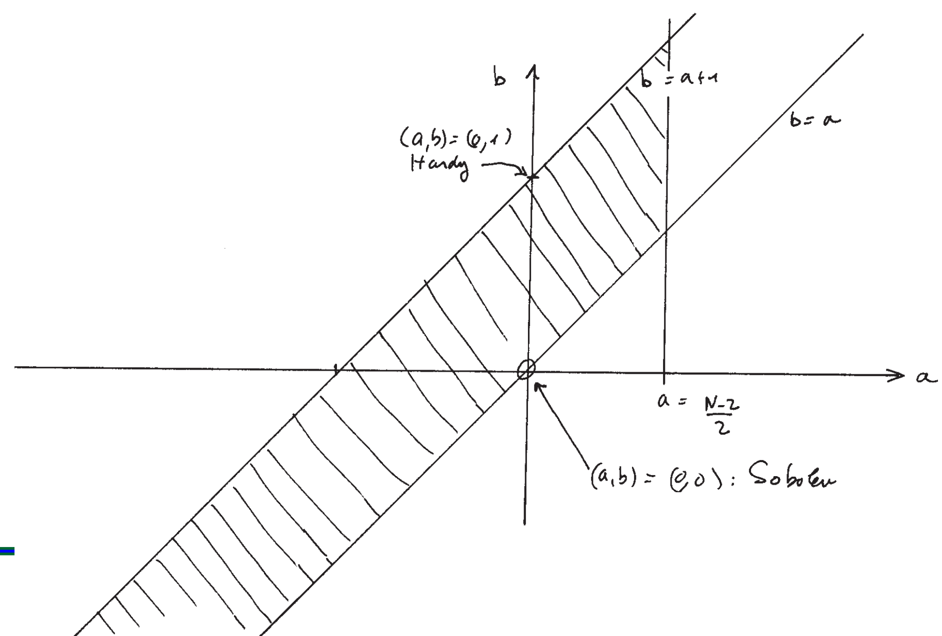
Caffarelli-Kohn-Nirenberg (CKN) inequalities

$$\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le C_{a,b}^N \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\, u \in \mathcal{D}_{a,b}$$

with $a \le b \le a+1$ if $N \ge 3$, $a < b \le a+1$ if $N = 2$, and $a \neq \frac{N-2}{2}$

$$p = \frac{2\,N}{N-2+2\,(b-a)}$$

$$\mathcal{D}_{a,b} := \left\{\,|x|^{-b}\,u \in L^p(\mathbb{R}^N, dx)\ :\ |x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^N, dx)\right\}$$



The symmetry issue

$$\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le C^N_{a,b} \int_{\mathbb{R}^N} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\; u \in \mathcal{D}_{a,b}$$

$C_{a,b}$ = best constant for general functions u
$C^*_{a,b}$ = best constant for radially symmetric functions u

$$C^*_{a,b} \le C_{a,b}$$

Up to scalar multiplication and dilation, the optimal radial function is

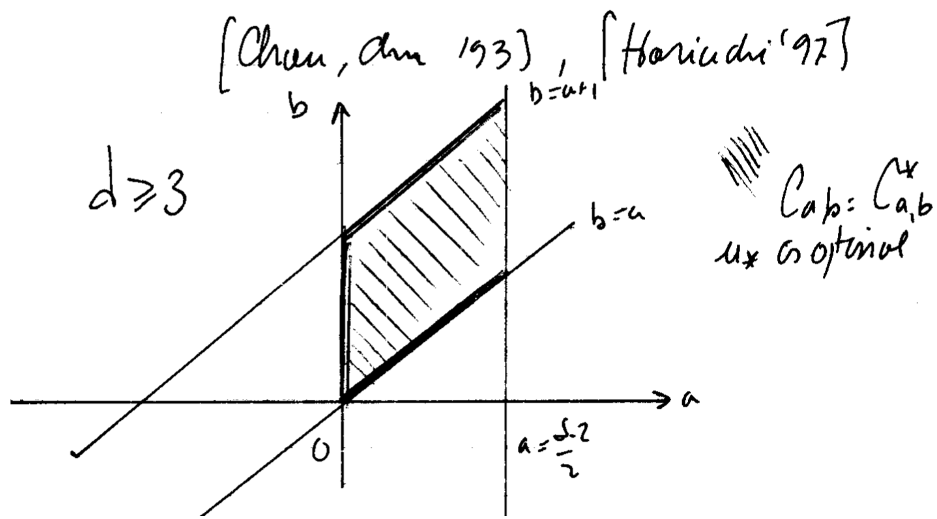
$$u^*_{a,b}(x) = \left(1 + |x|^{-\frac{2a\,(1+a-b)}{b-a}}\right)^{-\frac{b-a}{1+a-b}}$$

Questions: is optimality (equality) achieved ? do we have $u_{a,b} = u^*_{a,b}$?

Positive answers

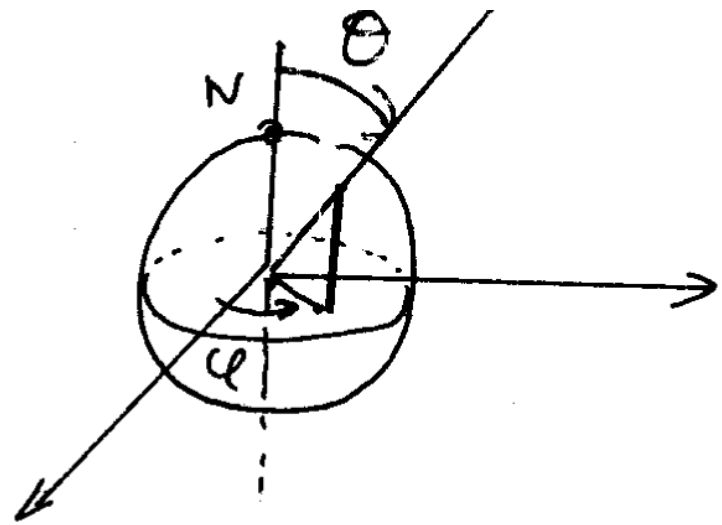
[Aubin, Talenti, Lieb, Chou-Chu, Lions, Catrina-Wang, ...]

- Extremals exist for $a < b < a+1$ and for $0 \leq a \leq \frac{N-2}{2}$, $a \leq b < a+1$, $N \geq 2$
- Optimal constants are never achieved for $b = a < 0$, $N \geq 3$ and for $b = a+1$, $N \geq 2$
- If $N \geq 3$, $0 \leq a < \frac{N-2}{2}$ and $a \leq b < a+1$, the extremal functions are radially symmetric ... $u(x) = |x|^a \, v(x)$ + Schwarz symmetrization



More results on symmetry

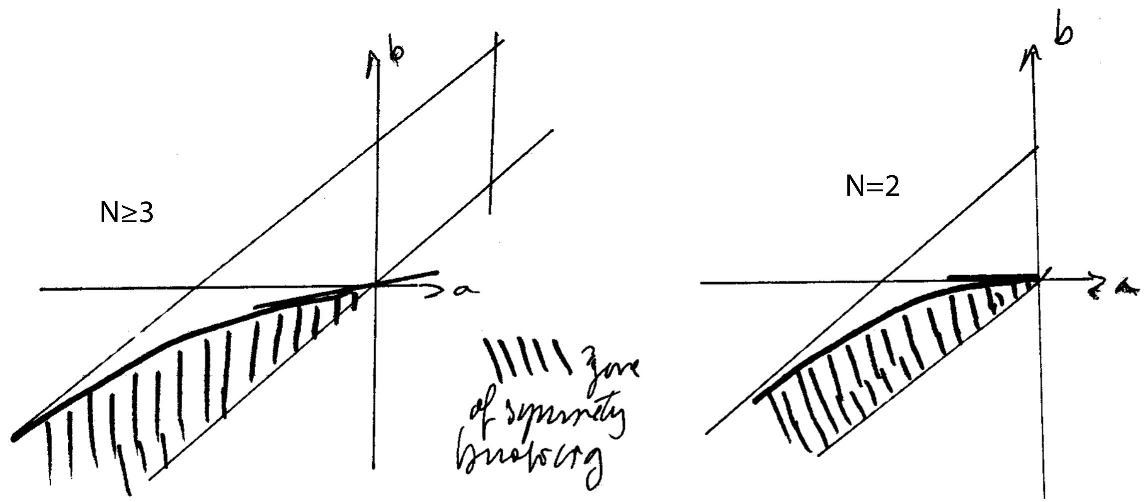
- Radial symmetry has also been established for $N \geq 3$, $a < 0$, $|a|$ small and $0 < b < a + 1$: [Lin-Wang, Smets-Willem]
- Schwarz foliated symmetry [Smets-Willem]



$N = 3$: optimality is achieved among solutions which depend only on the “latitude" θ and on r. Similar results hold in higher dimensions

Symmetry breaking

- [Catrina-Wang, Felli-Schneider] if $a<0$, $a\le b<b^{FS}(a)$, the extremal functions ARE NOT radially symmetric !



Zones of symmetry breaking in dark grey. Left: $N\ge 3$*. Right:* $N=2$

$$b^{FS}(a)=\frac{N(N-2-2a)}{2\sqrt{(N-2-2a)^2+4(N-1)}}-\frac{1}{2}\,(N-2-2a)$$

- [Catrina-Wang] As $a\to-\infty$, optimal functions look like some decentered optimal functions for some Gagliardo-Nirenberg interpolation inequalities (after some appropriate transformation)

Main result 1 [J.D.-Esteban-Tarantello]

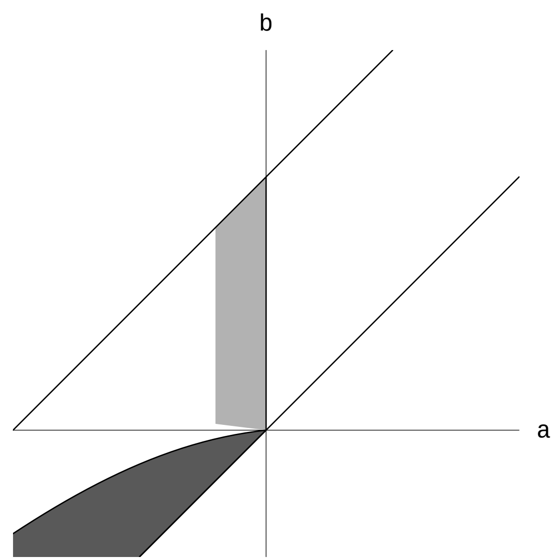
- For $N = 2$, radial symmetry can be proved when

$$-\eta < a < 0 \quad \text{and} \quad -\varepsilon(\eta)\, a \le b < a + 1$$

Theorem 3. *For all $\varepsilon > 0$ there exists $\eta > 0$ s.t. for $a < 0$, $|a| < \eta$ and*

(i) if $|a| > \frac{2}{p-\varepsilon}\,(1 + |a|^2)$, then $C_{a,b} > C^*_{a,b}$ (symmetry breaking)

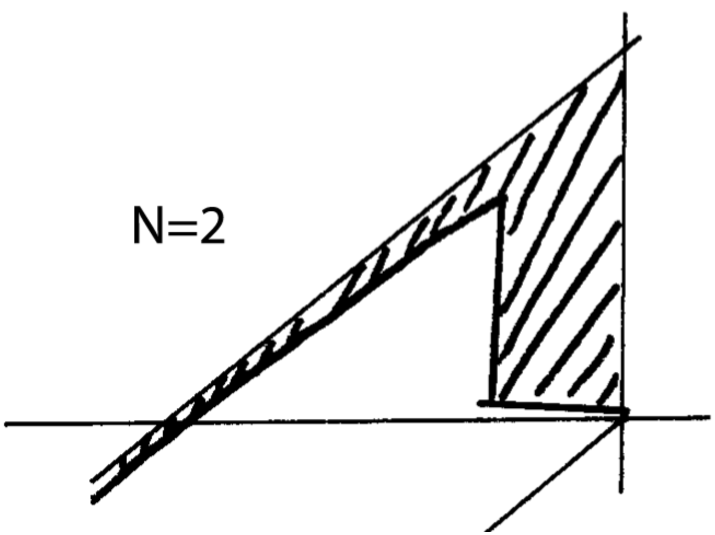
(ii) if $|a| < \frac{2}{p+\varepsilon}\,(1 + |a|^2)$, then $C_{a,b} = C^*_{a,b}$ and $u_{a,b} = u^*_{a,b}$

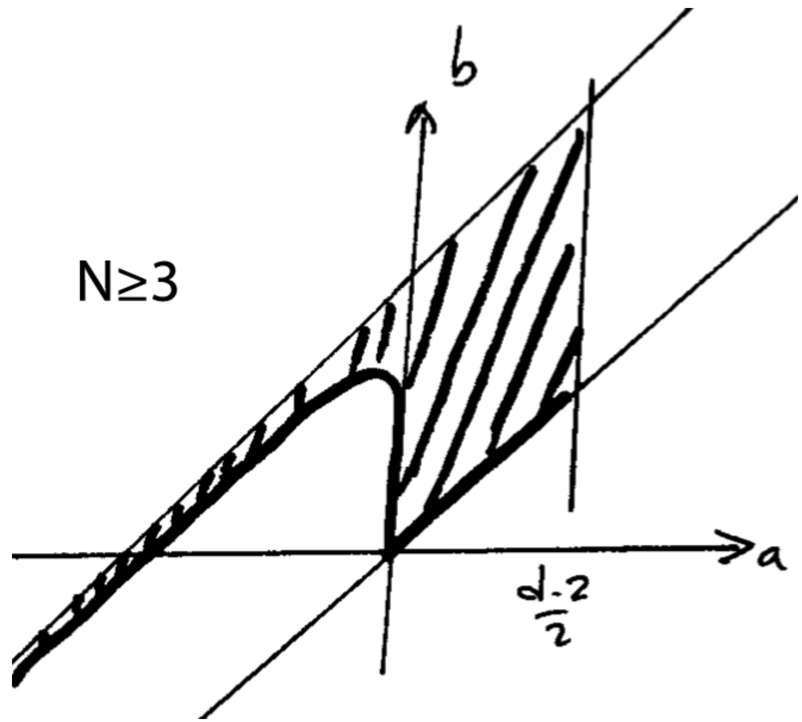


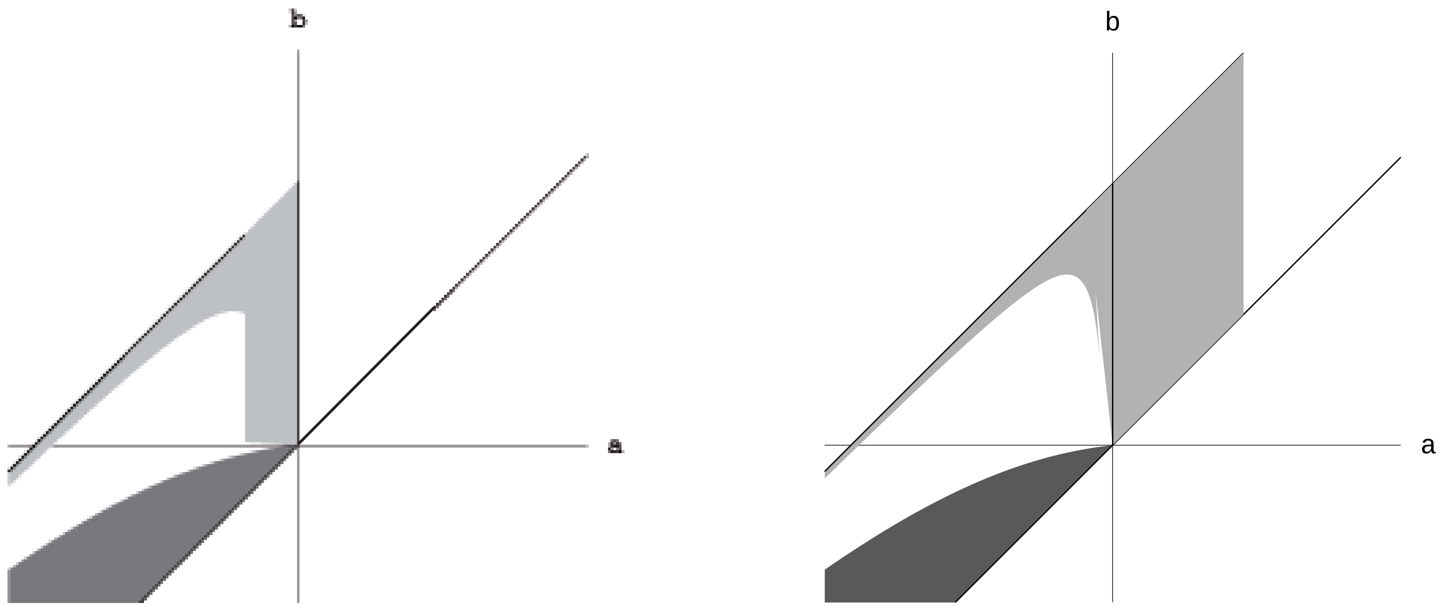
Main result 2 [J.D.-Esteban-Loss-Tarantello]

- For $N \geq 2$, radial symmetry can be proved when b is close to $a+1$

Theorem 4. *Let $N \geq 2$. For every $A < 0$, there exists $\varepsilon > 0$ such that the extremals are radially symmetric if $a+1-\varepsilon < b < a+1$ and $a \in (A,0)$. So they are given by $u^*_{a,b}$, up to a scalar multiplication and a dilation*







Zones of symmetry breaking in dark grey. Left: $N = 2$. *Right:* $N \geq 3$

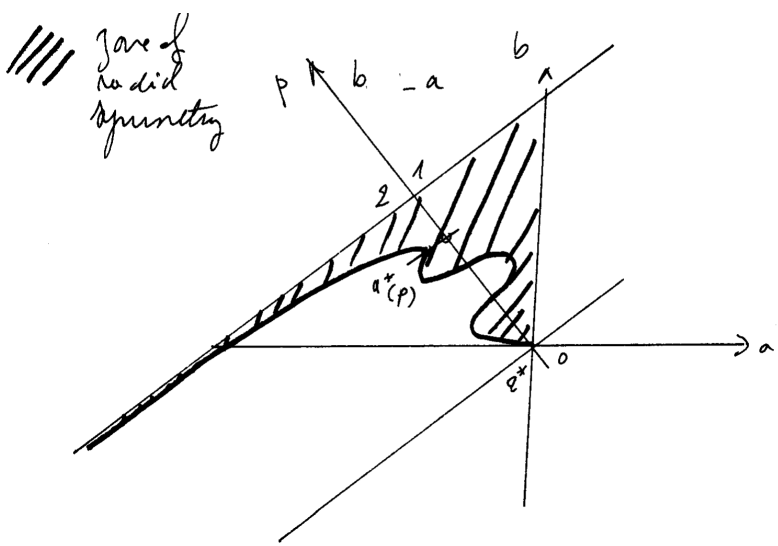
Main result 3 [J.D.-Esteban-Loss-Tarantello]

- The symmetry and the symmetry breaking zones are simply connected and separated by a continuous curve

Theorem 5. *For all* $N \geq 2$*, there exists a continuous function* $a^*\colon (2, 2^*) \longrightarrow (-\infty, 0)$ *such that* $\lim_{p\to 2^*_-} a^*(p) = 0$*,* $\lim_{p\to 2_+} a^*(p) = -\infty$ *and*

(i) If $(a,p) \in \left(a^*(p), \frac{N-2}{2}\right) \times (2, 2^*)$*, all extremals radially symmetric*

(ii) If $(a,p) \in (-\infty, a^*(p)) \times (2, 2^*)$*, none of the extremals is radially symmetric*



Conjecture. The curves obtained by Felli-Schneider and ours coincide

Emden-Fowler transformation and the cylinder $\mathcal{C} = \mathbb{R} \times S^{N-1}$

$$t = \log|x| \;, \quad \theta = \frac{x}{|x|} \in S^{N-1} \,, \quad w(t,\theta) = |x|^{-a}\, v(x) \;, \quad \Lambda = \frac{1}{4}\,(N-2-2a)^2$$

$$\|w\|^2_{L^p(\mathcal{C})} \le C_{\Lambda,p} \left[\|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\,\|w\|^2_{L^2(\mathcal{C})}\right]$$

$$\mathcal{E}_\Lambda[w] := \|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\,\|w\|^2_{L^2(\mathcal{C})}$$

$$C_{\Lambda,p}^{-1} := C_{a,b}^{-1} = \inf\left\{\mathcal{E}_\Lambda(w) \;:\; \|w\|^2_{L^p(\mathcal{C})} = 1\right\}$$

$$a < 0 \implies \Lambda > \frac{(N-2)^2}{4}$$

$$b - a \to 0 \iff p \to \frac{2N}{N-2}$$

$$b - (a+1) \to 0 \Leftrightarrow p \to 2_+$$

Symmetry breaking

Strategy of [Catrina-Wang, Felli-Schneider]

Expand $\mathcal{E}_\Lambda[w]$ around $w^*_{\Lambda,p}$ with w an appropriate orthogonal space to $w_{\Lambda,p}$. This amounts to study the spectrum of

$$-\Delta + \Lambda - (p-1)\,|w^*_{\Lambda,p}|^{p-2}$$

in $H^1(\mathcal{C})$, make en expansion in spherical harmonics and compute the lowest eigenvalue associated to the first non-constant spherical harmonic function

Alternative proof in dimension $N = 2$ close to $(a, b) = (0, 0)$: [J.D.-Esteban-Loss-Tarantello]

Auxiliary results for symmetry proofs

Multiplication by constants does not affect optimality (no more scaling invariance in $\mathcal{C}$): we normalize so that the optimal functions solve $-\Delta w + \Lambda w = w^{p-1}$, that is

$$\int_{\mathcal{C}} |\nabla w|^2 \, dx + \Lambda \int_{\mathcal{C}} |w|^2 \, dx = \int_{\mathcal{C}} |w|^p \, dx$$

With $1/C_{\Lambda,p} = \mathcal{E}[w]/\|w\|^2_{L^p(\mathcal{C})}$, this determines $\|w\|_{L^p(\mathcal{C})}$

Lemma 6. *Let $N \geq 2$, $p \in (2, 2^*)$. For any $\Lambda \neq 0$, we have*

$$\left(\mathsf{C}^N_{\Lambda,p}\right)^{-\frac{p}{p-2}} = \|w_{\Lambda,p}\|^p_{L^p(\mathcal{C})} \leq \|w^*_{\Lambda,p}\|^p_{L^p(\mathcal{C})} = 4\,|S^{N-1}|\,(2\,\Lambda\,p)^{\frac{p}{p-2}}\;\frac{c_p}{2\,p\,\sqrt{\Lambda}}$$

where $p \mapsto c_p$ is increasing and $\lim_{p\to 2_+} 2^{\frac{2\,p}{p-2}}\,\sqrt{p-2}\,c_p = \sqrt{2\pi}$

The extremals can be chosen to satisfy: $w_{\Lambda,p}$ depends only on r and the azimuthal angle θ, $\max_{\mathcal{C}} w_{\Lambda,p} = w_{\Lambda,p}(0,\theta_0)$ for some $\theta_0 \in S^{N-1}$ and $\partial_t w_{\Lambda,p} < 0$ for any $t > 0$

“Hardy” regime (b close to $a+1$, $N \geq 2$)

Proof of Result # 2: Let $(\Lambda_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ be such that

$$\lim_{n\to+\infty} \Lambda_n = \Lambda \geq (N-2)^2/4 \quad \text{and} \quad \lim_{n\to+\infty} p_n = 2_+$$

such that the corresponding global minimizer $w_n := w_{\Lambda_n,\,p_n}$ satisfies

$$\mathcal{F}_{\Lambda,p}[w_{\Lambda_n,\,p_n}] < \mathcal{F}_{\Lambda,p}[w^*_{\Lambda_n,\,p_n}] \quad \text{and} \quad -\Delta_y w_n + \Lambda_n\, w_n = w_n^{p-1} \quad \text{in} \quad \mathcal{C}$$

Define $c_n^2 := (\Lambda_n\, p_n)^{-\frac{p_n}{p_n-2}}\, 2^{\frac{p_n}{p_n-2}} \sqrt{p_n - 2}$ and $W_n := c_n\, w_n$. We have $\lim_{n\to+\infty} c_n^{2-p_n} = \Lambda$ and

$$\limsup_{n\to+\infty} \int_{\mathcal{C}} |\nabla W_n|^2\, dy + \Lambda_n \int_{\mathcal{C}} W_n^2\, dy = \limsup_{n\to+\infty} c_n^2 \int_{\mathcal{C}} w_n^{p_n}\, dy \leq |S^{N-1}|\, \sqrt{2\,\pi/\Lambda}$$

so that $(W_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathcal{C})$. By elliptic estimates, $W_n \to W$ and $-\Delta W + \Lambda W = \Lambda W \implies W \equiv 0$

“Hardy” regime (continued)

Let $\chi_n := \nabla_\theta w_n := \sin\theta^{2-d}\,\frac{\partial}{\partial\theta}\big(\sin\theta^{d-2}\,w_n\big)$. By differentiating $-\Delta W_n + \Lambda_n\,W_n = c_n^{2-p_n}\,W_n^{p_n-1}$ with respect to θ, we get

$$-\Delta\chi_n + \Lambda_n\,\chi_n = (p_n-1)\,c_n^{2-p_n}\,W_n^{p_n-2}\,\chi_n$$

$$0 = \int_{\mathcal C} |\nabla\chi_n|^2\,dy + \Lambda_n \int_{\mathcal C} |\chi_n|^2\,dy - (p_n-1)\,c_n^{2-p_n}\int_{\mathcal C} W_n^{p_n-2}\,|\chi_n|^2\,dy$$

Since $\int_{S^{N-1}} \chi_n\,d\theta = 0$

$$\int_{\mathcal C} |\nabla\chi_n|^2\,dy \geq (N-1)\int_{\mathcal C} |\chi_n|^2\,dy$$

by the Poincaré inequality. But W_n is bounded by $W_n(0,\theta_0)$, we get

$$0 \geq \Big(\underbrace{N-1+\Lambda_n-(p_n-1)\,c_n^{2-p_n}\,W_n(0,\theta_0)^{p_n-2}}_{\to 0 \text{ as } n\to\infty}\Big)\int_{\mathcal C} |\chi_n|^2\,dy$$

This proves that $\chi_n \equiv 0$ for n large enough $\square$

Symmetry and symmetry breaking regions are separated by

Proof of Main result # 3 ($N \geq 2$): let $w_\sigma(t,\theta) := w(\sigma\, t, \theta)$ for any $\sigma > 0$

$$\mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma) = \sigma^{1+2/p}\, \mathcal{F}_{\Lambda,p}(w) - \sigma^{-1+2/p}\,(\sigma^2 - 1)\, \frac{\int_{\mathcal{C}} |\nabla_\theta w|^2\, dy}{\left(\int_{\mathcal{C}} |w|^p\, dy\right)^{2/p}}$$

Lemma 7. *If $N \geq 2$, $\Lambda > 0$ and $p \in (2, 2^*)$*

(i) If $\mathsf{C}^N_{\Lambda,p} = \mathsf{C}^{N,*}_{\Lambda,p}$, then $C^N_{\lambda,p} = \mathsf{C}^{N,*}_{\lambda,p}$ and $w_{\lambda,p} = w^*_{\lambda,p}$, for any $\lambda \in (0, \Lambda)$

(ii) If there is a non radially symmetric extremal $w_{\Lambda,p}$, then $\mathsf{C}^N_{\lambda,p} > \mathsf{C}^{N,*}_{\lambda,p}$ for all $\lambda > \Lambda$

Symmetry and symmetry breaking regions are separated by

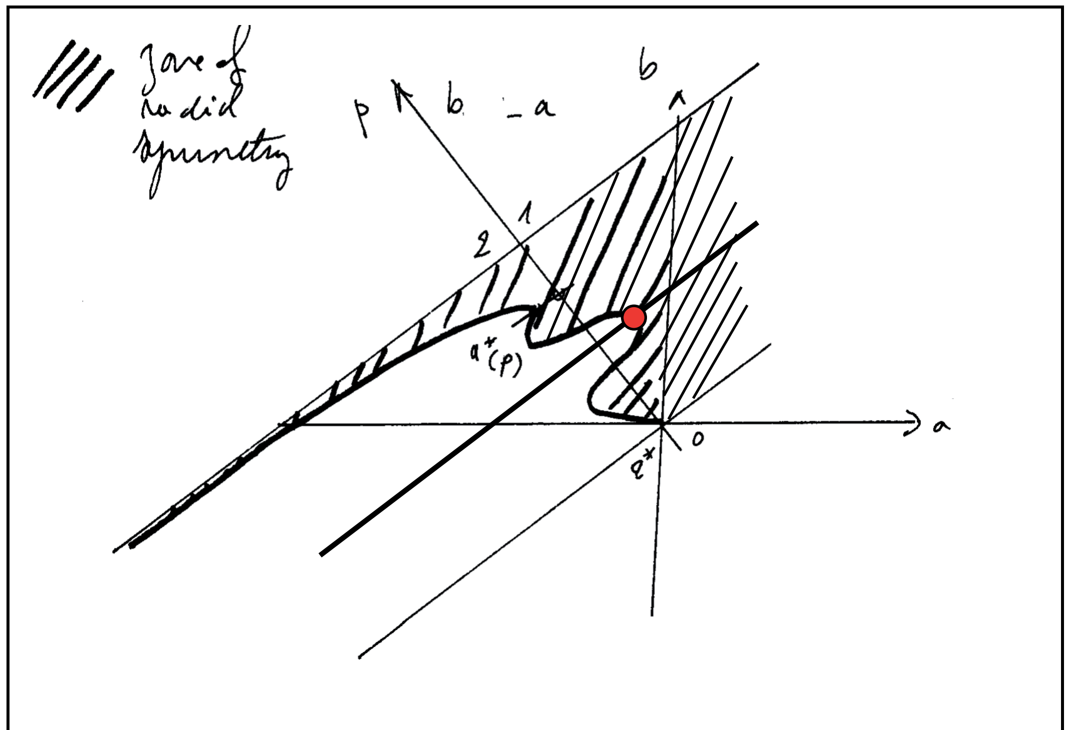
Corollary 8. *Let* $N \geq 2$*. For all* $p \in (2, 2^*)$*,* $\Lambda^*(p) \in (0, \Lambda^{\mathrm{FS}}(p)]$ *and*

(i) *If* $\lambda \in (0, \Lambda^*(p))$*, then* $w_{\lambda,p} = w^*_{\lambda,p}$ *and clearly,* $C^N_{\lambda,p} = C^{N,*}_{\lambda,p}$

(ii) *If* $\lambda = \Lambda^*(p)$*, then* $C^N_{\lambda,p} = C^{N,*}_{\lambda,p}$

(iii) *If* $\lambda > \Lambda^*(p)$*, then* $C^N_{\lambda,p} > C^{N,*}_{\lambda,p}$

Upper semicontinuity
is easy to prove
For continuity,
a delicate spectral
analysis is needed



Extensions and limit cases

- The case of dimension 2
- Other Caffarelli-Kohn-Nirenberg inequalities
- Logarithmic Hardy inequalities

Proof of Result # 1 in the case of dimension 2

Joint work with M. Esteban and G. Tarantello

The case of dimension 2 (1/3)

Assume that there exists $\varepsilon_0 \in (0,1)$ and, for all $n \in \mathbb{N}$, $a_n > 0$, $p_n > 2$, such that $\lim_{n\to+\infty} a_n = 0$, $a_n\, p_n < 2 - \varepsilon_0$ and $\mathcal{E}(w_n) < \mathcal{E}(w_n^*)$ with $w_n = w_{a_n,\,p_n}$ and $w_n^* = w^*_{a_n,\,p_n}$. We can assume

$$w_n(t,\theta) = w_n(-t,\theta)\,, \quad \frac{\partial w_n}{\partial t}(t,\theta) < 0 \quad \forall\, t > 0 \quad \text{and} \quad w_n(0,0) = \max_{\mathcal{C}} w_n$$

$$\begin{cases} -(\partial_t^2 w_n + \partial_\theta^2 w_n) + a^2\, w_n = w_n^{p-1} \quad \text{in} \quad \mathbb{R}\times[-\pi,\pi] \\ w_n > 0\,, \quad w_n(t,\cdot) \quad \text{is } 2\pi\text{-periodic} \quad \forall\, t \in \mathbb{R} \end{cases}$$

As $p_n \to +\infty$, for a subsequence, we have

$$\lim_{n\to+\infty} w_n(0,0) = 1\,, \ \lim_{n\to+\infty} \big[w_n(0,0)\big]^{p_n} = 0\,,$$

$$\lim_{n\to+\infty} p_n\, \big[w_n(0,0)\big]^{p_n-2} = \mu \in [1,+\infty)$$

The case of dimension 2 (2/3)

Define $V_n(t,\theta) = p_n\left(\frac{w_n(t,\theta)}{w_n(0,0)} - 1\right)$, $\alpha = -1 + \lim_{n\to\infty}(1-\varepsilon_n)\,a_n/\varepsilon_n$

$$-\Delta V_n = p_n\,(w_n(0,0))^{p_n-2}\left(1+\frac{V_n}{p_n}\right)^{p_n-1} - a_n^2\,p_n\left(1+\frac{V_n}{p_n}\right) \quad \text{in} \quad \mathcal{C}$$

$$V_n \le 0 = V_n(0,0)\ , \quad V_n(t,\cdot) \quad \text{is } 2\pi\text{-periodic}$$

$$p_n\,(w_n(0,0))^{p_n}\int_{\mathcal{C}}\left(1+\frac{V_n}{p_n}\right)^{p_n} dx = p_n\int_{\mathcal{C}}|w_n|^{p_n}\,dx$$

$$\le p_n\int_{\mathcal{C}}|w_n^*|^{p_n}\,dx = 8\pi\,(1+\alpha) < 8\pi$$

Elliptic estimates and Harnack's inequality imply that $V_n \to V$ locally and

$$-\Delta V = \mu\,e^V \quad \text{in} \quad \mathcal{C}\ , \quad \mu\int_{\mathcal{C}} e^V\,dx \le 8\pi\,(1+\alpha)$$

$$\max_{\mathcal{C}} V \le 0 = V(0,0)\ , \quad V(t,\cdot) \quad \text{is } 2\pi\text{-periodic} \quad \forall\, t\in\mathbb{R}$$

The case of dimension 2 (3/3)

Known results on Liouville's equation $-\Delta V = \mu\, e^V$ and $\mu \int_{\mathcal{C}} e^V\, dx \;\le\; 8\pi\,(1+\alpha)$ show that

$$\mu = 2\,(\alpha+1)^2 \quad \text{and} \quad V(t) = -2\log\left[\cosh((\alpha+1)\,t)\right]$$

With $\chi_n := \partial_\theta w_n$ such that $-\Delta\chi_n + a_n^2\,\chi_n = (p_n-1)\,\big(w_n(t,\theta)\big)^{p_n-2}\,\chi_n$

$$\|\nabla\chi_n\|_{L^2}^2 + a_n^2\,\|\chi_n\|_{L^2}^2 = (p_n-1)\int_{\mathcal{C}}\left(\tfrac{w_n(t,\theta)}{w_n(0,0)}\right)^{p_n-2}\chi_n^2\,dx \sim (p_n-1)\int_{\mathcal{C}} \sim e^V\,\chi_n^2\,dx$$

$$\begin{aligned} 0 = \;& \|\nabla\chi_n\|_2^2 + a_n^2\,\|\chi_n\|_{L^2}^2 - (p_n-1)\textstyle\int_{\mathcal{C}}\big(w_n(t,\theta)\big)^{p_n-2}\chi_n^2\,dx \\ \ge\;& \Big[1 + \underbrace{a_n^2}_{\to 0} - \underbrace{(\alpha+1)^2}_{<1} - \underbrace{(p_n-1)\big(w_n(0,0)\big)^{p_n-2}}_{\to\mu=2\,(\alpha+1)^2}\,\underbrace{r_n}_{\to 0}\Big]\,\|\chi_n\|_{L^2(\mathcal{C})}^2 \\ &+ \Big[2\,(\alpha+1)^2 - \underbrace{(p_n-1)\,(w_n(0,0))^{p_n-2}}_{\to\mu=2\,(\alpha+1)^2}\Big]\int_{\mathcal{C}}\frac{\chi_n^2}{\big(\cosh((\alpha+1)\,t)\big)^2}\,dx \end{aligned}$$

Generalized Caffarelli-Kohn-Nirenberg inequalities

Joint work with M. del Pino, S. Filippas and A. Tertikas

Other Caffarelli-Kohn-Nirenberg inequalities

Let $2^* = \infty$ if $N = 1$ or $N = 2$, $2^* = 2N/(N-2)$ if $N \geq 3$ and define

$$\vartheta(p, N) := \frac{N\,(p-2)}{2\,p}$$

Theorem 9. *[Caffarelli-Kohn-Nirenberg-84] Let $N \geq 1$. For any $\theta \in [\vartheta(p,N), 1]$, there exists a positive constant $\mathsf{C}(\theta, p, a)$ such that*

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta, p, a) \left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

Define

$$\Theta(a, p, d) := \frac{p-2}{32\,(d-1)\,p}\;\left[(p+2)^2\,(d^2 + 4\,a^2 - 4\,a\,(d-2)) - 4\,p\,(p+4)\,(d-1)\right]$$

$$a_-(p) := \frac{d-2}{2} - \frac{2\,(d-1)}{p+2}$$

Symmetry breaking for generalized CKN inequalities

Theorem 10. *Let $d \geq 2$, $2 < p < 2^*$ and $a < a_-(p)$. Then $\mathsf{C}(\theta, p, a) > \mathsf{C}^*(\theta, p, a)$ if either*

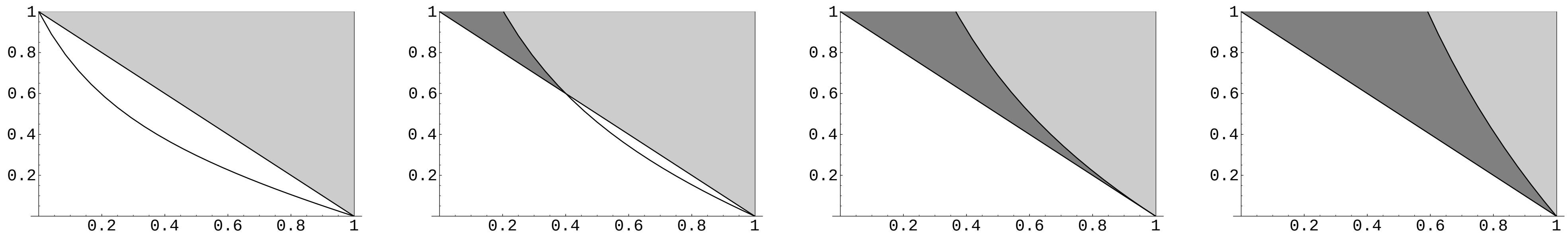
$$\vartheta(p,d) \leq \theta < \Theta(a,p,d) \quad \textit{when} \quad a \geq \frac{d-2}{2} - \frac{2\sqrt{d-1}}{\sqrt{(p-2)(p+2)}}$$

or

$$\vartheta(p,d) \leq \theta \leq 1 \quad \textit{when} \quad a < \frac{d-2}{2} - \frac{2\sqrt{d-1}}{\sqrt{(p-2)(p+2)}}$$

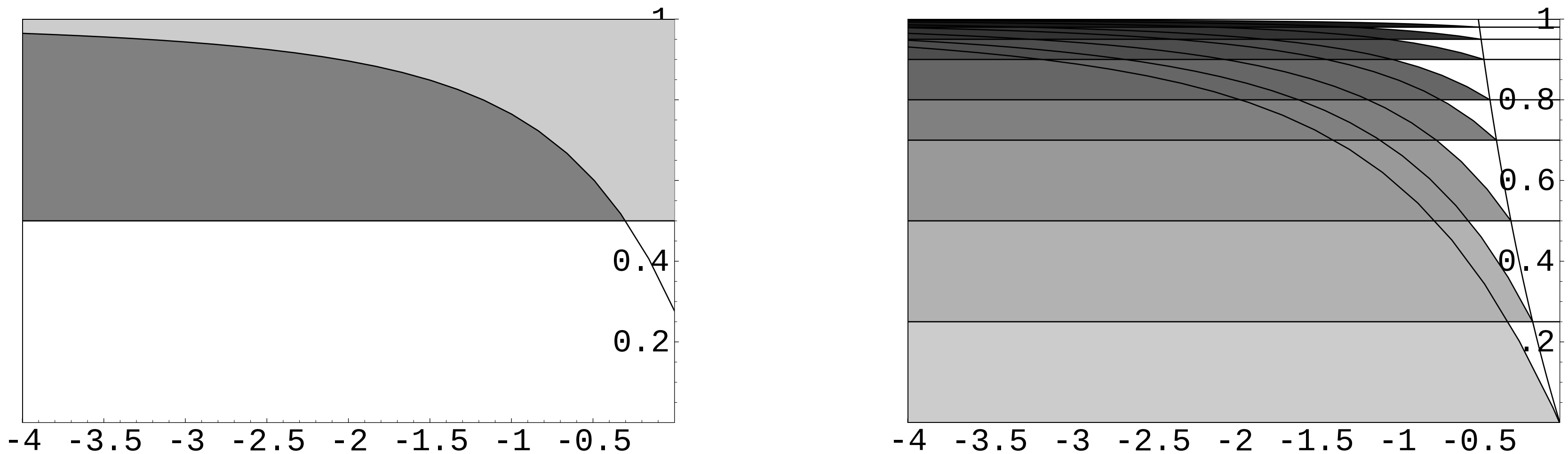
In other words, symmetry breaking occurs if a, θ and p are in any of the two above regions. Moreover, if $a < -1/2$, there exists $\varepsilon > 0$, $\gamma_1 > d/4$ and $\gamma_2 > \gamma_1$ such that symmetry breaking occurs if $\theta = \gamma\,(p-2)$ for any $\gamma \in (\gamma_1, \gamma_2)$ and any $p \in (2, 2+\varepsilon)$

Plots (1/2)



Plot of the admissible regions (gray areas) with symmetry breaking region established in Theorem 10 (dark grey) in (η, θ) coordinates, with $\eta := b - a$, for various values of a, in dimension $d = 3$: from left to right, $a = 0$, $a = -0.25$, $a = -0.5$ and $a = -1$. The two curves are $\eta \mapsto \vartheta(p, d) = 1 - \eta$ and $\eta \mapsto \Theta(a, p, d)$, for $p = 2\,d/(d - 2 + 2\,\eta)$. In the range $a \in (-1/2, 0)$, they intersect for $a = a_-(p)$, *i.e.* $\eta = 2\,a\,(1 - d)/(d + 2\,a)$. They are tangent at $(\eta, \theta) = (1, 0)$ for $a = -1/2$. The symmetry breaking region contains a cone attached to $(\eta, \theta) = (1, 0)$ for $a < -1/2$, which determines values of γ for which symmetry breaking occurs in the logarithmic Hardy inequality

Plots (2/2)



Left.– For a given value of $\theta \in (0,1]$, admissible values of the parameters for which the generalized CKN inequality holds are given by $\eta = b - a \geq 1 - \theta$ (grey areas) in terms of (a, η). According to Theorem 10, symmetry breaking occurs if $\theta < \Theta(a,p,d)$, which determines a region $\eta < g(a,\theta)$ (dark grey). Notice that $\eta < g(a,1)$ corresponds to the condition found by Felli and Schneider. The plot corresponds to $d = 3$ and $\theta = 0.5$

Right.– Regions of symmetry breaking, *i.e.* $1 - \theta \leq \eta < g(a,\theta)$, are shown for $\theta = 1$, 0.75, 0.5, 0.3, 0.2, 0.1, 0.05, 0.02. For each value of θ, the supremum value for which symmetry breaking has been established is $a = a_-(p)$ for $p = 2\,d/(d - 2\,\theta)$, which determines a curve $\eta = h(a)$ by requiring that $\theta = 1 - \eta$. The limit case $\eta = 0 = h(0)$ corresponds to the case studied by Felli and Schneider

Logarithmic Hardy inequalities

Joint work with M. del Pino, S. Filippas and A. Tertikas

Logarithmic Hardy inequalities

Theorem 11. *Let* $N \geq 3$*. There exists a constant* $\mathsf{C}_{\rm LH} \in (0, \mathsf{S}]$ *such that, for all* $u \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ *with* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\, dx = 1$*, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \log\left(|x|^{N-2}|u|^2\right) dx \leq \frac{N}{2} \log\left[\mathsf{C}_{\rm LH} \int_{\mathbb{R}^d} |\nabla u|^2\, dx\right]$$

Theorem 12. *Let* $N \geq 1$*. Suppose that* $a < (N-2)/2$*,* $\gamma \geq N/4$ *and* $\gamma > 1/2$ *if* $N = 2$*. Then there exists a positive constant* $\mathsf{C}_{\rm GLH}$ *such that, for any* $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$ *normalized by* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\, dx = 1$*, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}} \log\left(|x|^{N-2-2\,a}\, |u|^2\right) dx \leq 2\,\gamma\, \log\left[\mathsf{C}_{\rm GLH} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\, dx\right]$$

Logarithmic Hardy inequalities: radial case

Theorem 13. *Let* $N \geq 1$, $a < (N-2)/2$ *and* $\gamma \geq 1/4$. *If* $u = u(|x|) \in \mathcal{D}^{1,2}_a(\mathbb{R}^N)$ *is radially symmetric, and* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx = 1$, *then*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}} \log\left(|x|^{N-2-2\,a}\,|u|^2\right) dx \leq 2\,\gamma\,\log\left[\mathsf{C}^*_{\rm GLH}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

$$\mathsf{C}^*_{\rm GLH} = \frac{1}{\gamma}\,\frac{\left[\Gamma\left(\frac{N}{2}\right)\right]^{\frac{1}{2\,\gamma}}}{(8\,\pi^{N+1}\,e)^{\frac{1}{4\,\gamma}}}\left(\frac{4\,\gamma-1}{(N-2-2\,a)^2}\right)^{\frac{4\,\gamma-1}{4\,\gamma}} \quad \textit{if} \quad \gamma > \frac{1}{4}$$

$$\mathsf{C}^*_{\rm GLH} = 4\,\frac{\left[\Gamma\left(\frac{N}{2}\right)\right]^2}{8\,\pi^{N+1}\,e} \quad \textit{if} \quad \gamma = \frac{1}{4}$$

If $\gamma > \frac{1}{4}$, *equality is achieved by the function*

$$u = \frac{\tilde u}{\int_{\mathbb{R}^d} \frac{|\tilde u|^2}{|x|^2}\,dx} \quad \textit{where} \quad \tilde u(x) = |x|^{-\frac{N-2-2\,a}{2}} \exp\left(-\frac{(N-2-2\,a)^2}{4\,(4\,\gamma-1)}\left[\,\log|x|\,\right]^2\right)$$

Logarithmic Hardy inequalities: symmetry breaking

Theorem 14. *Let $N \geq 2$ and $a < -1/2$. Assume that $\gamma > 1/2$ if $N = 2$. If, in addition,*

$$\frac{N}{4} \leq \gamma < \frac{1}{4} + \frac{(N-2\,a-2)^2}{4\,(N-1)}$$

then the optimal constant $\mathsf{C}_{\rm GLH}$ is not achieved by a radial function and $\mathsf{C}_{\rm GLH} > \mathsf{C}^*_{\rm GLH}$

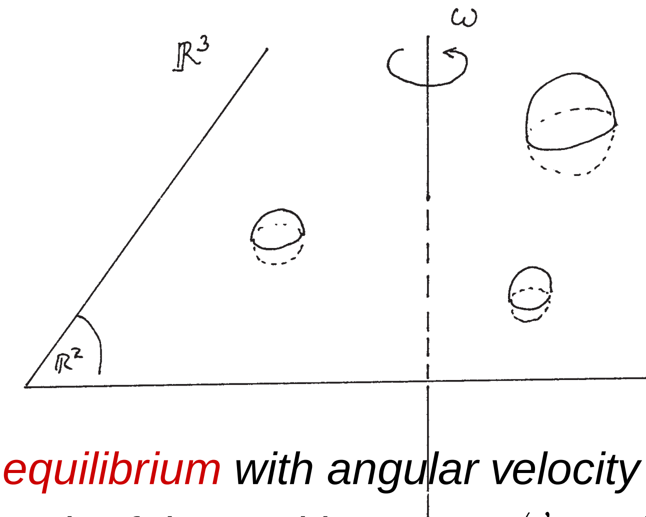
Symmetry breaking in gravitational models

An example of defocusing external potential
Joint work with J. Campos and M. del Pino

Relative equilibria in continuous stellar dynamics

Gravitational (non-relativistic) Vlasov-Poisson system in $\mathbb{R}^3$

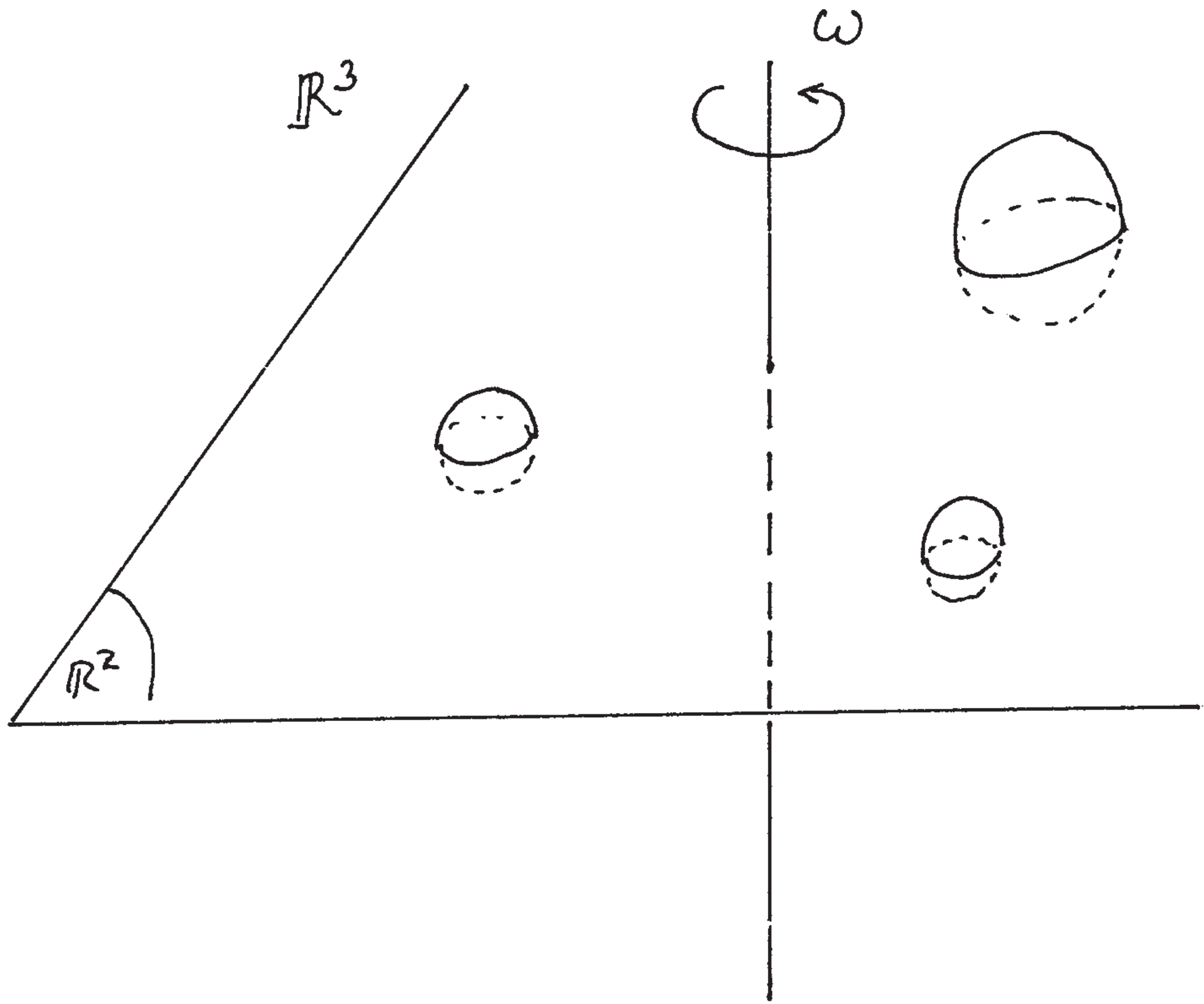
$$\left\{\begin{array}{l}\partial_t F + w \cdot \nabla_z F - \nabla_z \Phi \cdot \nabla_w F = 0 \\ \Delta \Phi = \displaystyle\int_{\mathbb{R}^3} F \, dw\end{array}\right.$$



Theorem 15. *For any $N \geq 2$, any $p \in (1,5)$, any positive numbers λ_1, λ_2, $\ldots \lambda_N$ and any $\omega > 0$ small enough, there is a solution F^ω which is a relative equilibrium with angular velocity ω whose support has N disjoint connected components, each of them with mass m_i^ω such that*

$$\lim_{\omega \to 0_+} m_i^\omega = \lambda_i^{(3-p)/2} \, m_* =: m_i$$

for some positive constant m_*. The center of mass $z_i^\omega(t)$ of each component is such that $\lim_{\omega \to 0_+} \omega^{2/3} \, z_i^\omega(t) =: z_i(t)$ is a relative equilibrium of the N-body Newton's equations with gravitational interaction


ω
$\mathbb{R}^3$
$\mathbb{R}^2$

Newton's equations and basic relative equilibria

$$m_i \frac{d^2 z_i}{dt^2} = \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{4\pi} \, \frac{z_j - z_i}{|z_j - z_i|^3}$$

Ansatz: the system is stationary in a reference frame rotating at constant angular velocity $\Omega = \omega\, \mathsf{e}_3$: $x' = (x^1, x^2, 0) = x - (x \cdot \mathsf{e}_3)\, \mathsf{e}_3$

$$x^3 = z^3 \,, \quad x^1 + i\, x^2 = e^{i\omega t}(z^1 + i\, z^2)$$

Newton's equations in a rotating frame

$$\tfrac{d^2 x_i}{dt^2} = \sum_{i \neq j = 1}^{N} \tfrac{m_j}{4\pi} \, \tfrac{x_j - x_i}{|x_j - x_i|^3} + \omega^2\, x_i' + 2\, \Omega \wedge \tfrac{dx_i}{dt}$$

Relative equilibria are critical points of the function

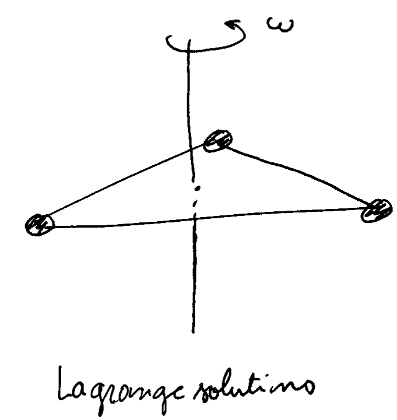
$$\mathcal{V}_\omega(x_1', x_2', \ldots x_N') := -\frac{1}{8\pi} \sum_{i \neq j = 1}^{N} \frac{m_i\, m_j}{|x_j' - x_i'|} - \frac{\omega^2}{2} \sum_{i=1}^{N} m_i\, |x_i'|^2$$

Relative equilibria: classification

- *Lagrange solution:* all masses m_i are equal to $m>0$ and x_i' are located at the summits of a regular polygon, whose radius is adjusted so that

$$\frac{d}{dr}\left[\frac{a_N}{4\pi}\,\frac{m}{r}+\frac{1}{2}\,\omega^2\,r^2\right]=0 \quad \text{with} \quad a_N:=\frac{1}{\sqrt{2}}\sum_{j=1}^{N-1}\frac{1}{\sqrt{1-\cos\left(2\pi j/N\right)}}$$

  [Perko-Walter]: all masses have to be equal
  Scale invariance: $r(N,\varepsilon^{3/2}\,\omega)=\frac{1}{\varepsilon}\,r(N,\omega)\quad\forall\,\varepsilon>0$
  If $\nabla\mathcal{V}_\omega(x_1',x_2',\dots x_N')=0$,
  then $\nabla\mathcal{V}_{\varepsilon^{3/2}\,\omega}(\varepsilon^{-1}\,x_1',\varepsilon^{-1}\,x_2',\dots\varepsilon^{-1}\,x_N')=0$
  the study of the critical points
  of $\mathcal{V}_\omega$ can be reduced to the case $\omega=1$



- [Palmore] For $N\geq 3$, there are (generically) at least $\mu_i(N):=\binom{N}{i}(N-1-i)\,(N-2)\,!$ distinct relative equilibria

Stationary solutions of the Vlasov-Poisson system ($\omega = 0$)

If $\omega = 0$, one can minimize the *free energy*

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\,dx\,dv + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2\,f\,dx\,dv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx$$

under the constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\,dx\,dv = M$ to get a stationary solution which is then dynamically stable

With $\beta(f) = \kappa\,f^q$, this amounts to look for an optimal function of the interpolation inequality

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\rho(x)\,\rho(y)}{|x-y|}\,dx\,dy \le C\,M^a\,\|f\|^b_{L^q(\mathbb{R}^3\times\mathbb{R}^3)}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2\,f\,dx\,dv\right)^{2-a-b}$$

with $\rho = \int_{\mathbb{R}^3} f\,dv$

[Guo, Rein, Schaeffer, Soler, Sánchez,...]

Relative equilibria of the Vlasov-Poisson system

If $\omega \neq 0$, one find the relative equilibria by minimizing the *free energy in the rotating reference frame*

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f)\,dx\,dv + \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(|v|^2 - \omega^2\,|x'|^2\right) f\,dx\,dv - \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\phi|^2\,dx$$

under various constraints:

- symmetry constraint (under rotation of an angle $2\pi/N$)
- mass constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} f\,dx\,dv = M$
- angular momentum constraint $\iint_{\mathbb{R}^3\times\mathbb{R}^3} |x|^2\,f\,dx\,dv = J$
- a localization constraint

Three-dimensional case: [McCann]
Flat case: [Rein, J.D.-Fernández]

Method

Let $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, fix $\lambda_1, \dots \lambda_N$ and $\omega > 0$, small: the problem is

$$\Delta\phi = \sum_{i=1}^{N} \rho_i \quad \text{in } \mathbb{R}^3\,, \quad \rho_i := \left(\tfrac{1}{2}\,\omega^2\,|x'|^2 - \lambda_i - \phi\right)_+^p \chi_i$$

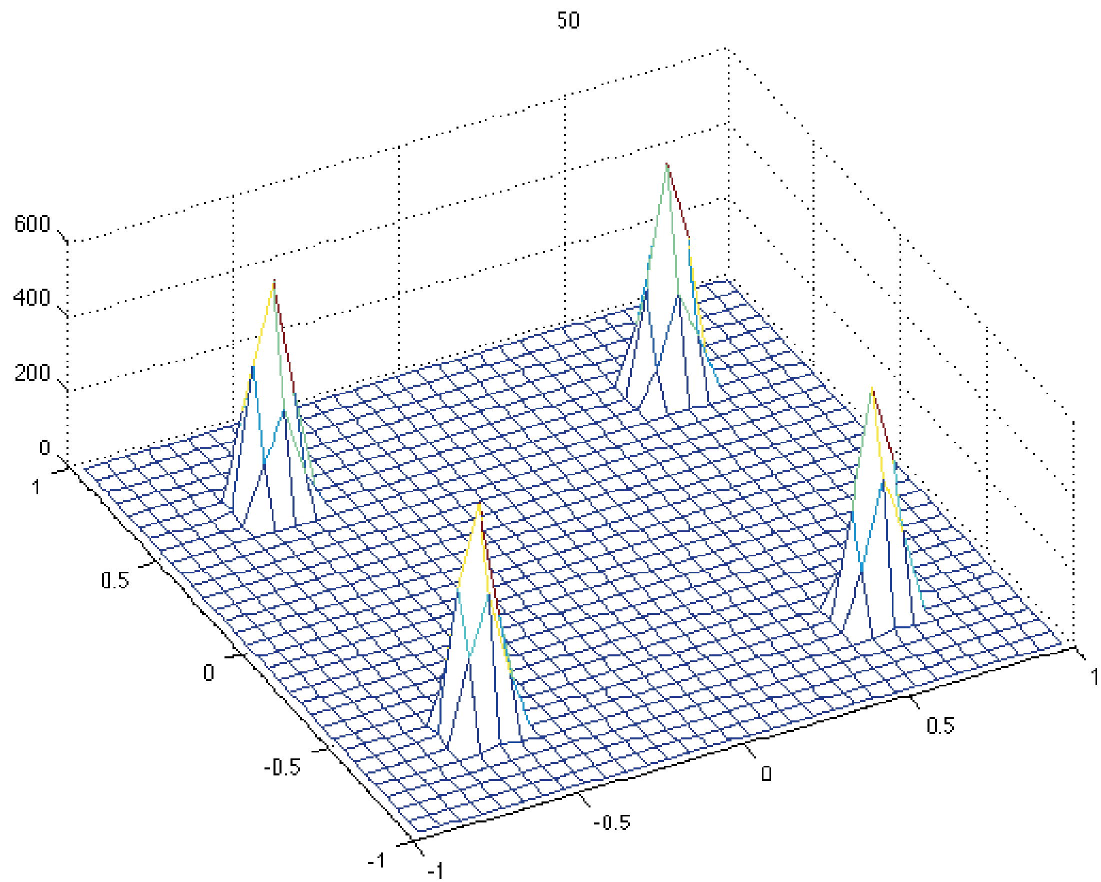
where χ_i denotes the characteristic function of K_i + Boundary condition $\lim_{|x|\to\infty} u(x) = 0$ + Mass and center of mass associated to each component by

$$m_i := \int_{\mathbb{R}^3} \rho_i\, dx \quad \text{and} \quad x_i := \frac{1}{m_i} \int_{\mathbb{R}^3} x\, \rho_i\, dx$$

Lyapunov-Schmidt method [Campos-del Pino, J.D.] applied to

$$\mathcal{J}[\phi] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla\phi|^2\, dx + \sum_{i=1}^{N} \left[\int_{K_i} \left(\lambda_i + \phi(x) - \frac{1}{2}\,\omega^2\,|x'|^2 \right)_+^p dx - m_i\,\lambda_i \right]$$

Numerical results



[J.D.-Salomon, work in progress]