

Relative equilibria in continuous stellar dynamics

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Relative equilibria: [J. Campos, M. del Pino, and J. Dolbeault. Relative equilibria in continuous stellar dynamics. Communications in Mathematical Physics, Doi:300:765788, 2010, Online First]

Flat systems: [J. Dolbeault and J. Fernández, Localized minimizers of flat rotating gravitational systems, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis, 25 (2008), pp. 1043-1071]

Symmetry breaking issues in functional inequalities:

● [J. Dolbeault and M.J. Esteban. About existence, symmetry and symmetry breaking for extremal functions of some interpolation functional inequalities, Preprint (2010)]

● [J. Dolbeault and M.Esteban. Extremal functions in some interpolation inequalities: Symmetry, symmetry breaking and estimates of the best constants , Preprint (2010)]

● [J. Dolbeault, M.J. Esteban, G. Tarantello, and A. Tertikas. Radial symmetry and symmetry breaking for some interpolation inequalities, Preprint (2010)]

● [J. Dolbeault and M.J. Esteban. Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities, Preprint (2010)]

● [J. Dolbeault, M. J. Esteban, M. Loss, and G. Tarantello. On the symmetry of extremals for the Caffarelli-Kohn-Nirenberg inequalities. Advanced Nonlinear Studies, 9:713-727, 2009]

● [J. Dolbeault, M. J. Esteban, and G. Tarantello, The role of Onofri type inequalities in the symmetry properties of extremals for Caffarelli-Kohn-Nirenberg inequalities, in two space dimensions, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), 7 (2008), pp. 313-341]

Introduction: N-body problems and mean-field kinetic equations

Outline

- ▶ A first statement: there are non radially symmetric critical points
- ▶ Relative equilibria: examples and (partial) classification for systems of point particles
- ▶ Kinetic equations: extending the notion of relative equilibria to continuum mechanics
- ▶ Results for kinetic / diffusion equations
- ▶ The variational approach: heuristics
- ▶ The variational approach: a sketch of the proofs
- ▶ Flat systems: results and numerical computation
- ▶ Concluding remarks: symmetry breaking

Gravitational (non-relativistic) Vlasov-Poisson system in \mathbb{R}^3

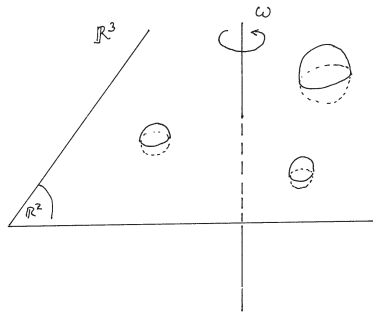
$$\begin{cases} \partial_t F + w \cdot \nabla_z F - \nabla_z \Phi \cdot \nabla_w F = 0 \\ \Delta \Phi = \int_{\mathbb{R}^3} F dw \end{cases} \quad (1)$$

Theorem

For any $N \geq 2$, any $p \in (1, 5)$, any positive numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ and any $\omega > 0$ small enough, there is a solution F^ω of (1) which is a **relative equilibrium** with angular velocity ω whose support has N disjoint connected components, each of them with mass m_i^ω such that

$$\lim_{\omega \rightarrow 0_+} m_i^\omega = \lambda_i^{(3-p)/2} m_* =: m_i$$

for some positive constant m_* . The center of mass $z_i^\omega(t)$ of each component is such that $\lim_{\omega \rightarrow 0_+} \omega^{2/3} z_i^\omega(t) =: z_i(t)$ is a relative equilibrium of the N -body Newton's equations with gravitational interaction



Systems of discrete particles: the N-body problem in gravitation

Solutions of the N-body problem in gravitation

... many solutions are known

- ▶ No stationary (time independent) solutions
- ▶ Periodic solutions in Hamiltonian dynamics: [Ekeland et al.]
- ▶ Choreographies: [Chenciner et al.], [Terracini et al.]

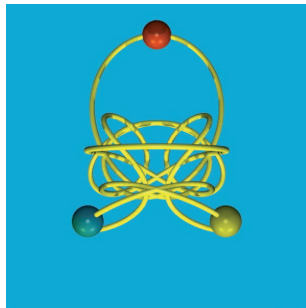
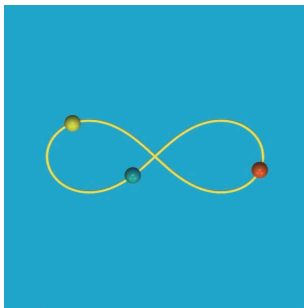


Figure: Choreographies, pictures taken from S. Terracini's web page

<http://www.matapp.unimib.it/~suster/files/index.html>

Consider N point particles with masses m_i located at $z_i(t) \in \mathbb{R}^3$ subject to Newton's equations

$$m_i \frac{d^2 z_i}{dt^2} = \sum_{i \neq j=1}^N \frac{m_i m_j}{4\pi} \frac{z_j - z_i}{|z_j - z_i|^3} \quad (2)$$

Ansatz: the system is stationary in a reference frame rotating at constant angular velocity $\Omega = \omega e_3$

Notation: $x' = (x^1, x^2, 0) = x - (x \cdot e_3) e_3$, a change of coordinates

$$x^3 = z^3, \quad x^1 + i x^2 = e^{i\omega t} (z^1 + i z^2)$$

provides Newton's equations in a rotating frame

$$\frac{d^2 x_i}{dt^2} = \sum_{i \neq j=1}^N \frac{m_j}{4\pi} \frac{x_j - x_i}{|x_j - x_i|^3} + \omega^2 x_i' + 2\Omega \wedge \frac{dx_i}{dt}$$

We look for stationary solutions in the rotating frame: **relative equilibria**

The configuration is *central* and planar: critical points of the function

$$\mathcal{V}_\omega(x'_1, x'_2, \dots, x'_N) := -\frac{1}{8\pi} \sum_{i \neq j=1}^N \frac{m_i m_j}{|x'_j - x'_i|} - \frac{\omega^2}{2} \sum_{i=1}^N m_i |x'_i|^2$$

- ▶ All masses m_i are equal to some $m > 0$ and x'_i are located at the summits of a regular polygon, where $r = |x'_i|$ is adjusted so that

$$\frac{d}{dr} \left[\frac{a_N}{4\pi} \frac{m}{r} + \frac{1}{2} \omega^2 r^2 \right] = 0 \quad \text{with} \quad a_N := \frac{1}{\sqrt{2}} \sum_{j=1}^{N-1} \frac{1}{\sqrt{1 - \cos(2\pi j/N)}}$$

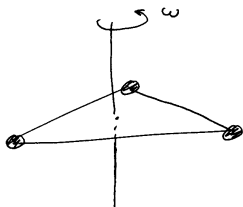
gives a *Lagrange solution* with $r = r(N, \omega) := \left(\frac{a_N m}{4\pi \omega^2} \right)^{1/3}$

[Perko-Walter]: all masses have to be equal if $N \geq 4$

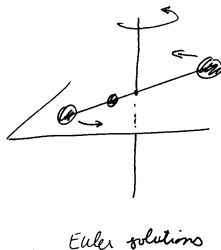
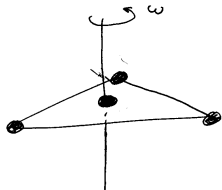
Scale invariance:

$$r(N, \varepsilon^{3/2} \omega) = \frac{1}{\varepsilon} r(N, \omega) \quad \forall \varepsilon > 0$$

If $\nabla \mathcal{V}_\omega(x'_1, x'_2, \dots, x'_N) = 0$,
 then $\nabla \mathcal{V}_{\varepsilon^{3/2} \omega}(\varepsilon^{-1} x'_1, \varepsilon^{-1} x'_2, \dots)$
 the study of the critical points
 of \mathcal{V}_ω can be
 reduced to the case $\omega = 1$



- ▶ $N - 1$ points particles of same mass are located at the summits of a regular centered polygon and one more point particle stands at the center (with not necessarily the same mass as the other ones). A solution is then found again by adjusting the size of the polygon
- ▶ The *Euler-Moulton solutions* are made of aligned points



Relative equilibria are critical points of the function $\mathcal{V}_\omega : (\mathbb{R}^2)^N \rightarrow \mathbb{R}$

$$\mathcal{V}_\omega(x'_1, x'_2, \dots, x'_N) := -\frac{1}{8\pi} \sum_{i \neq j=1}^N \frac{m_i m_j}{|x'_j - x'_i|} - \frac{1}{2} \omega^2 \sum_{i=1}^N m_i |x'_i|^2$$

Generic case: all masses are different

▶ $N = 2$:

$$|x_1 - x_2| = \left(\frac{M}{4\pi \omega^2} \right)^{1/3} \quad \text{and} \quad m_1 x_1 + m_2 x_2 = 0, \quad \text{with } M = m_1 + m_2$$

▶ $N = 3$:

- *Lagrange solutions*: masses are located at the vertices of an equilateral triangle, and the distance between each point is $(M/(4\pi \omega^2))^{1/3}$ with $M = m_1 + m_2 + m_3$: two classes of solutions corresponding to the two orientations of the triangle when labeled by the masses

- *Euler solutions* are made of aligned points and provide three classes of critical points, one for each ordering of the masses on the line

- $N \geq 4$: solutions made of aligned points are *Moulton's solutions*
- $N \geq 4$: Lagrange solutions (all particles are located at the vertices of a regular N -polygon) exists if and only if all masses are equal

• **Standard variational setting [Smale]**: for $m = (m_1, \dots, m_N) \in \mathbb{R}_+^N$, consider the manifold $(q_1, \dots, q_N) \in \mathbb{R}^{2N}$ such that

$$\sum_{i=1}^N m_i q_i = 0, \quad \frac{1}{2} \sum_{i=1}^N m_i |q_i|^2 = 1, \quad q_i \neq q_j \text{ if } i \neq j$$

quotiented by the equivalence classes associated to the invariances: rotations and scalings

$\dim(\mathcal{S}_m) = 2N - 3$, relative equilibria are critical points on \mathcal{S}_m of the potential

$$U_m(q_1, \dots, q_N) = -\frac{1}{8\pi} \sum_{i \neq j=1}^N \frac{m_i m_j}{|q_j - q_i|}$$

For $N \geq 4$, various classification results have been achieved by [Palmore]

- ▶ For $N \geq 3$, the index of a relative equilibrium is always greater or equal than $N - 2$. This bound is achieved by Moulton's solutions
- ▶ For $N \geq 3$, there are at least $\mu_i(N) := \binom{N}{i}(N - 1 - i)(N - 2)!$ distinct relative equilibria in \mathcal{S}_m of index $2N - 4 - i$ if U_m is a Morse function. As a consequence, there are at least

$$\sum_{i=0}^{N-2} \mu_i(N) = [2^{N-1}(N - 2) + 1](N - 2)!$$

distinct relative equilibria in \mathcal{S}_m if U_m is a Morse function

- ▶ For every $N \geq 3$ and for almost all masses $m \in \mathbb{R}_+^N$, U_m is a Morse function
- ▶ There are only finitely many classes of relative equilibria for every $N \geq 3$ and for almost all masses $m \in \mathbb{R}_+^N$
- ▶ If $N \geq 4$, the set of masses for which there exist degenerate classes of relative equilibria has positive k -dimensional Hausdorff measure if $0 \leq k \leq N - 1$

The kinetic problem

Gravitational Vlasov-Poisson system (with centrifugal and Coriolis forces):

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f - \omega^2 x' \cdot \nabla_v f + 2 \Omega \wedge v \cdot \nabla_v f = 0$$

$$\Delta \phi = \rho := \int_{\mathbb{R}^3} f \, dv$$

Boundary conditions: $\phi = -\frac{1}{4\pi|\cdot|} * \rho$

Change of coordinates: $f(t, x, v) = F(t, z, w)$, $\phi(t, x) = \Phi(t, z)$

$$x = \exp(\omega t A) z, \quad v = \Omega \wedge x + \exp(\omega t A) w \quad \text{with} \quad A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

For some arbitrary convex function β , critical points of the *free energy*

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f) \, dx \, dv + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (|v|^2 - \omega^2 |x'|^2) f \, dx \, dv - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx$$

give stationary solutions under the constraint $\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f \, dx \, dv = M$

For $\omega \neq 0$: no minimizers

[Binney-Tremaine] A typical example of such a function is

$$\beta(f) = \kappa f^q$$

A critical point of \mathcal{F} such that $\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f \, dx \, dv = M$ is given by

$$f(x, v) = \gamma \left(\lambda + \frac{1}{2} |v|^2 + \phi(x) - \frac{1}{2} \omega^2 |x'|^2 \right)$$

where $\gamma(s) = (\beta')^{-1}(-s)$: $\gamma(s) = (-s)_+^{1/(q-1)}$

The problem is reduced to solve a non-linear Poisson equation

$$\Delta \phi = g \left(\lambda + \phi(x) - \frac{1}{2} \omega^2 |x'|^2 \right) \chi_{\text{supp}(\rho)}$$

$$g(\mu) := \int_{\mathbb{R}^3} \gamma \left(\mu + \frac{1}{2} |v|^2 \right) \, dv$$

Variational approach:

$$\mathcal{J}[\phi] := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx + \int_{\mathbb{R}^3} G \left(\lambda + \phi(x) - \frac{1}{2} \omega^2 |x'|^2 \right) \, dx - \int_{\mathbb{R}^3} \lambda \rho \, dx$$

where $\lambda = \lambda[x, \phi]$ is now a functional which is constant on each connected component K_i of the support of $\rho(x)$

The total mass is $M = \sum_{i=1}^N m_i$

$$\int_{K_i} g \left(\lambda_i + \phi(x) - \frac{1}{2} \omega^2 |x'|^2 \right) dx = m_i$$

$$\mathcal{J}[\phi] = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 dx + \sum_{i=1}^N \left[\int_{K_i} G \left(\lambda_i + \phi(x) - \frac{1}{2} \omega^2 |x'|^2 \right) dx - m_i \lambda_i \right]$$

Heuristics. The various components K_i are far away from each other so that the dynamics of their center of mass is described by the N -body point particles system, at first order. On each component K_i , the solution is a perturbation of an isolated minimizer of \mathcal{F} (without angular rotation) under the constraint that the mass is equal to m_i . **Alternatively**, we consider a critical point of \mathcal{J} obtained as the perturbation of a superposition of single components critical points of \mathcal{J} of mass m_i , provided the centers of mass x_i of each of the components are close enough of a critical point of \mathcal{V}_ω , with $\omega > 0$, small

$\omega = 0$: [Guo et al.], [Lemou-Méhats-Raphaël], [Rein et al.], [Sánchez et al.], [Soler et al.], [Schaeffer], [Wolansky], [JD-Fernández] + [Kurth]

The first result

The spatial density $\rho^\omega := \int_{\mathbb{R}^3} f^\omega dv$ has exactly N disjoint connected components K_i^ω and

$$m_i^\omega = \int_{K_i^\omega} \rho^\omega dx, \quad z_i^\omega(t) = \exp(-\omega t A) x_i^\omega \quad \text{where} \quad x_i^\omega := \frac{1}{m_i^\omega} \int_{K_i^\omega} x \rho^\omega dx$$

$$\rho_i(x) := \frac{1}{\lambda_i^p} \rho^\omega \left(\lambda_i^{(1-p)/2} (x + x_i^\omega) \right) \chi_{K_i^\omega} \left(\lambda_i^{(1-p)/2} (x + x_i^\omega) \right)$$

converges to a density function $\rho_* = (w-1)_+^p$ given by

$$-\Delta w = (w-1)_+^p \quad \text{in } \mathbb{R}^3, \quad \lim_{|x| \rightarrow \infty} w(x) = 0$$

Theorem

For any $N \geq 2$, any $p \in (1, 5)$, any positive numbers $\lambda_1, \lambda_2, \dots, \lambda_N$ and any $\omega > 0$ small enough, there is a **relative equilibrium** solution F^ω s.t.

$$\lim_{\omega \rightarrow 0_+} m_i^\omega = \lambda_i^{(3-p)/2} m_* =: m_i$$

for $m_* = \int_{\mathbb{R}^3} \rho_* dx$. The center of mass $z_i^\omega(t)$ of each component is such that $\lim_{\omega \rightarrow 0_+} \omega^{2/3} z_i^\omega(t) =: z_i(t)$ is a relative equilibrium of the N -body problem (Newton's equations)

Let $x = (x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, fix $\lambda_1, \dots, \lambda_N$ and $\omega > 0$, small: the problem is

$$-\Delta u = \sum_{i=1}^N \rho_i \quad \text{in } \mathbb{R}^3, \quad \rho_i := \left(u - \lambda_i + \frac{1}{2} \omega^2 |x'|^2\right)_+^p \chi_i \quad (3)$$

where χ_i denotes the characteristic function of K_i

Boundary condition $\lim_{|x| \rightarrow \infty} u(x) = 0$

Mass and center of mass associated to each component by

$$m_i := \int_{\mathbb{R}^3} \rho_i \, dx \quad \text{and} \quad x_i := \frac{1}{m_i} \int_{\mathbb{R}^3} x \rho_i \, dx$$

We shall say that two solutions u_1 and u_2 are equivalent if there is a rotation $R \in \text{SO}(2) \times \{\text{Id}\}$, *i.e.* a rotation in the plane orthogonal to the direction of rotation, such that $u_2(x) = u_1(Rx)$ for any $x \in \mathbb{R}^3$. We shall say that u_1 and u_2 are **distinct** if they are not equivalent

Theorem

For ω small enough, and for almost every positive $(\lambda_1, \dots, \lambda_N) \in (0, \infty)^N$, (3) has at least $[2^{N-1}(N-2) + 1](N-2)!$ distinct solutions which continuously depend on ω

If u^ω is such a solution, there are points $\xi_1^\omega, \dots, \xi_N^\omega \in \mathbb{R}^3$ such that as $\omega \rightarrow 0$, $|\xi_j^\omega - \xi_i^\omega| \rightarrow \infty$ for any $j \neq i$ and $u^\omega(\cdot + \xi_i^\omega)$ locally converges to the unique radial nonnegative solution of

$$-\Delta w = (w - \lambda_i)_+^p \quad \text{in } \mathbb{R}^3$$

For $\omega > 0$ small enough, the support of ρ^ω has N connected components

$$\lim_{\omega \rightarrow 0} m_i^\omega = \lambda_i^{(3-p)/2} m_* =: m_i \quad \text{and} \quad \lim_{\omega \rightarrow 0} \omega^{2/3} \xi_i^\omega := \xi_i$$

and (ξ_1, \dots, ξ_N) is a relative equilibrium with masses $(m_i)_{1 \leq i \leq N}$

The scaling invariance is recovered only in the limit $\omega \rightarrow 0_+$

$$-\Delta w = (w - 1)_+^p \quad \text{in } \mathbb{R}^3 \quad (4)$$

Lemma

Under the condition $\lim_{|x| \rightarrow \infty} w(x) = 0$, Equation (4) has a unique solution, up to translations, which is positive and radially symmetric. It is a non-degenerate critical point of $\frac{1}{2} \int_{\mathbb{R}^3} |\nabla w|^2 dx + \frac{1}{p+1} \int_{\mathbb{R}^3} (w - 1)_+^{p+1} dx$

$$w_i(x) := w^{\lambda_i}(x - \xi_i), \quad W_\xi := \sum_{i=1}^N w_i$$

Compatibility condition: for a large, fixed $\mu > 0$, and all small $\omega > 0$,

$$|\xi_i| < \mu \omega^{-\frac{2}{3}}, \quad |\xi_i - \xi_j| > \mu^{-1} \omega^{-\frac{2}{3}} \quad (5)$$

Ansatz: we look for a solution of (3) of the form

$$u = W_\xi + \phi$$

with $\text{supp}(w^{\lambda_i} - \lambda_i)_+ \subset B(0, R)$ for all $i = 1, \dots, N$ for $R > 0$ large

$$\Delta \phi + \sum_{i=1}^N p (W_\xi - \lambda_i + \frac{1}{2} \omega^2 |x'|^2)_+^{p-1} \chi_i \phi = -E - N[\phi]$$

The nonlinear perturbation problem

We want to solve

$$\Delta\phi + \sum_{i=1}^N p \left(W_\xi - \lambda_i + \frac{1}{2} \omega^2 |x'|^2 \right)_+^{p-1} \chi_i \phi = -E - N[\phi]$$

where

$$E := \Delta W_\xi + \sum_{i=1}^N \left(W_\xi - \lambda_i + \frac{1}{2} \omega^2 |x'|^2 \right)_+^p \chi_i$$

$$N[\phi] := \sum_{i=1}^N \left[\left(W_\xi - \lambda_i + \frac{1}{2} \omega^2 |x'|^2 + \phi \right)_+^p - \left(W_\xi - \lambda_i + \frac{1}{2} \omega^2 |x'|^2 \right)_+^p - p \left(W_\xi - \lambda_i + \frac{1}{2} \omega^2 |x'|^2 \right)_+^{p-1} \phi \right] \chi_i$$

The variational scheme

- ▶ A linear theory
- ▶ The projected nonlinear problem (Lagrange multipliers)
- ▶ The variational reduction
- ▶ A variational approach in finite dimension

[Floer-Weinstein 1986] + many others...

$$\|\phi\|_* = \sup_{x \in \mathbb{R}^3} \left(\sum_{i=1}^N |x - \xi_i| + 1 \right) |\phi(x)|, \quad \|h\|_{**} = \sup_{x \in \mathbb{R}^3} \left(\sum_{i=1}^N |x - \xi_i|^4 + 1 \right) |h(x)|$$

Consider the *projected problem*

$$L[\phi] = h + \sum_{i=1}^N \sum_{j=1}^3 c_{ij} Z_{ij} \chi_i$$

where $Z_{ij} := \partial_{x_j} w_i$, subject to orthogonality conditions

$$\int_{\mathbb{R}^3} \phi Z_{ij} \chi_i dx = 0 \quad \forall i, j = 1, 2, \dots, N$$

Lemma

Assume that (5) holds. Given h with $\|h\|_{**} < +\infty$, there is a unique solution $\phi =: T[h]$ and there exists a positive constant C , which is independent of ξ such that, for $\omega > 0$ small enough,

$$\|\phi\|_* \leq C \|h\|_{**}$$

The projected nonlinear problem

Find ϕ with $\|\phi\|_* < +\infty$, solution of

$$L[\phi] = -E - N[\phi] + \sum_{i=1}^N \sum_{j=1}^3 c_{ij} Z_{ij} \chi_i$$

such that $\phi(x) \rightarrow 0$ as $|x| \rightarrow +\infty$ and

$$\int_{\mathbb{R}^3} \phi Z_{ij} \chi_i dx = 0 \quad \text{for all } i, j.$$

To do this analysis we have to measure the size of the error E . We recall that

$$\begin{aligned} E &= \sum_{i=1}^N \left[(w_i + \sum_{j \neq i} w_j - \lambda_i + \frac{1}{2} \omega^2 |x'|^2)_+^p - (w_i - \lambda_i)_+^p \right] \chi_i \\ &= \sum_{i=1}^N p(w_i - \lambda_i + t(\sum_{j \neq i} w_j + \frac{1}{2} \omega^2 |x'|^2))_+^{p-1} (\sum_{j \neq i} w_j + \frac{1}{2} \omega^2 |x'|^2) \chi_i \end{aligned}$$

for some $t \in (0, 1)$

$$|E| \leq C \sum_{i=1}^N \left[\sum_{j \neq i} \frac{1}{|\xi_i - \xi_j|} + \frac{1}{2} \omega^2 |\xi_i|^2 \right] \chi_i \leq C \omega^{\frac{2}{3}} \sum_{i=1}^N \chi_i$$

Thus $\|E\|_{**} \leq C \omega^{\frac{2}{3}}$. Moreover, for $\|\phi\|_* \leq 1$,

$$|N[\phi]| \leq C \sum_{i=1}^N |\phi|^\gamma \chi_i, \quad \gamma = \min\{p, 2\}$$

$\|N[\phi]\|_{**} \leq C \|\phi\|_*^\gamma$ and $\|N(\phi_1) - N(\phi_2)\|_{**} \leq o(1) \|\phi_1 - \phi_2\|_*^\gamma$

We look for a fixed point $\phi = \mathcal{A}[\phi] := -T[E + N[\phi]]$ on the region

$$\mathcal{B} = \left\{ \phi : \|\phi\|_* \leq K \omega^{\frac{2}{3}} \right\}$$

Lemma

$\exists ! \phi_\xi = \phi(\xi_1, \dots, \xi_k)$ which depends continuously on its parameters for the $\|\cdot\|_*$ -norm and $\|\phi_\xi\|_* \leq C \omega^{\frac{2}{3}}$,

$$\phi_{e^{\theta A} \xi} = \phi_\xi(e^{-\theta A} \cdot) \quad \text{and} \quad c_i(e^{\theta A} \xi) = e^{-\theta A} c_i.$$

Lemma

We have that $c_{ij} = 0$ for all i, j if and only if the k -tuple (ξ_1, \dots, ξ_N) is a critical point of the functional

$$(\xi_1, \dots, \xi_N) \mapsto \Lambda(\xi_1, \dots, \xi_N) := J(W_\xi + \phi_\xi)$$

A Taylor expansion

$$J(W_\xi) = J(W_\xi + \phi_\xi) - DJ(W_\xi + \phi_\xi)[\phi_\xi] + \frac{1}{2} \int_0^1 D^2 J(W_\xi + (1-t)\phi_\xi)[\phi_\xi]^2 dt$$

$$D^2 J(W_\xi + (1-t)\phi_\xi)[\phi_\xi]^2 = O(\omega^{\frac{4}{3}})$$

$$\Lambda(\xi) = \sum_{i=1}^N \lambda_i^{5-p} e_* - \left[\frac{1}{2} \sum_{i \neq j} \frac{m_i m_j}{|\xi_i - \xi_j|} + \frac{1}{2} \omega^2 \sum_{i=1}^N m_i |\xi_i|^2 \right] + O(\omega^{\frac{4}{3}})$$

In the region

$$\mathfrak{B} = \left\{ (\xi_1, \dots, \xi_k) : |\xi_i - \xi_j| > \rho \omega^{-\frac{2}{3}}, \quad |\xi_i| < \rho^{-1} \omega^{-\frac{2}{3}} \text{ for all } i, j \right\}$$

where $\rho > 0$ is chosen small enough and fixed, we have that

$$\sup_{\mathfrak{B}} \Lambda > \sup_{\partial \mathfrak{B}} \Lambda$$

so that this functional has a local maximum somewhere in \mathfrak{B} . Hence a critical point of this functional does exist in \mathfrak{B} □

Lemma

For any $\lambda_i > 0$, ξ_i , we have found a critical point of Λ , i.e. a solution of

$$L[\phi] = -E - N[\phi]$$

$$U_m(q_1, \dots, q_N) = -\frac{1}{8\pi} \sum_{i \neq j=1}^N \frac{m_i m_j}{|q_j - q_i|}$$

is a Morse function on \mathcal{S}_m . Take a local system of coordinates $(\eta_1, \dots, \eta_{2N-4})$ on a neighborhood of a critical point \bar{q} and for $\alpha > 0$, let

$$\xi(\alpha, p, \eta) = (\alpha(q_1(\eta) + p), \dots, \alpha(q_N(\eta) + p))$$

$$\Phi(\alpha, p, \eta) = \omega^{-\frac{2}{3}} \Lambda(\xi(\omega^{-\frac{2}{3}} \alpha, p, q(\eta)))$$

$$\begin{aligned} \nabla \Phi(\alpha, p, \eta) = \nabla \left(-\frac{1}{2} \sum_{i \neq j=1}^N \frac{m_i m_j}{\alpha |q_j(\eta) - q_i(\eta)|} - \frac{1}{2} \alpha^2 \sum_{i=1}^N m_i |q_i(\eta)|^2 \right. \\ \left. + \frac{1}{2} \alpha^2 |p|^2 \sum_{i=1}^N m_i \right) + O(\omega^{\frac{2}{3}}) \end{aligned}$$

$(\bar{\lambda}^{\frac{1}{3}}, 0, \bar{\eta})$ is nondegenerate, so the local degree $\deg(\nabla \tilde{\Phi}, \mathcal{U}, 0)$ is well defined and nonzero: there exists a critical point (α^*, p^*, η^*) as $\omega \rightarrow 0$

A flat model: theory and numerical results

[JD-Fernández] Written in cartesian coordinates, the equation is

$$\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f = 0$$
$$\phi = -\frac{1}{4\pi |x|} * \int_{\mathbb{R}^2} f \, dv$$

where $a \wedge b := a^\perp \cdot b = a_1 b_2 - a_2 b_1$ and $x, v \in \mathbb{R}^2$

Definition

A **localized minimizer** is a critical point ρ of \mathcal{G}_ω which is compactly supported in a ball $B(0, R - \varepsilon)$ for some $R > 0$ and $\varepsilon \in (0, R)$, and which is a minimizer of \mathcal{G}_ω restricted to the set

$$\{\rho \in L^1_+(\mathbb{R}^2) : \text{supp}(\rho) \subset B(0, R) \text{ and } \int_{\mathbb{R}^2} \rho \, dx = M\}$$

Theorem

For any $M > 0$, there exists $\omega_*(M) = \omega_* > 0$ and $\omega^*(M) = \omega^* > 0$ with $\omega_* \leq \omega^*$ such that

(i) If $\omega \in [-\omega_*, \omega_*]$, the *reduced free energy functional*

$$\mathcal{G}_\omega[\rho] := \frac{\kappa}{m-1} \int_{\mathbb{R}^2} \rho^m dx - \frac{\omega^2}{2} \int_{\mathbb{R}^2} |x|^2 \rho dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy$$

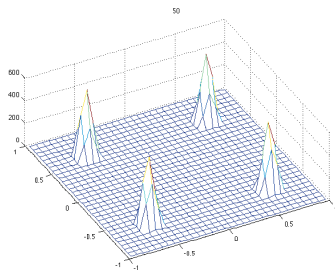
admits a localized minimizer

(ii) If $|\omega| > \omega^*$, $\mathcal{G}_\omega[\rho]$ *admits no localized minimizer*

More detailed results in the radial case. How does symmetry breaking occur ?

Goal: investigate the energy landscape [JD-Fernández-Salomon]

- ▶ Local minimizers (under appropriate constraints) have a very small basin of attraction
- ▶ Compact support has to be enforced at each step
- ▶ Iteration method inspired by mean-field models in quantum mechanics work, with similar difficulties: the Cancès-LeBris method of relaxation has to be introduced to achieve convergence



Polytropic case: $\beta(f) = \kappa f^q$, without angular velocity: minimizing the free energy

$$\mathcal{F}[f] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f) \, dx \, dv + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, dx \, dv - \frac{1}{2} \int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx$$

amounts to control the potential energy and find the optimal function in

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx \leq C \|f\|_1^\alpha \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f) \, dx \, dv \right)^{\beta(2-\alpha)} \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, dx \, dv \right)^{\beta(2-\alpha)}$$

● **Explanation:** With $\rho(x) = \int_{\mathbb{R}^3} f(x, v) \, dv$, **HLS inequality**

$$\int_{\mathbb{R}^3} |\nabla \phi|^2 \, dx = \int_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x) \rho(y)}{4\pi |x - y|} \, dx \, dy \leq C_{HLS} \|\rho\|_1^\alpha \|\rho\|_q^{2-\alpha}$$

+ **interpolation** $\|\rho\|_q \leq C_{p,q} \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \beta(f) \, dx \, dv \right)^\beta \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f \, dx \, dv \right)^\beta$

● **Relative equilibria:** The constraint $\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x'|^2 f \, dx \, dv = \text{Const}$
 (+ localization in x-space) may break the symmetry

Conclusion: symmetry breaking and stability

- ▶ In nonlinear PDEs, **symmetry breaking** usually occurs because of a competition between the nonlinearity and an external potential
- ▶ A classical example is the (PDE) Hénon problem
- ▶ The case covered by the theorem of [Gidas-Ni-Nirenberg] is the trivial one: the nonlinearity and an external potential cooperate
- ▶ Symmetry breaking is usually achieved by eigenvalue considerations
- ▶ Here we have an example based on multiscale analysis: this is new !
- ▶ Main issue (especially in gravitation) is **dynamical / orbital stability**
Constrained (localized) minimization and mass transport methods [McCann] but new ideas are required
- ▶ Building other examples of periodic solutions (choreographies ?) would be an intermediate step