Large mass self-similar solutions of the parabolic-parabolic Keller–Segel model of chemotaxis

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Modélisation et EDP non linéaires

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Outline

Parabolic-elliptic model: preliminaries

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- The asymptotic behaviour of the solutions of the Keller-Segel model

Parabolic-parabolic model

- introduction: self-similar solutions, parametrization, etc
- *a priori* estimates
- cumulated densities
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Parabolic-elliptic model: preliminaries

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2\,,\ t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2\,,\ t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y|\, u(t,y)\, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2}\, u(t,y)\, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x)\, dx = 0$

Blow-up

$M = \int_{\mathbb{R}^2} n_0\, dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2\, n_0\, dx < \infty$: blow-up in finite time

a solution u of

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u\, \nabla v)$$

satisfies

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2\, u(t,x)\; dx$$

$$= - \int_{\mathbb{R}^2} 2x\, \Delta u\; dx + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \underbrace{\tfrac{2x \cdot (y-x)}{|x-y|^2}\, u(t,x)\, u(t,y)\; dx\, dy}_{\tfrac{(x-y)\cdot(y-x)}{|x-y|^2}\, u(t,x)\, u(t,y)\; dx\, dy}$$

$$= 4\, M - \frac{M^2}{2\pi} < 0 \quad \text{if} \quad M > 8\pi$$

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \leq 8\pi$: global existence [Jäger, Luckhaus], [JD, Perthame], [Blanchet, JD, Perthame], [Blanchet, Carrillo, Masmoudi]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot \left[u \left(\nabla \left(\log u \right) - \nabla v \right) \right]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u \, v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = - \int_{\mathbb{R}^2} u \left| \nabla \left(\log u \right) - \nabla v \right|^2 \, dx$$

Log HLS inequality [Carlen, Loss]: F is bounded from below if $M < 8\pi$

The dimension $d = 2$

- In dimension d, the norm $L^{d/2}(\mathbb{R}^d)$ is critical. If $d = 2$, the mass is critical
- Scale invariance: if (u, v) is a solution in $\mathbb{R}^2$ of the parabolic-elliptic Keller and Segel system, then

$$\Big(\lambda^2\, u(\lambda^2\, t, \lambda\, x)\,,\; v(\lambda^2\, t, \lambda\, x)\Big)$$

  is also a solution
- For $M < 8\pi$, the solution vanishes as $t \to \infty$, but saying that "diffusion dominates" is not correct: to see this, study "intermediate asymptotics"

The existence setting

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \, , \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \, , \; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\,dx) \;, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \;, \quad M := \int_{\mathbb{R}^2} n_0(x)\,dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\,dx$ for any $t \geq 0$, see [Jäger-Luckhaus], [Blanchet, JD, Perthame]

$v = -\frac{1}{2\pi} \log |\cdot| * u$

Time-dependent rescaling

$$u(x,t) = \frac{1}{R^2(t)}\, n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x,t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1+2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\ t > 0 \\ c = -\dfrac{1}{2\pi}\, \log|\cdot| * n & x \in \mathbb{R}^2\,,\ t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[Blanchet, JD, Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means "intermediate asymptotics" in original variables:

$$\|u(x,t) - \tfrac{1}{R^2(t)}\, n_\infty\left(\tfrac{x}{R(t)}, \tau(t)\right)\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \, \frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty \,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- Radial symmetry [Naito]
- Uniqueness [Biler, Karch, Laurençot, Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [Blanchet, JD, Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[Joseph, Lundgreen] [JD, Stańczy]

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla \Big[n \, (\log n - x + \nabla c) \Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \; dx + \int_{\mathbb{R}^2} \frac{1}{2} \, |x|^2 \, n \; dx - \frac{1}{2} \int_{\mathbb{R}^2} n \, c \; dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = - \int_{\mathbb{R}^2} n \, |\nabla \, (\log n) + x - \nabla c|^2 \; dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2 \, n(\lambda \, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \; dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2}-1}{2} \, |x|^2 \, n \; dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) \, n(y) \; \log \frac{1}{\lambda} \; dx \, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi} \right)}_{>0 \text{ if } M<8\pi} \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 \, n \; dx$$

Parabolic-elliptic case: large time asymptotics

Theorem 1. *There exists a positive constant* M^* *such that, for any initial data* $n_0 \in L^2(n_\infty^{-1}\,dx)$ *of mass* $M < M^*$ *satisfying the above assumptions, there is a unique solution* $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau,\infty)\times\mathbb{R}^2)$ *for any* $\tau > 0$

Moreover, there are two positive constants, C *and* δ, *such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \,\frac{dx}{n_\infty} \leq C\,e^{-\,\delta\,t} \quad \forall\, t > 0$$

As a function of M, δ *is such that* $\lim_{M\to 0_+} \delta(M) = 1$

The condition $M \leq 8\,\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- *Spectral gap* of a linearised operator $\mathcal{L}$

Introduction to the parabolic-parabolic Keller-Segel model

The parabolic-parabolic Keller-Segel system

$$n_t = \Delta n - \nabla \cdot (n \nabla c)$$
$$\tau c_t = \Delta c + n$$

Some (open) mathematical issues:

- global in time existence *versus* finite time blowup
- large time behaviour
- behaviour near blow-up and measure-valued solutions

Goal: the study of self-similar solutions
... the total mass is conserved

$$M := \int_{\mathbb{R}^2} n(t,x)\,\mathrm{d}x = \int_{\mathbb{R}^2} n_0(x)\,\mathrm{d}x$$

... there are self-similar solutions with an arbitrary large mass if $\tau > 0$

Some results

- [Raczyński] solutions of the parabolic-parabolic Keller–Segel system converge to those of parabolic-elliptic system when $\tau \searrow 0$ if M is small enough
- [Blanchet, JD, Perthame] when $\tau = 0$ in (1), $M = 8\,\pi$ is a threshold for existence *versus* blowup
- [Blanchet, Karch, Laurençot, Nadzieja], [Blanchet, Carrillo, Masmoudi] $\tau = 0$ and $M = 8\,\pi$
- [Calvez,Corrias] $\tau > 0$, $M < 8\,\pi$, solutions globally exist

does explosion occur as soon as $M > 8\,\pi$, for instance under some additional assumptions like a smallness condition on $\int_{\mathbb{R}^2} |x|^2\, n_0(x)\, \mathrm{d}x$? If $M = 8\,\pi$, $\tau = 0$, there is an infinite number of steady states. If $\tau > 0$?

... Existence of *positive forward self-similar solutions* with *a large total mass* $M \geq 8\,\pi$

Duhamel approach

[Biler 98], [Naito 06]: look for mild solutions of

$$n(t,\cdot) = \mathrm{e}^{(t-t_0)\Delta}\, n(t_0,\cdot) - \int_{t_0}^{t} \left(\nabla \mathrm{e}^{(t-s)\Delta}\right) \cdot \big(n(s,\cdot)\, \nabla c(s,\cdot)\big)\, \mathrm{d}s$$

$$c(t,\cdot) = \mathrm{e}^{\frac{t-t_0}{\tau}\Delta}\, c(t_0,\cdot) + \frac{1}{\tau}\int_{t_0}^{t} \mathrm{e}^{\frac{t-s}{\tau}\Delta}\, n(s,\cdot)\, \mathrm{d}s$$

for any $t > t_0 \geq 0$ ($\tau = 1$)

Method: self-similar solutions are obtained by a fixed point theorem under a smallness condition on M are required in order to apply a contraction mapping principle

PDE approach

$$n(t,x) = \frac{1}{t}\,u\left(\frac{x}{\sqrt{t}}\right) \quad \text{and} \quad c(t,x) = v\left(\frac{x}{\sqrt{t}}\right)$$

With $\xi = x/\sqrt{t}$, the equations for self-similar solutions are

$$\Delta u - \nabla\cdot\left(u\,\nabla v - \frac{1}{2}\,\xi\,u\right) = 0$$
$$\Delta v + \frac{\tau}{2}\,\xi\cdot\nabla v + u = 0$$

Functional space: u and $v \in C^2_0(\mathbb{R}^2) \approx C^2(\mathbb{R}^2)$ such that

$$\lim_{|\xi|\to\infty} u(\xi) = 0 \quad \text{and} \quad \lim_{|\xi|\to\infty} v(\xi) = 0$$

First equation: $\nabla \cdot \left[\mathrm{e}^{v}\, \mathrm{e}^{-|\xi|^2/4}\, \nabla \left(u\, \mathrm{e}^{-v}\, \mathrm{e}^{|\xi|^2/4} \right) \right] = 0$

[Naito, Suzuki, Yoshida 02] u, v, and $|\nabla v|$ are bounded; there exists a constant σ such that

$$u(\xi) = \sigma\, \mathrm{e}^{v(\xi)} \mathrm{e}^{-\frac{|\xi|^2}{4}}$$

The problem is reduced to find a family of nonlinear elliptic equations for v

$$\Delta v + \frac{\tau}{2}\, \xi \cdot \nabla v + \sigma\, \mathrm{e}^{v}\, \mathrm{e}^{-\frac{|\xi|^2}{4}} = 0$$

parametrized by $\sigma > 0$, with $u \in L^1(\mathbb{R}^2)$. By the maximum principle

$$v(\xi) \le C\, \mathrm{e}^{-\min\{1,\tau\} \frac{|\xi|^2}{4}}$$

with $C \min\{1, \tau\} \ge \sigma\, \mathrm{e}^{\|v\|_\infty}$, $v \in L^1(\mathbb{R}^2)$ and $M = \sigma \int_{\mathbb{R}^2} \mathrm{e}^{v(\xi)}\, \mathrm{e}^{-\frac{|\xi|^2}{4}}\, \mathrm{d}\xi$

... a nonlocal problem if parametrized by M

Variational approaches

[Yoshida 01], [Muramoto, Naito, Yoshida 00]
Solutions of

$$\nabla \cdot \left(\mathrm{e}^{\frac{\tau}{4}|\xi|^2} \nabla v \right) + \sigma \, \mathrm{e}^{v} \, \mathrm{e}^{\frac{\tau-1}{4}|\xi|^2} = 0$$

can be found by varaitional approaches in the weighted functional space

$$H^1 \left(\mathbb{R}^2 ; e^{\frac{\tau}{4}|\xi|^2} \, \mathrm{d}\xi \right)$$

If $\tau \in (0, 2)$ and $0 < \sigma < \sigma^*$, for some $\sigma^* > 0$, solutions exist, are positive and belong to $C_0^2(\mathbb{R}^2)$

Radial symmetry

[Naito, Suzuki, Yoshida 02] By the moving planes technique: any positive solution $v \in C_0^2(\mathbb{R}^2)$ must be radially symmetric and

$$u' - u\,v' + \frac{1}{2}\,r\,u = 0$$

$$v'' + \left(\frac{1}{r} + \frac{\tau}{2}\,r\right)\,v' + u = 0$$

where u and v are considered as functions of the radial variable $r = |\xi|$

$$u(r) = \sigma\,\mathrm{e}^{v(r)}\,\mathrm{e}^{-r^2/4},$$

$$v'' + \left(\frac{1}{r} + \frac{\tau}{2}\,r\right)v' + \sigma\,\mathrm{e}^{v}\,\mathrm{e}^{-r^2/4} = 0$$

[Mizutani, Muramoto, Yoshida 99] there exists a positive decreasing solution if $v'(0) = 0$ and $\int_0^\infty r\,v(r)\,\mathrm{d}r < \infty$, $\sigma\,\frac{\log\tau}{\tau-1} < 1/\mathrm{e}$

Equivalent boundary conditions

Natural boundary conditions

$$v'(0) = 0 \quad \text{and} \quad \lim_{r\to\infty} v(r) = 0$$

are equivalent to

$$w'' + \left(\frac{1}{r} + \frac{\tau}{2}\,r\right) w' + \mathrm{e}^{w}\,\mathrm{e}^{-r^2/4} = 0$$

$$w'(0) = 0 \quad \text{and} \quad w(0) = s$$

for some shooting parameter $s \in \mathbb{R}$ with

$$v(r) = w(r;s) - \lim_{r\to\infty} w(r;s)$$

but the range of $M(s) = 2\,\pi \int_0^\infty \mathrm{e}^{w(r;s)}\,\mathrm{e}^{-r^2/4}\,r\,\mathrm{d}r$ is not explicit

Main result

Theorem 2. *For any $M > 0$, there exists $\tilde{\tau}(M) \geq 0$ such that there is at least one solution for any $\tau \geq \tilde{\tau}(M)$*

- *If $M < 8\pi$, $\tilde{\tau}(M) = 0$*
- *If $M > 8\pi$, $\tilde{\tau}(M)$ is positive and there are at least two solutions (except for the maximal possible value of M)*

All solutions are radial, non-increasing, with fast decay at infinity and uniquely determined by $a := \frac{1}{2}\,u(0)$, which defines $M = M(a,\tau)$

- $\lim_{a\to\infty} M(a,\tau) = 8\pi$ *and as $a \to \infty$, $\frac{u}{8\pi}$ concentrates into a Dirac delta distribution*

Large mass positive forward self-similar solutions

The diffusion of c for positive large τ and some $M > 8\,\pi$ may prevent the blowup of the solutions of the parabolic-parabolic Keller–Segel system. This is a major difference with the parabolic-elliptic case $\tau = 0$, for which the response of c to the variations of n being instantaneous, any smooth solution with mass $M > 8\,\pi$ must concentrate and blow up in finite time

There are self-similar solutions with a an arbitrary large mass if τ is large enough

A positive classical solution v solves

$$\left(r\,\mathrm{e}^{\tau\,r^2/4}\,v'\right)' + \sigma\,r\,\mathrm{e}^{(\tau-1)\,r^2/4}\,\mathrm{e}^{v} = 0$$

which, after an integration on $(0,r)$, gives

$$v'(r) = -\frac{\sigma}{r}\,\mathrm{e}^{-\tau\,r^2/4}\int_0^r \mathrm{e}^{(\tau-1)\,z^2/4}\,\mathrm{e}^{v(z)}\,z\,\mathrm{d}z$$

Hence $v(z) \leq v(0)$ for any $z \geq 0$ and, for $\tau \neq 1$

$$v'(r) \geq -\frac{\sigma}{r}\,\mathrm{e}^{-\tau\,r^2/4}\,\mathrm{e}^{v(0)}\int_0^r \mathrm{e}^{(\tau-1)\,z^2/4}\,z\,\mathrm{d}z = -\frac{2}{\tau-1}\,\frac{\sigma}{r}\,\mathrm{e}^{v(0)}\left(\mathrm{e}^{-r^2/4} - \mathrm{e}^{-\tau\,r^2/4}\right)$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty \left(\mathrm{e}^{-r^2/4} - \mathrm{e}^{-\tau\,r^2/4}\right)\frac{2\,\mathrm{d}r}{r} = \int_0^\infty \mathrm{e}^{-\tau\,r^2/4}\,\frac{r}{2}\,\mathrm{d}r = \frac{1}{\tau}$$

There are self-similar solutions with an arbitrary large mass

If $\tau \neq 1$ (extension to $\tau = 1$ is easy), we get the first estimate

$$v(0) \leq \underbrace{\sigma\, \mathrm{e}^{v(0)}}_{=u(0)=2\,a}\; I(\tau) \quad \text{with} \quad I(\tau) := \frac{\log \tau}{\tau - 1}$$

and for any $r \in \mathbb{R}_+$, any $\tau > 0$,

$$0 = \lim_{z \to \infty} v(z) \leq v(r) \leq v(0) \leq 2\,a\,I(\tau)$$

$$M = 2\,\pi\,\sigma \int_0^\infty \mathrm{e}^{v(r)}\,\mathrm{e}^{-r^2/4}\,r\,\mathrm{d}r \geq 2\,\pi\,\sigma \int_0^\infty \mathrm{e}^{-r^2/4}\,r\,\mathrm{d}r = 4\,\pi\,\sigma \geq \widetilde{M}(a,\tau)$$

where $\widetilde{M}(a,\tau) := 8\,\pi\,a\,\mathrm{e}^{-2\,a\,I(\tau)}$ achieves its maximum at $a_*(\tau) := \frac{1}{2\,I(\tau)}$

$$\max_{a>0} M(a,\tau) \geq \widetilde{M}(a_*(\tau),\tau) = \frac{4\,\pi}{\mathrm{e}\,I(\tau)} \to \infty \quad \text{as} \quad \tau \to 0_+$$

Further consequences

- the solution $u(r) = \sigma\,\mathrm{e}^{v(r)}\mathrm{e}^{-r^2/4}$ has mass $M > 8\,\pi$ if $\frac{4\,\pi}{\mathrm{e}\,I(\tau)} > 8\,\pi$
  $I(\bar\tau) = \frac{1}{2\,\mathrm{e}}$, means $\bar\tau \approx 16.1109$
- for any $\tau > \bar\tau$ the density u corresponding to $a = a_*(\tau)$ satisfies
  $u(0) = 2\,a_*(\tau) > 2\,\mathrm{e}$
- by monotonicity $\mathrm{e}^{-v(0)} - \lim_{r\to\infty}\mathrm{e}^{-v(r)} \leq -\,\sigma\,I(\tau)$ for any $\tau > 0$

$$1 - \mathrm{e}^{v(0)} \leq -\,\sigma\,I(\tau)\,\mathrm{e}^{v(0)} = -\,2\,a\,I(\tau)$$

  gives the estimate

$$v(0) > \log(2\,a\,I(\tau) + 1) \to \infty \quad \text{as} \quad a \to +\infty$$

- σ takes arbitrarily large values for τ large enough

$$2\,a\,\mathrm{e}^{-2\,a\,I(\tau)} \leq \sigma \leq \min\left\{\frac{M}{4\,\pi}\,,\,\frac{2\,a}{2\,a\,I(\tau)+1}\right\}$$

Remarks

- Estimates on σ are new
- [Naito, Suzuki, Yoshida 02] analyzed the continuous map $s \mapsto \sigma(s)$ and proved that $\lim_{s\to\pm\infty} \sigma(s) = 0$ so that σ is bounded by $\sigma^* = \sigma(s^*)$, for some $s^* \in \mathbb{R}$, and there is no solution for $\sigma > \sigma^*$, at least one solution for $\sigma = \sigma^*$ and (at least) two distinct solutions for $0 < \sigma < \sigma^*$. However, estimates on σ (or σ^*) were missing

Cumulated densities

Cumulated densities: definition

$$\phi(y) := \frac{1}{2\,\pi} \int_{B(0,\sqrt{y})} u(\xi)\,\mathrm{d}\xi = \int_0^{\sqrt{y}} r\,u(r)\,\mathrm{d}r$$

$$\psi(y) := \frac{1}{2\,\pi} \int_{B(0,\sqrt{y})} v(\xi)\,\mathrm{d}\xi = \int_0^{\sqrt{y}} r\,v(r)\,\mathrm{d}r$$

Using the relations

$$\phi'(y) = \frac{1}{2}\,u\left(\sqrt{y}\right) \quad \text{and} \quad \phi''(y) = \frac{1}{4\,\sqrt{y}}\,u'\left(\sqrt{y}\right)$$

$$\psi'(y) = \frac{1}{2}\,v\left(\sqrt{y}\right) \quad \text{and} \quad \psi''(y) = \frac{1}{4\,\sqrt{y}}\,v'\left(\sqrt{y}\right)$$

ϕ and ψ solve

$$\phi'' + \frac{1}{4}\,\phi' - 2\,\phi'\psi'' = 0$$

$$4\,y\,\psi'' + \tau\,y\,\psi' - \tau\,\psi + \phi = 0$$

As in [Naito, Suzuki, Yoshida 02], [Biler 06], with

$$4\,(y\,\psi' - \psi)' + \tau\,(y\,\psi' - \psi) + \phi = 0$$

$$S(y) := 4\,(\psi(y) - y\,\psi'(y))' = -4\,y\,\psi''(y) = -\sqrt{y}\,v'(\sqrt{y})$$

the system becomes

$$\phi'' + \frac{1}{4}\,\phi' + \frac{1}{2\,y}\,\phi' S = 0$$
$$S' + \frac{\tau}{4}\,S = \phi'$$

A single non-local integro-differential equation

The last formulation of the ODE system can be equivalently written as a single integro-differential equation, hence nonlocal, for ϕ'

$$\phi'' + \frac{1}{4}\,\phi' + \frac{1}{2\,y}\,\phi'\,\mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{\tau\,z/4}\,\phi'(z)\,\mathrm{d}z = 0$$

using

$$S(y) = \mathrm{e}^{-\tau\,y/4}\int_0^y \mathrm{e}^{\tau\,z/4}\,\phi'(z)\,\mathrm{d}z$$

and as a single, local but nonlinear second order ODE for S

$$S'' + \frac{1}{4}\,(\tau+1)\,S' + \frac{\tau}{16}\,S + \frac{1}{2\,y}\left(S\,S' + \frac{\tau}{4}\,S^2\right) = 0$$

Corresponding initial conditions are

$$\phi(0) = 0\,, \quad \phi'(0) = a > 0 \quad \text{and} \quad S(0) = 0$$

Reparametrization

$$\phi(\infty) := \lim_{y\to\infty} \phi(y) = \frac{M(a,\tau)}{2\,\pi}$$

The problem is now formulated in terms of a shooting parameter problem with shooting parameter $a = u(0)/2$

$$2\,a = \mathrm{e}^s\,, \quad s = v(0) + \log\sigma$$

$$v(0) = v(\sqrt{y}) + \frac{1}{2}\int_0^y \frac{S(z)}{z}\,\mathrm{d}z$$

and the boundary condition $\lim_{r\to\infty} v(r) = 0$ is equivalent to

$$v(0) = \frac{1}{2}\int_0^\infty \frac{S(z)}{z}\,\mathrm{d}z$$

We also have: $\sigma = \lim_{r\to\infty} u(r)\,\mathrm{e}^{r^2/4} = 2\lim_{y\to\infty} \phi'(y)\,\mathrm{e}^{\,y/4}$

Detailed results

Mass estimates

Theorem 3. *For any* $(a,\tau)\in\mathbb{R}_+^2$ *there exists a unique positive solution* (ϕ,S) *such that* $\phi\in C^2(0,\infty)\cap C^1[0,\infty)$ *and* $S\in C^1[0,\infty)$. *The maps* $a\in\mathbb{R}_+\mapsto(\phi,S)$ *and* $a\in\mathbb{R}_+\mapsto M(a,\tau)\in\mathbb{R}_+$ *are continuous and*

$$g(a,\tau)\le\frac{M(a,\tau)}{2\,\pi}\le f(a,\tau)$$

where

$$f(a,\tau)=\begin{cases}\min\{4,4\,a\} & \textit{if}\quad \tau\in\left(0,\frac{1}{2}\right]\\ \min\left\{4\,a,\frac{2}{3}\,\pi^2\right\} & \textit{if}\quad \tau\in\left(\frac{1}{2},1\right]\\ \min\left\{4\,a,\frac{2}{3}\,\pi^2\,\tau,4\,(\tau+1)\right\} & \textit{if}\quad \tau>1\end{cases}$$

and

$$g(a,\tau)=\begin{cases}\max\left\{4\,a\,\mathrm{e}^{-2\,a\,\frac{\log\tau}{\tau-1}},\frac{4\,a\,\tau}{a+\tau}\right\} & \textit{if}\quad \tau\in(0,1]\\ \max\left\{4\,a\,\mathrm{e}^{-2\,a\,\frac{\log\tau}{\tau-1}},\frac{4\,a}{a+1}\right\} & \textit{if}\quad \tau>1\end{cases}$$

Concentration

Theorem 4. *Given any fixed* $\tau > 0$*, for any positive sequence* $\{a_k\}$ *such that* $a_k \to \infty$ *as* $k \to \infty$*, there exists a sequence of positive self-similar solutions* $(u_k, v_k) \in (C_0^2(\mathbb{R}^2))^2$ *such that* $u_k(0) = 2\,a_k$*,* $v_k'(0) = 0$ *and*

$$u_k \rightharpoonup 8\,\pi\,\delta_0 \quad \text{as} \quad k \to \infty$$

in the sense of weak convergence of measures.

$$\lim_{k\to\infty} \int_{\mathbb{R}^2} u_k \,\mathrm{d}x = 8\,\pi \quad \text{and} \quad \lim_{k\to\infty} \|v_k\|_{L^\infty(\mathbb{R}^2)} \;=\; \infty$$

This result has already been proved in [Naito, Suzuki, Yoshida 02] using a classical result by Brezis and Merle; in the cumulated densities formulation, we obtain a direct proof

Multiplicity

Theorem 5. *For any fixed $\tau > 0$ there exists $M^* = M^*(\tau) \geq 8\,\pi$ such that, with*

$$\phi(0) = 0\,, \quad \lim_{y\to\infty} \phi(y) = \frac{M}{2\,\pi}\,, \quad S(0) = 0$$

the problem has no positive solution $(\phi, S) \in C^2[0,\infty) \times C^1[0,\infty)$ if $M > M^*$ and has at least one positive solution $(\phi, S) \in C^2[0,\infty) \times C^1[0,\infty)$ in the following cases:

(i) $M \in (0, M^*]$ *if* $M^* > 8\,\pi$

(ii) $M \in (0, M^*)$ *if* $M^* = 8\,\pi$

Moreover, there exist $1/2 < \tau^* \leq \tau^{**}$ such that M^* satisfies: $M^* = 8\,\pi$ if $0 < \tau \leq \tau^*$ and $M^* > 8\,\pi$ if $\tau > \tau^{**}$

When $M^* > 8\,\pi$, there are at least two positive solutions for any $M \in (8\,\pi, M^*)$. When $M^* = 8\,\pi$, it is still an open question to decide if there is a positive solution such that $M = M^*$ or to prove a uniqueness result for any $M \in (0, 8\,\pi)$

The function $M(a,\tau)$ depends non-monotonously on a

Numerical results

Plots in direct variables and in the cumulated densities framework

Bifurcation diagrams

Taylor expansions around $s = 0_+$: for $\varepsilon > 0$ small enough,

$$w(\varepsilon\,;s) \approx s - \frac{1}{4}\,\varepsilon^2\,\mathrm{e}^s \quad \text{and} \quad w'(\varepsilon\,;s) \approx -\frac{1}{2}\,\varepsilon\,\mathrm{e}^s$$

and we obtain $M(s)$ by solving $M'(r) = 2\,\pi\,\mathrm{e}^{w(r;s)}\,\mathrm{e}^{-r^2/4}\,r$ with the approximate initial condition $M(\varepsilon) = \pi\,\varepsilon^2\,\mathrm{e}^s$

Similarly
$S' \sim \phi' \sim a$ on $(0,\varepsilon)$ and so

$$S(y) = a\,y + \mathcal{O}(\varepsilon^2) \quad \text{and} \quad \phi''(y) \sim -\frac{a}{4}\,(1 + 2\,a) + \mathcal{O}(\varepsilon)$$

In practice we solve the equations on $(\varepsilon, y_{\max})$ with the initial data

$$\phi'(\varepsilon) = a - \frac{a}{4}\,(1 + 2\,a)\,\varepsilon\,, \quad \phi(\varepsilon) = a\,\varepsilon - \frac{a}{8}\,(1 + 2\,a)\,\varepsilon^2 \quad \text{and} \quad S(\varepsilon) = a\,\varepsilon$$

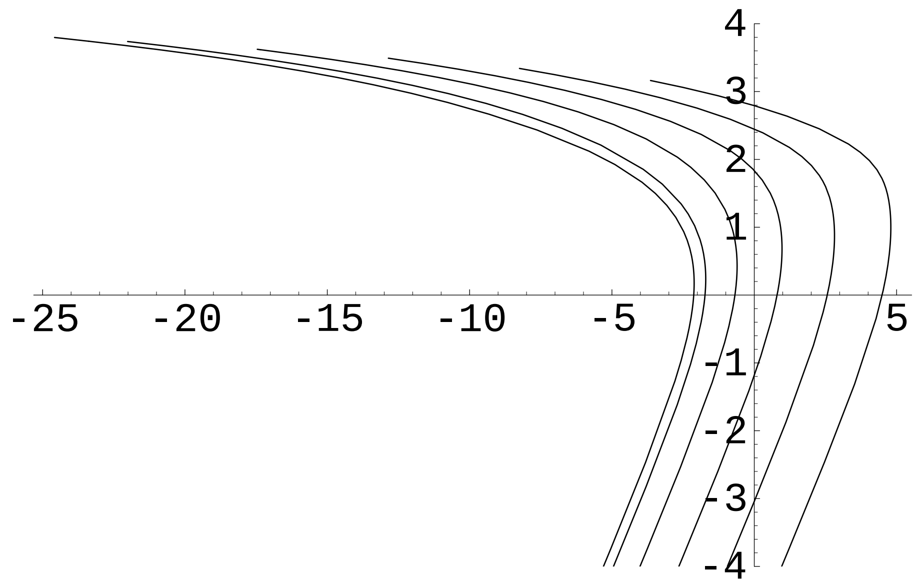


Figure 1: The set of all positive solutions of $\Delta v_\sigma + \frac{\tau}{2}\,\xi \cdot \nabla v_\sigma + \sigma\, \mathrm{e}^{v_\sigma}\, \mathrm{e}^{-|\xi|^2/4} = 0$ in $C_0^2(\mathbb{R}^2)$, where $\sigma = \sigma(s) = \mathrm{e}^{w(\infty;s)}$, is represented by the multivalued diagram $s \mapsto (\log \sigma, \log v_\sigma(0))$ for $\tau = 10^\alpha$, $\alpha = -2, -1, \ldots, 3$. Recall that the solutions v_σ are radial and decreasing so that $v_\sigma(0) = \|v_\sigma\|_{L^\infty(\mathbb{R}^2)}$. We observe that $\max_{s\in\mathbb{R}} \log \sigma(s)$ appears as an increasing function of τ.

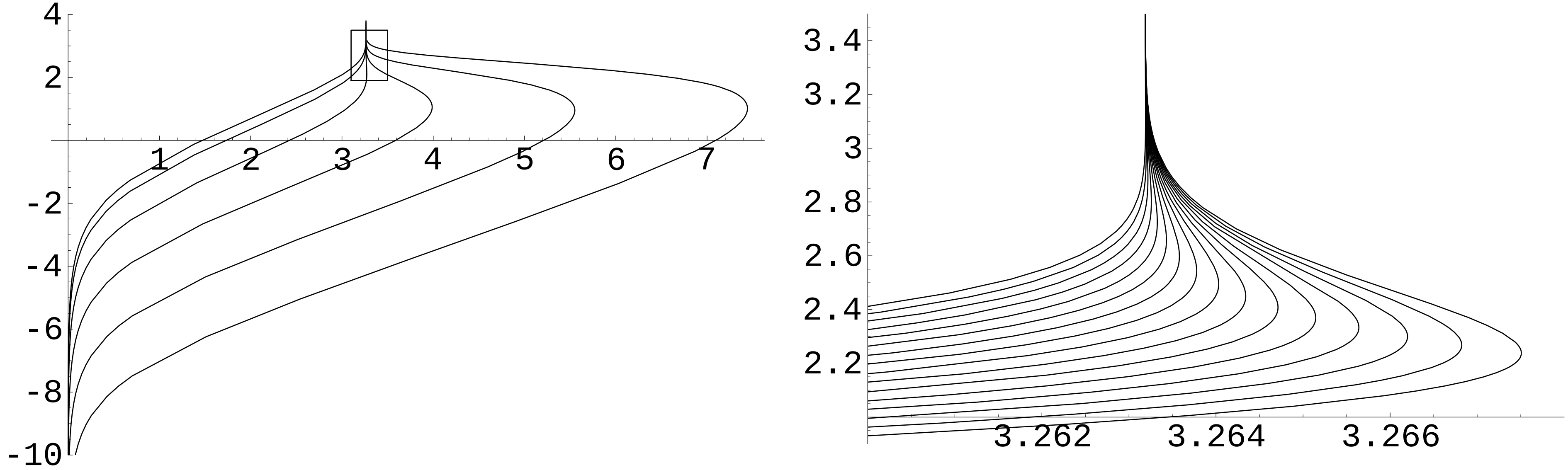
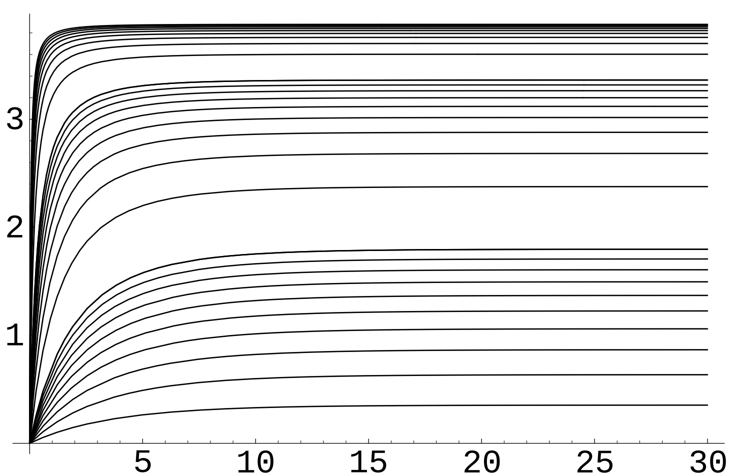


Figure 2: *Left:* The set of all positive solutions of $\Delta v_\sigma + \frac{\tau}{2}\,\xi\cdot\nabla v_\sigma + \sigma\,\mathrm{e}^{v_\sigma}\,\mathrm{e}^{-|\xi|^2/4} = 0$ in $C_0^2(\mathbb{R}^2)$ is now represented by the diagram $s \mapsto (\log(1+M(s)), \log v_\sigma(0))$ for $\tau = 10^\alpha$, $\alpha = -2$, -1, ..., 3. We observe that $\max_{s\in\mathbb{R}} M(s)$ appears as an increasing function of τ.

Right: The plot is an enlargement of the rectangle of Fig. 2 (left), with $\tau = 0.60$, 0.62, 0.64, ..., 0.90. Numerically, the first solution with mass larger than $8\,\pi$ appears for $\tau \in (0.62, 0.64)$, which is far (below) from the theoretical bound. This is not easy to read on the above figure, but it can be shown graphically by enlarging it enough.



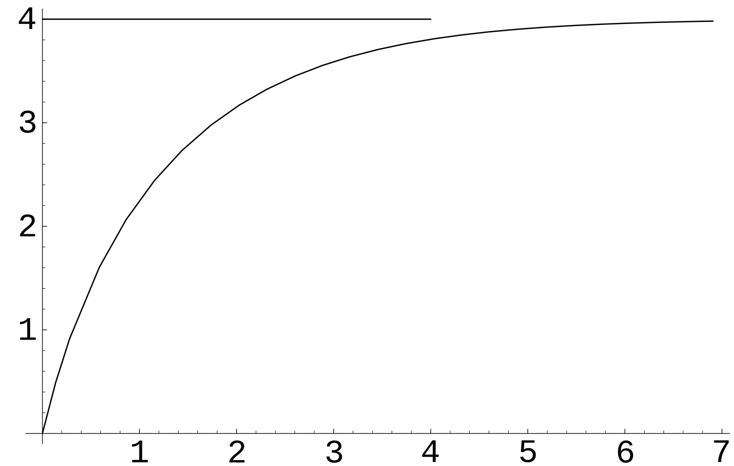


Figure 3: Left: Plots of ϕ for $\phi'(0) = a$, with $a = 10^b\, c$, $b = -1,\, 0,\, 1$, $c \in \{1, \ldots, 10\}$ for $\tau = 0.1$. Right: Plot of $b \mapsto \phi(y_{\max})$ in the logarithmic scale, with $\phi'(0) = a$, $a = \mathrm{e}^b - 1$, $y_{\max} = 30$.

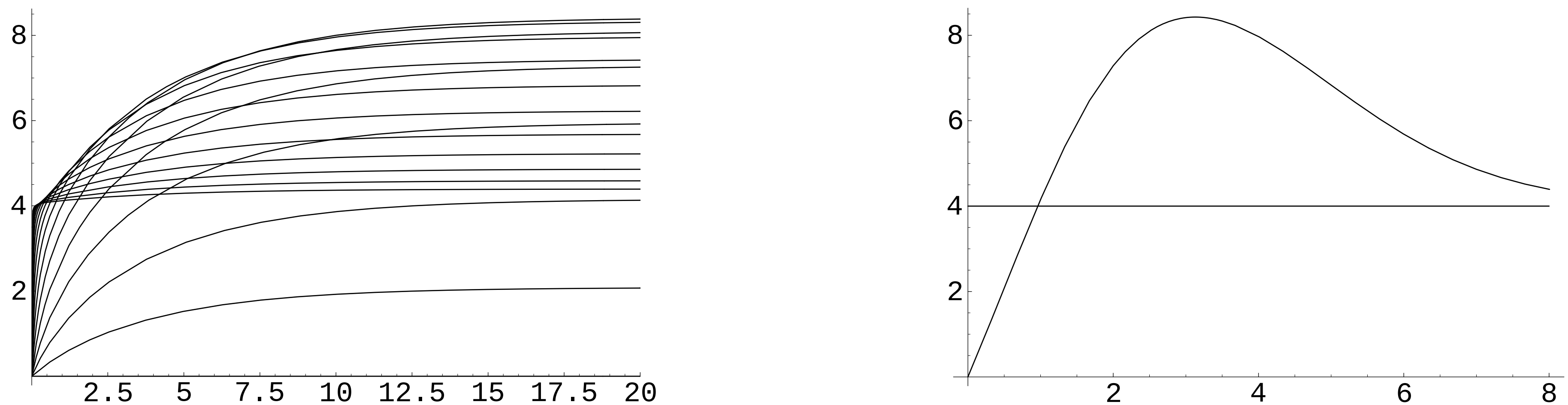


Figure 4: Left: Plots of ϕ for $\phi'(0) = \mathrm{e}^{\alpha}$, with $\alpha = 1, 2, \ldots, 20$ for $\tau = 10$. Right: Plot of $\phi(y_{\max})$ as a function of b (in the logarithmic scale), with $\phi'(0) = a$, $a = \mathrm{e}^{b} - 1$. Here $\tau = 10$, $y_{\max} = 30$.

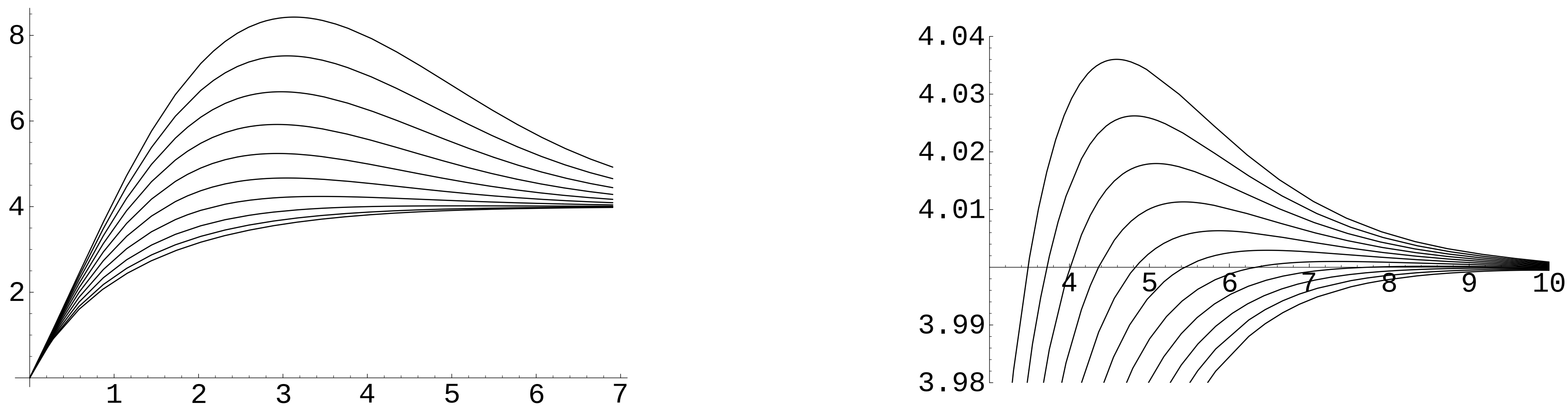


Figure 5: Left: The value of mass $\phi(\infty) = M(a,\tau)/(2\,\pi)$ in the logarithmic scale as a function of a, for $\tau = 0.1\,k^2$ with $k = 1$, 2, ..., 10. Right: An enlargement around the value $M(a,\tau)/(2\,\pi) = 4$ in the logarithmic scale as a function of a, for $\tau = 0.50$, 0.55, 0.60, ..., 1.00.

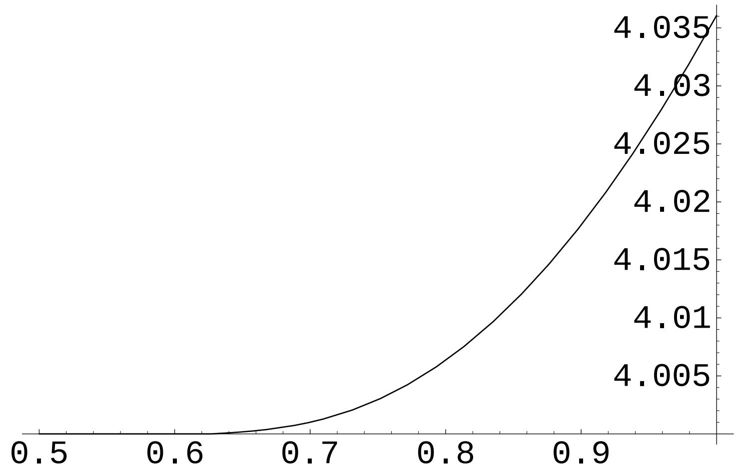


Figure 6: The value of the maximal (in terms of a) mass $\phi(\infty) = M_*(\tau)/(2\pi)$ as a function of τ. Numerically, the first solution with mass larger than 8π appears for $\tau \in (0.62, 0.64)$, as already noticed at the level of Fig. 2 (right). This is again not easy to read on the above figure, but it can be shown graphically by enlarging it enough.

Conclusions

- more than an example of a family of solutions
- self-similar solutions are likely to be attracting a whole class of solutions, although this is still an open question for the parabolic-parabolic Keller–Segel model with large mass (see [Naito, 06] for a result for small mass solutions)
- how to determine the basin of attraction of these self-similar solutions ? Not as simple as in the parabolic-elliptic case. We can conjecture that blowup occurs for mass large enough and even, maybe, as soon as the total mass of the system is above $8\,\pi$ if initial data are sufficiently concentrated.

How do the estimates of such a simple model extend to more realistic ones ?

Thank you for your attention !