

Méthodes d'entropie pour des diffusions linéaires et non-linéaires

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I — Entropy methods for linear diffusions

The logarithmic Sobolev inequality

Convex Sobolev inequalities

I-A. Entropy method for getting the intermediate asymptotics of the heat equation

Consider the heat equation:

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n, t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0 dx = 1 \end{cases} \quad (1)$$

As $t \rightarrow +\infty$, $u(x, t) \sim \mathcal{U}(x, t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$. What is the (optimal) rate of convergence of $\|u(\cdot, t) - \mathcal{U}(\cdot, t)\|_{L^1(\mathbb{R}^n)}$?

The time dependent rescaling

$$u(x, t) = \frac{1}{R^n(t)} v \left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0) \right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{n/2} e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R}$ $R(0) = 1$ $\tau(0) = 0$:

$$R(t) = \sqrt{1 + 2t}, \quad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi v) & \xi \in \mathbb{R}^n, \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0 dx = 1 \end{cases} \quad (2)$$

Entropy (relative to the stationary solution v_∞):

$$\Sigma[v] := \int_{\mathbf{R}^n} v \log \left(\frac{v}{v_\infty} \right) dx$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau} \Sigma[v(\cdot, \tau)] = - \int_{\mathbf{R}^n} v \left| \nabla \log \left(\frac{v}{v_\infty} \right) \right|^2 dx =: -I[v(\cdot, \tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\int_{\mathbf{R}^n} v \log v dx + n \left(1 + \frac{1}{2} \log(2\pi) \right) \leq \frac{1}{2} \int_{\mathbf{R}^n} \frac{|\nabla v|^2}{v} dx$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2} e^{-|\xi|^2/2}$

$$\implies \Sigma[v(\cdot, \tau)] \leq \frac{1}{2} I[v(\cdot, \tau)]$$

$$\Sigma[v(\cdot, \tau)] \leq e^{-2\tau} \Sigma[u_0] = e^{-2\tau} \int_{\mathbf{R}^n} u_0 \log \left(\frac{u_0}{v_\infty} \right) dx$$

- Csiszár-Kullback inequality: Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbf{R}^n} v \, dx = \int_{\mathbf{R}^n} \bar{v} \, dx =: M > 0$

$$\int_{\mathbf{R}^n} v \log \left(\frac{v}{\bar{v}} \right) \, dx \geq \frac{1}{4M} \|v - \bar{v}\|_{L^1(\mathbf{R}^n)}^2$$

$$\implies \|v - v_\infty\|_{L^1(\mathbf{R}^n)}^2 \leq 4M \Sigma[u_0] e^{-2\tau}$$

$$\tau(t) = \log \sqrt{1 + 2t}$$

$$\|u(\cdot, t) - u_\infty(\cdot, t)\|_{L^1(\mathbf{R}^n)}^2 \leq \frac{4}{1 + 2t} \Sigma[u_0]$$

$$u_\infty(x, t) = \frac{1}{R^n(t)} v_\infty \left(\frac{x}{R(t)}, \tau(t) \right)$$

The proof of the Csiszár-Kullback inequality is given by a Taylor development at second order.

Euclidean logarithmic Sobolev inequality: other formulations

1) independent from the dimension [Gross, 75]

$$\int_{\mathbf{R}^n} w \log w \, d\mu(x) \leq \frac{1}{2} \int_{\mathbf{R}^n} w |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x) \, dx$.

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbf{R}^n} w^2 \log w^2 \, dx \leq \frac{n}{2} \log \left(\frac{2}{\pi n e} \int_{\mathbf{R}^n} |\nabla w|^2 \, dx \right)$$

for any $w \in H^1(\mathbf{R}^n)$ such that $\int w^2 \, dx = 1$

Proof: take $w = \sqrt{\frac{v}{M v_\infty}}$ and optimize for $w_\lambda(x) = \lambda^{n/2} w(\lambda x)$

$$\begin{aligned} & \int_{\mathbf{R}^n} |\nabla w_\lambda|^2 dx - \int_{\mathbf{R}^n} w_\lambda^2 \log w_\lambda^2 dx \\ &= \lambda^2 \int_{\mathbf{R}^n} |\nabla w|^2 dx - \int_{\mathbf{R}^n} w^2 \log w^2 dx - n \log \lambda \int_{\mathbf{R}^n} w^2 dx \end{aligned}$$

□

Entropy-entropy production method: a proof of the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau} (I[v(\cdot, \tau)] - 2\Sigma[v(\cdot, \tau)]) = -C \sum_{i,j=1}^n \int_{\mathbf{R}^n} \left| w_{ij} + a \frac{w_i w_j}{w} + b w \delta_{ij} \right|^2 dx < 0$$

for some $C > 0$, $a, b \in \mathbb{R}$. Here $w = \sqrt{v}$.

$$I[v(\cdot, \tau)] - 2\Sigma[v(\cdot, \tau)] \searrow I[v_\infty] - 2\Sigma[v_\infty] = 0$$

$$\implies \forall u_0, \quad I[u_0] - 2\Sigma[u_0] \geq I[v(\cdot, \tau)] - 2\Sigma[v(\cdot, \tau)] \geq 0 \text{ for } \tau > 0$$

I-B. Entropy-entropy production method: improvements of convex Sobolev inequalities

goal: large time behavior of parabolic equations:

$$\begin{cases} v_t = \operatorname{div}_x [D(x) (\nabla_x v + v \nabla_x A(x))] = \operatorname{div} [e^{-A} \nabla (v e^A)] \\ v(x, t = 0) = v_0(x) \in L^1_+(\mathbb{R}^n) \end{cases} \quad t > 0, x \in \mathbb{R}^n \quad (3)$$

$A(x)$... given 'potential'

$v_\infty(x) = e^{-A(x)} \in L^1$... (unique) steady state

mass conservation: $\int_{\mathbb{R}^d} v(t) dx = \int_{\mathbb{R}^d} v_\infty dx = 1$

questions: exponential rate ? connection to logarithmic Sobolev inequalities ? [Bakry-Emery '84, Gross '75, Toscani '96, ...]

• ENTROPY-ENTROPY PRODUCTION METHOD

[Bakry, Emery, 84]

[Arnold, Markowich, Toscani, Unterreiter, 01]

relative entropy of $v(x)$ w.r.t. $v_\infty(x)$:

$$\Sigma[v|v_\infty] := \int_{\mathbf{R}^d} \psi\left(\frac{v}{v_\infty}\right) v_\infty dx \geq 0$$

with $\psi(w) \geq 0$ for $w \geq 0$, convex

$$\psi(1) = \psi'(1) = 0$$

$$(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$$

Examples:

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(v|v_\infty) = \int v \ln\left(\frac{v}{v_\infty}\right) dx \dots$ physical entropy

$\psi_p = w^p - p(w-1) - 1$, $1 < p \leq 2$, $\Sigma_2(v|v_\infty) = \int_{\mathbf{R}^d} (v - v_\infty)^2 v_\infty^{-1} dx$

EXPONENTIAL DECAY OF ENTROPY PRODUCTION

$$I(v(t)|v_\infty) := \frac{d}{dt} \Sigma[v(t)|v_\infty] = - \int \psi''\left(\frac{v}{v_\infty}\right) \underbrace{\left|\nabla \frac{v}{v_\infty}\right|^2}_{=:u} v_\infty dx \leq 0$$

Assume: $D \equiv 1$, $\underbrace{\frac{\partial^2 A}{\partial x^2}}_{\text{Hessian}} \geq \lambda_1 Id$, $\lambda_1 > 0$ ($A(x) \dots$ unif. convex)

entropy production rate:

$$\begin{aligned} I' &= 2 \int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot uv_\infty dx + \underbrace{2 \int \text{Tr}(XY)v_\infty dx}_{\geq 0} \\ &\geq -2\lambda_1 I \end{aligned}$$

with

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij} \left(\frac{\partial u_i}{\partial x_j}\right)^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow |I(t)| \leq e^{-2\lambda_1 t} |I(t=0)| \quad t > 0$$

$\forall v_0$ with $|I(v_0|v_\infty)| < \infty$

EXPONENTIAL DECAY OF RELATIVE ENTROPY

$$\begin{aligned} \text{know:} \quad I' &\geq -2\lambda_1 \underbrace{I}_{=\Sigma'} \quad , \quad \int_t^\infty \dots dt \\ \Rightarrow \quad \Sigma' = I &\leq -2\lambda_1 \Sigma \end{aligned} \quad (4)$$

Theorem 1 [Bakry, Emery], [Arnold, Markowich, Toscani, Unterreiter]

$$\begin{aligned} \frac{\partial^2 A}{\partial x^2} &\geq \lambda_1 Id \quad (\text{“Bakry–Emery condition”}), \quad \Sigma[v_0|v_\infty] < \infty \\ \Rightarrow \Sigma[v(t)|v_\infty] &\leq \Sigma[v_0|v_\infty] e^{-2\lambda_1 t}, \quad t > 0 \end{aligned}$$

$$\|v(t) - v_\infty\|_{L^1}^2 \leq C \Sigma[v(t)|v_\infty] \dots \text{Csiszár-Kullback}$$

CONVEX SOBOLEV INEQUALITIES

Entropy–entropy production estimate (4) for $A(x) = -\ln v_\infty$ uniformly convex:

$$\Sigma[v|v_\infty] \leq \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

Example 1: logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln\left(\frac{v}{v_\infty}\right) dx \leq \frac{1}{2\lambda_1} \int v |\nabla \ln\left(\frac{v}{v_\infty}\right)|^2 dx$$

$$\forall v, v_\infty \in L^1_+(\mathbb{R}^n), \int v dx = \int v_\infty dx = 1$$

logarithmic Sobolev inequality – “entropy version”

Set $f^2 = \frac{v}{v_\infty} \Rightarrow$

$$\int f^2 \ln f^2 dv_\infty \leq \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

$$\forall f \in L^2(\mathbb{R}^n, dv_\infty), \int f^2 dv_\infty = 1$$

logarithmic Sobolev inequality— dv_∞ measure version [Gross '75]

Example 2: non-logarithmic entropies:

$$\psi_p(w) = w^p - p(w - 1) - 1, \quad 1 < p \leq 2$$

$$(B_p) \quad \frac{p}{p-1} \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty \right)^p \right] \leq \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

from (4) with $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} dv_\infty}$ $\forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, v_\infty dx)$

Poincaré-type inequality [Beckner '89], $(B_p) \Rightarrow (B_2), \quad 1 < p \leq 2$

• REFINED CONVEX SOBOLEV INEQUALITIES

Estimate of entropy production rate / entropy production:

$$\begin{aligned}
 I' &= 2 \int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty dx + \underbrace{2 \int \text{Tr}(XY) v_\infty dx}_{\geq 0} \\
 &\geq -2\lambda_1 I
 \end{aligned}$$

[Arnold, JD]: Observation for $\psi_p(w) = w^p - p(w-1) - 1$, $1 < p < 2$:

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''(\frac{v}{v_\infty}) \\ \psi'''(\frac{v}{v_\infty}) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} > 0$$

- Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 Id \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$, $\kappa = \frac{2-p}{p} < 1$

$$\Rightarrow \boxed{k(\Sigma[v|v_\infty]) \leq \frac{1}{2\lambda_1} |\Sigma'|} = \frac{1}{2\lambda_1} \int \psi''\left(\frac{v}{v_\infty}\right) \left|\nabla \frac{v}{v_\infty}\right|^2 dv_\infty$$

“refined convex Sobolev inequality” with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty \Rightarrow$

Theorem 2

$$\begin{aligned} \frac{1}{2} \left(\frac{p}{p-1}\right)^2 \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^{2(p-1)} \left(\int f^2 dv_\infty\right)^{\frac{2-p}{p}} \right] \\ \leq \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, dv_\infty) \end{aligned}$$

“refined Beckner inequality” [Arnold, JD '00]

$$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$$

I-C. An example of application: the flashing ratchet. Long time behavior and dynamical systems interpretation

[J.D., David Kinderlehrer, Michał Kowalczyk]

Flashing ratchet: a simple model for a molecular motor (Brownian motors, molecular ratchets, or Brownian ratchets)

Diffusion tends to spread and dissipate density / transport concentrates density at specific sites determined by the energy landscape: unidirectional transport of mass.

Fokker-Planck type problem

$$\begin{aligned} u_t &= (u_x + \psi_x u)_x, & (x, t) &\in \Omega \times (0, \infty), \\ u_x + \psi_x u &= 0, & (x, t) &\in \partial\Omega \times (0, \infty), \\ u(x, 0) &= u_0(x), & x &\in \Omega, \end{aligned} \tag{5}$$

$$u_0 > 0, \int_{\Omega} u_0 = 1, \psi = \psi(x, t)$$

PERIODIC STATE AND ASYMPTOTIC BEHAVIOUR

Theorem 1 *Let $\psi \in L^\infty([0, T) \times \Omega)$ be a T -periodic potential and assume that there exists a finite partition of $[0, T)$ into intervals $[T_i, T_{i+1})$, $i = 0, \dots, n$ with $T_0 = 0$, $T_n = T$ such that $\psi_{[T_i, T_{i+1})} \in L^\infty([T_i, T_{i+1}), W^{1, \infty}(\Omega))$. Then there exists a unique nonnegative T -periodic solution U to (5) such that $\int_\Omega U(x, t) dx = 1$ for any $t \in [0, T)$.*

$$\sigma_q(u) = \begin{cases} \frac{u^q - 1}{q - 1} & \text{if } q > 1, \\ u \ln u & \text{if } q = 1. \end{cases}$$

Entropy and entropy production :

$$\Sigma_q[u|v] = \int_\Omega \left[\sigma_q \left(\frac{u}{v} \right) - \sigma'_q(1) \left(\frac{u}{v} - 1 \right) \right] v dx$$

$$I_q[u|v] = \int_\Omega \sigma''_q \left(\frac{u}{v} \right) \left| \nabla \left(\frac{u}{v} \right) \right|^2 v dx,$$

Theorem 2 *Let u_1, u_2 be any two solutions to (5).*

$$\Sigma_q[u_1(t)|u_2(t)] \leq e^{-C_q t} \Sigma_q[u_1(0)|u_2(0)],$$

Proposition 3 *Ω is a bounded domain in \mathbb{R}^d with C^1 boundary. Let u and v be two nonnegative functions in $L^1 \cap L^q(\Omega)$ if $q \in (1, 2]$ and in $L^1(\Omega)$ with $u \log u$ and $u \log v$ in $L^1(\Omega)$ ($q = 1$).*

$$\Sigma_q[u|v] \geq 2^{-2/q} q \left[\max \left(\|u\|_{L^q(\Omega)}^{2-q}, \|v\|_{L^q(\Omega)}^{2-q} \right) \right]^{-1} \|u - v\|_{L^q(\Omega)}^2.$$

Corollary 4 *Let $q \in [1, 2]$. Any solution of (5) with initial data $u_0 \in L^1 \cap L^q(0, 1)$ $u_0 \log u_0 \in L^1(0, 1)$ if $q = 1$, converges to $\|u_0\|_{L^1} U(x, t)$, (periodic solution):*

$$\|u(x, t) - \|u_0\|_{L^1} U(x, t)\|_{L^q(0,1;dx)} \leq K e^{-C_{q,\psi} t} \quad \forall t \geq 0 \quad 20$$

$$\begin{aligned}
\frac{d}{dt} \Sigma_1[u|u_\psi] &= \int_{\Omega} \left[1 + \log \left(\frac{u}{u_\psi} \right) \right] u_t \, dx - \int_{\Omega} \frac{u}{u_\psi} u_{\psi,t} \, dx \\
&= -I_1[u|u_\psi] - \int_{\Omega} \frac{u}{u_\psi} u_{\psi,t} \, dx
\end{aligned}$$

Lemma 5 *Let $u \geq 0$ be a solution to (5) such that $\int_{\Omega} u \, dx = 1$. With the above notations, the following estimate holds:*

$$\frac{d}{dt} \Sigma_1[u|u_\psi] \leq -C_\psi \Sigma_1[u|u_\psi] + K_\psi. \tag{6}$$

Fixed-point for the map $\mathcal{T}(u(\cdot, 0)) = u(\cdot, T)$ in

$$\mathcal{Y} = \{u \in H^1(\Omega) \mid u \geq 0, \|u\|_{L^1(\Omega)} = 1, \Sigma_1[u|u_0(\cdot, 0)] \leq K_\psi/C_\psi\}.$$

Flashing potentials: same on each time interval.

Let \mathcal{X} be the set of bounded nonnegative functions u in $L^1 \cap L^q(\Omega)$ (resp. in $L^1(\Omega)$ with $u \log u$ in $L^1(\Omega)$) if $q \in (1, 2]$ (resp. if $q = 1$) such that $\int_{\Omega} u \, dx = 1$.

Theorem 6 *Assume that $v \in \mathcal{X}$ with $0 < m := \inf_{\Omega} v \leq v \leq \sup_{\Omega} v =: M < \infty$. For any $q \in [1, 2]$*

$$\mathcal{J} = \frac{q}{q-1} \inf_{\substack{u \in \mathcal{X} \\ u \neq v \text{ a.e.}}} \frac{I_q[u|v]}{\Sigma_q[u|v]} \quad \text{if } q > 1, \quad \mathcal{J} = \inf_{\substack{u \in \mathcal{X} \\ u \neq v \text{ a.e.}}} \frac{I_1[u|v]}{\Sigma_1[u|v]} \quad \text{if } q = 1 \quad (7)$$

can be estimated by

$$\mathcal{J} \geq 4 \lambda_1(\Omega) \frac{m}{M} \quad (8)$$

where $\lambda_1(\Omega)$ is Poincaré's constant of Ω (with weight 1).

Relation between entropy and entropy production: exponential decay of the relative entropy.

DISSIPATION PRINCIPLE

[Jordan, Kinderlehrer, Otto], [Chipot, Kinderlehrer, Kowalczyk]

Wasserstein distance between Borel probability measures μ, μ^* :

$$d(\mu, \mu^*)^2 = \inf_{p \in \mathcal{P}(\mu, \mu^*)} \int_{\Omega \times \Omega} |x - \xi|^2 p(dx d\xi),$$

$\phi : \Omega \rightarrow \Omega$, $\phi(0) = 0$, $\phi(1) = 1$, strictly increasing continuous

$$\int_{\Omega} \zeta f d\xi = \int_{\Omega} \zeta(\phi(x)) f^*(x) dx, \quad \text{for any } \zeta \in C^0(\Omega).$$

$f = F'$ is the **push forward** of $f^* = (F^*)'$, ϕ is the transfer function. In particular if $\zeta = \chi_{[0,x]}$, then

$$\int_0^{\phi(x)} f(\xi) d\xi = F(\phi(x)) = \int_0^x f^*(x') dx' = F^*(x) \implies \phi = F^{-1} \circ F^* .$$

Wasserstein distance: $d(f, f^*)^2 = \int_{\Omega} |x - \phi(x)|^2 f^*(x) dx$.

[Benamou and Brenier]: convex duality. Differentiating with respect to t and x yields

$$f_\xi(\phi(x, t), t) \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} + f_t(\phi(x, t), t) \frac{\partial \phi}{\partial x} + f(\phi(x, t), t) \frac{\partial^2 \phi}{\partial x \partial t} = 0.$$

We implicitly define a velocity ν by $\nu(\phi, t) = \frac{\partial \phi}{\partial t}$. Using $\frac{\partial^2 \phi}{\partial x \partial t} = \nu_\xi(\phi, t) \frac{\partial \phi}{\partial x}$ we find a continuity equation for $f(x, t)$:

$$f_t + (\nu f)_x = 0, \quad \text{in } \Omega \times (0, \tau).$$

$$d(f^{**}, f^*)^2 = \tau \min_{\nu} \int_0^\tau \int_\Omega \nu(x, t)^2 f(x, t) \, dx dt,$$

where the minimum is taken over all velocities ν such that

$$\begin{aligned} f_t + (\nu f)_x &= 0, & \text{in } \Omega \times (0, \tau), \\ f(x, 0) &= f^*, \quad f(x, \tau) = f^{**}(x) & x \in \Omega. \end{aligned}$$

Free energy functional:

$$F(u) = \int_{\Omega} (\psi u + \sigma u \log u) dx.$$

[Kinderlehrer, Otto, Jordan]: Determine $u^{(k)}$ such that

$$\frac{1}{2}d(u^{(k-1)}, u^{(k)})^2 + \tau F(u^{(k)}) = \min_u \left[\frac{1}{2}d(u^{(k-1)}, u)^2 + \tau F(u) \right]. \quad (9)$$

Then $u_{\tau}(x, t) := u^{(k)}(x)$ if $t \in [k\tau, (k+1)\tau)$, $x \in \Omega$.

(1) There exists a unique solution to the above scheme.

(2) As $\tau \rightarrow 0$, u_{τ} converges strongly in $L^1((0, t) \times \Omega)$ to the unique solution to (5).

Observe that in the limit $\tau \rightarrow 0$, $\nu(x, t) = -(\sigma \log u + \psi)_x$.

After [Kinderlehrer and Walkington], a new numerical scheme, based on the spatial discretization of

$$U_t = u(\log u + \psi)_x.$$

II — Porous media / fast diffusion equation

[coll. Manuel del Pino (Universidad de Chile)]

$$\begin{aligned}
u_t &= \Delta u^m \quad \text{in } \mathbb{R}^n \\
u|_{t=0} &= u_0 \geq 0 \\
u_0(1 + |x|^2) &\in L^1, \quad u_0^m \in L^1
\end{aligned} \tag{10}$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0 dx = 1$, the self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-n/(2-n(1-m))})$ as $t \rightarrow +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t, \cdot) - \mathcal{U}(t, \cdot)\|_{L^\infty} = o(t^{-n/(2-n(1-m))})$$

Rescaling: Take $u(t, x) = R^{-n}(t) v(\tau(t), x/R(t))$ where

$$\dot{R} = R^{n(1-m)-1}, \quad R(0) = 1, \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x v), \quad v|_{\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: *Entropy*

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0$$

$$\frac{d}{d\tau} \Sigma[v] = -I[v], \quad I[v] = \int v \left| \frac{\nabla v^{m-1}}{v} + x \right|^2 dx$$

Stationary solution: C s.t. $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m} |x|^2 \right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$.

$$\Sigma[v] = \int \psi \left(\frac{v^m}{v_\infty^m} \right) v_\infty^{m-1} dx$$

with $\psi(t) = \frac{mt^{1/m}-1}{1-m} + 1$

Theorem 7 $m \in [\frac{n-1}{n}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2 \Sigma[v]$$

$$p = \frac{1}{2m-1}, v = w^{2p}$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$

$$\frac{1}{2} \left(\frac{2m}{2m-1} \right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - n \right) \int |w|^{1+p} dx + K \geq 0$$

$m = \frac{n-1}{n}$: Sobolev, $m \rightarrow 1$: logarithmic Sobolev

[Del Pino, J.D.], [Carrillo, Toscani], [Otto]

Optimal constants for Gagliardo-Nirenberg inequalities [Del Pino, J.D.]

$$1 < p \leq \frac{n}{n-2} \text{ for } n \geq 3$$

$$\|w\|_{2p} \leq A \|\nabla w\|_2^\theta \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi n} \right)^{\frac{\theta}{2}} \left(\frac{2y-n}{2y} \right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{n}{2})} \right)^{\frac{\theta}{n}}$$
$$\theta = \frac{n(p-1)}{p(n+2-(n-2)p)}, \quad y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$. Uses [Serrin-Pucci], [Serrin-Tang].

$\Sigma[v] \leq \Sigma[u_0] e^{-2\tau} +$ Ciszár-Kullback inequalities

\Rightarrow Intermediate asymptotics [Del Pino, J.D.]

(i) $\frac{n-1}{n} < m < 1$ if $n \geq 3$

$$\limsup_{t \rightarrow +\infty} t^{\frac{1-n(1-m)}{2-n(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t \rightarrow +\infty} t^{\frac{1+n(m-1)}{2+n(m-1)}} \| [u - u_\infty] u_\infty^{m-1} \|_{L^1} < +\infty$$

*III – Entropy methods for nonlinear
diffusions*

The logarithmic Sobolev inequality in $W^{1,p}$

[coll. Manuel del Pino (Universidad de Chile), Ivan Gentil (?)]

1. **Comparison techniques:** [Friedman-Kamin, 1980], [Kamin-Vazquez, 1988], ...
[Vazquez, 2002]

2. **Entropy-Entropy production methods:**

- *probability theory:* [Bakry], [Emery], [Ledoux], [Coulhon], ...

- *linear diffusions:* [Toscani], [Arnold-Markowich-Toscani-Unterreiter],
[Otto-Kinderlehrer-Jordan]

- *nonlinear diffusions:* [Carrillo-Toscani], [Del Pino, JD], [Otto], [Juengel-Markowich-Toscani], [Carrillo-Juengel-Markowich-Toscani-Unterreiter], [Markowich-Lederman], [Carrillo-Vazquez]

+ *applications (...)*

3. **Variational approach:** [Gross], [Aubin], [Talenti], [Weissler], [Carlen],
[Carlen-Loss], [Beckner], [Del Pino, JD]

[4. **Mass transportation methods**] [Cordero-Erausquin, Gangbo, Houdré],
[Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]

OPTIMAL CONSTANTS FOR GAGLIARDO-NIRENBERG INEQ.

[Del Pino, J.D.]

Theorem 8 $1 < p < n$, $1 < a \leq \frac{p(n-1)}{n-p}$, $b = p \frac{a-1}{p-1}$

$$\|w\|_b \leq S \|\nabla w\|_p^\theta \|w\|_a^{1-\theta} \quad \text{if } a > p$$

$$\|w\|_a \leq S \|\nabla w\|_p^\theta \|w\|_b^{1-\theta} \quad \text{if } a < p$$

$$\text{Equality if } w(x) = A \left(1 + B |x|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{a-p}}$$

$$a > p: \theta = \frac{(q-p)n}{(q-1)(np - (n-p)q)}$$

$$a < p: \theta = \frac{(p-q)n}{q(n(p-q) + p(q-1))}$$

Proof based on [Serrin, Tang]

NONHOMOGENEOUS VERSION – GAGLIARDO-NIRENBERG INEQ.

$b = \frac{p(p-1)}{p^2-p-1}$, $a = bq$, $v = w^b$. For $p \neq 2$, let

$$\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p dx - \frac{1}{q} \left(\frac{n}{1-\kappa_p} + \frac{p}{p-2} \right) \int v^q dx$$

$$\kappa_p = \frac{1}{p} (p-1)^{\frac{p-1}{p}}$$

Corollary 3 $n \geq 2$, $(2n+1)/(n+1) \leq p < n$. $\forall v$ s.t. $\|v\|_{L^1} = \|v_\infty\|_{L^1}$

$$\mathcal{F}[v] \geq \mathcal{F}[v_\infty]$$

The **optimal L^p -Euclidean logarithmic Sobolev inequality** (an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 9 *If $\|u\|_{L^p} = 1$, then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log [\mathcal{L}_p \int |\nabla u|^p \, dx]$$

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} \left(\frac{\sigma}{p}\right)^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logarithmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

For some purposes, it is sometimes more convenient to use this inequality in a non homogeneous form, which is based upon the fact that

$$\inf_{\mu > 0} \left[\frac{n}{p} \log \left(\frac{n}{p\mu} \right) + \mu \frac{\|\nabla w\|_p^p}{\|w\|_p^p} \right] = n \log \left(\frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{n}{p} .$$

Corollary 10 *For any $w \in W^{1,p}(\mathbb{R}^n)$, $w \neq 0$, for any $\mu > 0$,*

$$p \int |w|^p \log \left(\frac{|w|}{\|w\|_p} \right) dx + \frac{n}{p} \log \left(\frac{p\mu e}{n \mathcal{L}_p} \right) \int |w|^p dx \leq \mu \int |\nabla w|^p dx .$$

Consequences

III-A. Existence and uniqueness: [M. Del Pino, J.D., I. Gentil]

Cauchy problem for $u_t = \Delta_p(u^{1/(p-1)}) \quad (x, t) \in \mathbb{R}^n \times \mathbb{R}^+$

III-B. Applications to nonlinear diffusions: [M. Del Pino, J.D.] intermediate asymptotics for $u_t = \Delta_p u^m$

III-C. Hypercontractivity, Ultracontractivity, Large deviations: [M. Del Pino, J.D., I. Gentil] Connections with $u_t + |\nabla v|^p = 0$

III-A. Existence and uniqueness

[Manuel Del Pino, J.D., Ivan Gentil] Consider the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \geq 0 \end{cases} \quad (11)$$

$\Delta_p u^m = \operatorname{div}(|\nabla u^m|^{p-2} \nabla u^m)$ is 1-homogeneous $\iff m = 1/(p-1)$.

Notations: $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q dx)^{1/q}$, $q \neq 0$. $p^* = p/(p-1)$, $p > 1$.

Theorem 11 *Let $p > 1$, $f \in L^1(\mathbb{R}^n)$ s.t. $|x|^{p^*} f, f \log f \in L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}_t^+, L^1)$ of (11) with initial data f , such that $u^{1/p} \in L_{\text{loc}}^1(\mathbb{R}_t^+, W_{\text{loc}}^{1,p})$.*

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]

[Bernis, 88], [Ishige, 96]

The *a priori* estimate on the entropy term $\int u \log u dx$ plays a crucial role in the proof.

(11) is 1-homogenous: we assume that $\int f dx = 1$. u is a solution of (11) if and only if v is a solution of

$$\begin{cases} v_\tau = \Delta_p v^{1/(p-1)} + \nabla_\xi(\xi v) & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, \tau = 0) = f \end{cases} \quad (12)$$

provided u and v are related by the transformation

$$u(x, t) = \frac{1}{R(t)^n} v(\xi, \tau), \quad \xi = \frac{x}{R(t)}, \quad \tau(t) = \log R(t), \quad R(t) = (1 + pt)^{1/p}$$

[DePino, J.D., 01]. Let

$$v_\infty(\xi) = \pi^{-\frac{n}{2}} \left(\frac{p}{\sigma}\right)^{n/p^*} \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{n}{p^*} + 1)} \exp\left(-\frac{p}{\sigma} |x|^{p^*}\right), \quad \sigma = (p^*)^2$$

$\forall \mu > 0$, μv_∞ is a nonnegative solution of the stationary equation

$$\Delta_p v^{1/(p-1)} + \nabla_\xi(\xi v) = 0$$

In the original variables, t and x : consider $u_\infty = \frac{1}{R(t)^n} v_\infty\left(\frac{x}{R(t)}, \log R(t)\right)$.

$$\int u \log\left(\frac{u}{u_\infty}\right) dx = \int u \log u \, dx + (p-1)(R(t))^{-p^*} \int |x|^{p^*} u \, dx + \sigma(t) \int u \, dx$$

Note that:

$$\frac{d}{dt} \int u \log u \, dx = -\frac{1}{p-1} \int |p^* \nabla u^{1/p}|^p \, dx .$$

Lemma 12 [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz, Saa, 87]

On the space $\{u \in L^1(\mathbb{R}^n) : u^{1/p} \in W^{1,p}(\mathbb{R}^n)\}$, the functional $F[u] := \int |\nabla u^\alpha|^p \, dx$ is convex for any $p > 1$, $\alpha \in [\frac{1}{p}, 1]$.

From $(p-1)\nabla u^{1/(p-1)} = p u^{1/(p(p-1))} \nabla u^{1/p}$, we get by Hölder's inequality (with Hölder exponents p and p^*)

$$\|\nabla u^{1/(p-1)}\|_{p-1} \leq p^* \|u\|_1^{1/(p(p-1))} \|\nabla u^{1/p}\|_p$$

Remark 13 *The entropy decays exponentially since*

$$\frac{d}{dt} \int u \log \left(\frac{u}{\int u dx} \right) dx = -\frac{1}{p-1} \int |p^* \nabla u^{1/p}|^p dx, \text{ and}$$

For any $w \in W^{1,p}(\mathbb{R}^n)$, $w \neq 0$, for any $\mu > 0$,

$$p \int |w|^p \log \left(\frac{|w|}{\|w\|_p} \right) dx + \frac{n}{p} \log \left(\frac{p \mu e}{n \mathcal{L}_p} \right) \int |w|^p dx \leq \mu \int |\nabla w|^p dx.$$

applied with $w = u^{1/p}$, $\mu = \frac{n \mathcal{L}_p}{p e}$, gives

$$\frac{d}{dt} \int u \log \left(\frac{u}{\int u dx} \right) dx \leq -\frac{(p^*)^{p+1} e}{n \mathcal{L}_p} \int u \log \left(\frac{u}{\int u dx} \right) dx .$$

Uniqueness. Consider two solutions u_1 and u_2 of (11).

$$\begin{aligned}
 & \frac{d}{dt} \int u_1 \log \left(\frac{u_1}{u_2} \right) dx \\
 &= \int \left(1 + \log \left(\frac{u_1}{u_2} \right) \right) (u_1)_t dx - \int \left(\frac{u_1}{u_2} \right) (u_2)_t dx \\
 &= -(p-1)^{-(p-1)} \int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2} \right] \cdot \left[\left| \frac{\nabla u_1}{u_1} \right|^{p-2} \frac{\nabla u_1}{u_1} - \left| \frac{\nabla u_2}{u_2} \right|^{p-2} \frac{\nabla u_2}{u_2} \right] dx .
 \end{aligned}$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4 \|f\|_1} \|u_1(\cdot, t) - u_2(\cdot, t)\|_1^2 \leq \int u_1(\cdot, t) \log \left(\frac{u_1(\cdot, t)}{u_2(\cdot, t)} \right) dx \leq \int f \log \left(\frac{f}{f} \right) dx = 0$$

by the Csiszár-Kullback inequality.

III-B. Optimal constants, Optimal rates

[Manuel Del Pino, J.D.]

Intermediate asymptotics for:

$$u_t = \Delta_p u^m$$

Convergence to a stationary solution for:

$$v_t = \Delta_p v^m + \nabla(x v)$$

Let $q = 1 + m - (p - 1)^{-1}$. Whether q is bigger or smaller than 1 determines two different regimes like for $m = 1$.

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \left[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty) \right] dx$$

$$\sigma(s) = \frac{s^q - s}{q - 1} \text{ if } q \neq 1$$

$$\sigma(s) = s \log s \text{ if } q = 1 \text{ (} p = 2 \text{)}$$

[Del Pino, J.D.] Intermediate asymptotics of $u_t = \Delta_p u^m$

Theorem 14 $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ and $q = 1 + m - \frac{1}{p-1}$

$$(i) \quad \|u(t, \cdot) - u_\infty(t, \cdot)\|_q \leq K R^{-(\frac{\alpha}{2} + n(1 - \frac{1}{q}))}$$

$$(ii) \quad \|u^q(t, \cdot) - u_\infty^q(t, \cdot)\|_{1/q} \leq K R^{-\frac{\alpha}{2}}$$

$$(i): \frac{1}{p-1} \leq m \leq \frac{p}{p-1} \quad (ii): \frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$$

$$\alpha = (1 - \frac{1}{p} (p-1)^{\frac{p-1}{p}}) \frac{p}{p-1}, \quad R = (1 + \gamma t)^{1/\gamma}, \quad \gamma = (mn + 1)(p-1) - (n-1)$$

$$u_\infty(t, x) = \frac{1}{R^n} v_\infty(\log R, \frac{x}{R})$$

$$v_\infty(x) = (C - \frac{p-1}{mp} (q-1) |x|^{\frac{p}{p-1}})_+^{1/(q-1)} \text{ if } m \neq \frac{1}{p-1}$$

$$v_\infty(x) = C e^{-(p-1)^2 |x|^{p/(p-1)}/p} \text{ if } m = (p-1)^{-1}.$$

Use $v_t = \Delta_p v^m + \nabla \cdot (x v)$

$$w = v^{(mp+q-(m+1))/p}, \quad a = b q = p \frac{m(p-1)+p-2}{mp(p-1)-1}.$$

Case $q \neq 1$: apply one of the optimal Gagliardo-Nirenberg inequalities.

Case $q = 1$: apply the optimal L^p -Euclidean logarithmic Sobolev inequality.

$$\frac{d\Sigma}{dt} \leq -C \Sigma .$$

Csiszár-Kullback inequality: an extension [Cáceres-Carrillo-JD]

Lemma 15 *Let f and g be two nonnegative functions in $L^q(\Omega)$ for a given domain Ω in \mathbb{R}^n . Assume that $q \in (1, 2]$. Then*

$$\int_{\Omega} \left[\sigma\left(\frac{f}{g}\right) - \sigma'(1)\left(\frac{f}{g} - 1\right) \right] g^q dx \geq \frac{q}{2} \max(\|f\|_{L^q(\Omega)}^{q-2}, \|g\|_{L^q(\Omega)}^{q-2}) \|f - g\|_{L^q(\Omega)}^2$$

III-C. Hypercontractivity, Ultracontractivity, Large deviations

[Manuel Del Pino, J.D., Ivan Gentil]

Understanding the regularizing properties of

$$u_t = \Delta_p u^{1/(p-1)}$$

Theorem 16 *Let $\alpha, \beta \in [1, +\infty]$ with $\beta \geq \alpha$. Under the same assumptions as in the existence Theorem, if moreover $f \in L^\alpha(\mathbb{R}^n)$, any solution with initial data f satisfies the estimate*

$$\|u(\cdot, t)\|_\beta \leq \|f\|_\alpha A(n, p, \alpha, \beta) t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}} \quad \forall t > 0$$

with $A(n, p, \alpha, \beta) = (\mathcal{C}_1 (\beta - \alpha))^{\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}} \mathcal{C}_2^{\frac{n}{p}}$, $\mathcal{C}_1 = n \mathcal{L}_p e^{p-1} \frac{(p-1)^{p-1}}{p^{p+1}}$,

$$\mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}} \beta^{\frac{1-p}{\beta} - \frac{1}{\alpha} + 1}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}} \alpha^{\frac{1-p}{\alpha} - \frac{1}{\beta} + 1}}. \quad \text{Case } p = 2, \mathcal{L}_2 = \frac{2}{\pi n e}, \text{ [Gross 75]}$$

As a special case of Theorem 16, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

Corollary 17 *Consider a solution u with a nonnegative initial data $f \in L^1(\mathbb{R}^n)$. Then for any $t > 0$*

$$\|u(\cdot, t)\|_\infty \leq \|f\|_1 \left(\frac{C_1}{t} \right)^{\frac{n}{p}} .$$

Case $p = 2$, [Varopoulos 85]

Proof. Take a nonnegative function $u \in L^q(\mathbb{R}^n)$ with $u^q \log u$ in $L^1(\mathbb{R}^n)$. It is straightforward that

$$\frac{d}{dq} \int u^q dx = \int u^q \log u dx . \quad (13)$$

Consider now a solution $u_t = \Delta_p u^{1/(p-1)}$. For a given $q \in [1, +\infty)$,

$$\frac{d}{dt} \int u^q dx = -\frac{q(q-1)}{(p-1)^{p-1}} \int u^{q-p} |\nabla u|^p dx . \quad (14)$$

Assume that q depends on t and let $F(t) = \|u(\cdot, t)\|_{q(t)}$. Let $' = \frac{d}{dt}$. A combination of (13) and (14) gives

$$\frac{F'}{F} = \frac{q'}{q^2} \left[\int \frac{u^q}{F^q} \log \left(\frac{u^q}{F^q} \right) dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}} \frac{1}{F^q} \int u^{q-p} |\nabla u|^p dx \right] .$$

Since $\int u^{q-p} |\nabla u|^p dx = \left(\frac{p}{q}\right)^p \int |\nabla u^{q/p}|^p dx$, Corollary 10 applied with $w = u^{q/p}$,

$$\mu = \frac{(q-1)p^p}{q' q^{p-2} (p-1)^{p-1}}$$

gives for any $t \geq 0$

$$F(t) \leq F(0) e^{A(t)} \quad \text{with } A(t) = \frac{n}{p} \int_0^t \frac{q'}{q^2} \log \left(\mathcal{K}_p \frac{q^{p-2} q'}{q-1} \right) ds$$

$$\text{and } \mathcal{K}_p = \frac{n \mathcal{L}_p (p-1)^{p-1}}{e^{p^{p+1}}}.$$

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$q'' q = 2 q'^2 \iff q(t) = \frac{1}{at + b}.$$

Take $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha - \beta}{\alpha\beta}$ and $b = \frac{1}{\alpha}$.

LARGE DEVIATIONS AND HAMILTON-JACOBI EQUATIONS

Consider a solution of

$$\begin{cases} v_t + \frac{1}{p} |\nabla v|^p = \frac{1}{p-1} p^{\frac{2-p}{p-1}} \varepsilon^{p^*} \Delta_p v & (x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, t=0) = g \end{cases} \quad (15)$$

Lemma 18 *Let $\varepsilon > 0$. Then v is a C^2 solution of (15) iff*

$$u = e^{-\frac{1}{\lambda \varepsilon^{p^*}} v} \quad \text{with } \lambda = \frac{p^{\frac{1}{p-1}}}{p-1}$$

is a C^2 positive solution of

$$u_t = \varepsilon^p \Delta_p(u^{1/(p-1)})$$

with initial data $f = e^{-\frac{1}{\lambda \varepsilon^{p^}} g}$.*

Conclusion: The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p dx = 1$,

$$\int |w|^p \log |w| dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \leq \|f\|_\alpha A(n, p, \alpha, \beta) t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p} g\|_\beta \leq \|e^g\|_\alpha B(n, p, \alpha, \beta) t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$