The logarithmic Sobolev inequality in $W^{1,p}$ and related questions

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Part I — Existence and uniqueness: [M. Del Pino, J.D., I. Gentil] Cauchy problem for $u_t = \Delta_p(u^{1/(p-1)}) \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$

Part II — Optimal constants, Optimal rates — A variational approach, with applications to nonlinear diffusions: [M. Del Pino, J.D.] optimal Gagliardo-Nirenberg inequalities, optimal logarithmic Sobolev inequality in $W^{1,p}$

$$\int |u|^p \log |u|\, dx \le \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p\, dx\right]$$

$$\int |u|^p\, dx = 1\,, \quad \mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)}\right]^{\frac{p}{n}}$$

intermediate asymptotics for $u_t = \Delta_p u^m$

Part III — Hypercontractivity, Ultracontractivity, Large deviations: [M. Del Pino, J.D., I. Gentil] Connections with $u_t + |\nabla v|^p = 0$

Part IV — Optimal transport

Part I — Existence and uniqueness

[Manuel Del Pino, J.D., Ivan Gentil]

$$u_t = \Delta_p(u^{1/(p-1)}) \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$$

Existence and uniqueness

Consider the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \geq 0 & \end{cases} \tag{1}$$

$\Delta_p u^m = \mathrm{div}\left(|\nabla u^m|^{p-2} \nabla u^m\right)$ is 1-homogeneous $\Longleftrightarrow m = 1/(p-1)$.
Notations: $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q \, dx)^{1/q}$, $q \neq 0$. $p^* = p/(p-1)$, $p > 1$.

Theorem 1 *Let $p > 1$, $f \in L^1(\mathbb{R}^n)$ s.t. $|x|^{p^*} f$, $f \log f \in L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}_t^+, L^1)$ of (1) with initial data f, such that $u^{1/p} \in L^1_{\mathrm{loc}}(\mathbb{R}_t^+, W^{1,p}_{\mathrm{loc}})$.*

The *a priori* estimate on the *entropy term* $\int u \log u \, dx$ plays a crucial role in the proof.

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]

(1) is 1-homogenous: we assume that $\int f\,dx = 1$. u is a solution of (1) if and only if v is a solution of

$$\begin{cases} v_\tau = \Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, \tau = 0) = f \end{cases} \tag{2}$$

provided u and v are related by the transformation

$$u(x,t) = \frac{1}{R(t)^n}\, v(\xi, \tau)\,,\ \xi = \frac{x}{R(t)}\,,\ \tau(t) = \log R(t)\,,\ R(t) = (1 + p\,t)^{1/p}$$

[DelPino, J.D., 01]. Let

$$v_\infty(\xi) = \pi^{-\frac{n}{2}} (\frac{p}{\sigma})^{n/p^*} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{p^*}+1)}\ \exp(-\frac{p}{\sigma}\,|x|^{p^*})\ , \quad \sigma = (p^*)^2$$

$\forall \mu > 0$, $\mu\, v_\infty$ is a nonnegative solution of the stationary equation

$$\Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) = 0$$

$$\begin{cases} v_\tau = \nabla_\xi \left[v \left(\left| \frac{\nabla_\xi v}{v} \right|^{p-2} \frac{\nabla_\xi v}{v} - \left| \frac{\nabla_\xi v_\infty}{v_\infty} \right|^{p-2} \frac{\nabla_\xi v_\infty}{v_\infty} \right) \right] & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, \tau = 0) = f \end{cases}$$

Regularization of the initial data f:

$$f^{\varepsilon_0} = N_{\varepsilon_0}^{-1} \chi_{\varepsilon_0} * \min(f_0 + \varepsilon_0 v_\infty, {\varepsilon_0}^{-1} v_\infty) \,, \quad \varepsilon_0 \in (0,1)$$

where $\chi_{\varepsilon_0} = {\varepsilon_0}^{-n} \chi(\cdot / \varepsilon_0)$ is a regularizing function. The normalization constant N_{ε_0} is chosen such that $\int f^{\varepsilon_0} \, dx = 1$.
Regularized equation

$$\begin{cases} v_\tau = \nabla \left[v \left(\left[(1-\varepsilon) \left| \frac{\nabla_\xi v}{v + \eta v_\infty} \right|^2 + \frac{\varepsilon}{(1+\eta^2)} \left| \frac{\nabla_\xi v_\infty}{v_\infty} \right|^2 \right]^{\frac{p}{2}-1} \frac{\nabla_\xi v}{v} - \left| \frac{\nabla_\xi v_\infty}{(1+\eta) v_\infty} \right|^{p-2} \frac{\nabla_\xi v_\infty}{v_\infty} \right) \right] \\ v(\cdot, \tau = 0) = f^{\varepsilon_0} \end{cases}$$

for some ϵ and $\eta > 0$. Note that v_∞ is a stationary solution.
Solution: $v_{\varepsilon,\eta}^{\varepsilon_0}$

Apply the standard theory [Ladyzenskaja, Solonnikov, Uralceva, 67] to $v_\tau = \nabla_\xi \cdot [a(\xi, v, \nabla_\xi v)]$ first on a bounded domain (a large centered ball B_R of radius R, with Dirichlet boundary conditions $v = v_\infty$ on ∂B_R) and then let $R \to +\infty$.

The solution is smooth and the Maximum Principle applies. The functions $\varepsilon_0 N_{\varepsilon_0}^{-1} v_\infty$ and $(\varepsilon_0 N_{\varepsilon_0})^{-1} v_\infty$ are respectively lower and upper stationary solutions:

$$\frac{\varepsilon_0}{N_{\varepsilon_0}} v_\infty(\xi) \le v_{\varepsilon,\eta}^{\varepsilon_0}(\tau,\xi) \le \frac{1}{\varepsilon_0 \, N_{\varepsilon_0}} v_\infty(\xi) \quad \forall\,(\xi,\tau) \in \mathbb{R}^n \times \mathbb{R}^+$$

uniformly with respect to ε, $\eta > 0$: let $\eta \to 0$ and keep the above estimate. A similar uniform in ε and η (but local in ξ) estimate holds for $(v_{\varepsilon,\eta}^{\varepsilon_0})^{-1} \left| \nabla_\xi v_{\varepsilon,\eta}^{\varepsilon_0} \right|$.

Entropy estimate:

$$\frac{d}{d\tau}\int v^{\varepsilon_0}_{\varepsilon,0}\log\left(\frac{v^{\varepsilon_0}_{\varepsilon,0}}{v_\infty}\right)d\xi = -\int\left[\frac{\nabla_\xi v^{\varepsilon_0}_{\varepsilon,0}}{v^{\varepsilon_0}_{\varepsilon,0}}-\frac{\nabla_\xi v_\infty}{v_\infty}\right]$$

$$\cdot\left[v^{\varepsilon_0}_{\varepsilon,0}\left(\left[(1-\varepsilon)\left|\frac{\nabla_\xi v^{\varepsilon_0}_{\varepsilon,0}}{v^{\varepsilon_0}_{\varepsilon,0}}\right|^2+\varepsilon\left|\frac{\nabla_\xi v_\infty}{v_\infty}\right|^2\right]^{\frac{p}{2}-1}\frac{\nabla_\xi v^{\varepsilon_0}_{\varepsilon,0}}{v^{\varepsilon_0}_{\varepsilon,0}}-\left|\frac{\nabla_\xi v_\infty}{v_\infty}\right|^{p-2}\frac{\nabla_\xi v_\infty}{v_\infty}\right)\right]d\xi$$

($v^{\varepsilon_0}_{\varepsilon,0}$ converges to v_∞ as $\tau\to+\infty$). Let $\varepsilon\to 0$:

$$\int v^{\varepsilon_0}\log\left(\frac{v^{\varepsilon_0}}{v_\infty}\right)d\xi \le \int f^{\varepsilon_0}\log\left(\frac{f^{\varepsilon_0}}{v_\infty}\right)d\xi$$
$$-\int_0^\tau\int v^{\varepsilon_0}\left(\frac{\nabla v^{\varepsilon_0}}{v^{\varepsilon_0}}-\frac{\nabla v_\infty}{v_\infty}\right)\cdot\left(\left|\frac{\nabla v^{\varepsilon_0}}{v^{\varepsilon_0}}\right|^{p-2}\frac{\nabla v^{\varepsilon_0}}{v^{\varepsilon_0}}-\left|\frac{\nabla v_\infty}{v_\infty}\right|^{p-2}\frac{\nabla v_\infty}{v_\infty}\right)d\xi\,d\tau\,,$$

$$\begin{cases} v^{\varepsilon_0}{}_\tau = \nabla_\xi \left[v^{\varepsilon_0} \left(\left| \frac{\nabla_\xi v^{\varepsilon_0}}{v^{\varepsilon_0}} \right|^{p-2} \frac{\nabla_\xi v^{\varepsilon_0}}{v^{\varepsilon_0}} - \left| \frac{\nabla_\xi v_\infty}{v_\infty} \right|^{p-2} \frac{\nabla_\xi v_\infty}{v_\infty} \right) \right] \\ v^{\varepsilon_0}(\cdot, \tau = 0) = f^{\varepsilon_0} \to f \quad \text{as } \varepsilon_0 \to 0 \end{cases}$$

Go back to the original variables, t and x: consider $u_\infty = \frac{1}{R(t)^n} v_\infty(\frac{x}{R(t)}, \log R(t))$.

$$\int u \log\left(\frac{u}{u_\infty}\right) dx = \int u \log u \, dx + (p-1)(R(t))^{-p^*} \int |x|^{p^*} u \, dx + \sigma(t) \int u \, dx$$

Pass to the limit as $\varepsilon_0 \to 0$ in the entropy inequality, *i.e.*,

$$\frac{d}{dt} \int u^{\varepsilon_0} \log u^{\varepsilon_0} \, dx = -\frac{1}{p-1} \int \left| p^* \nabla (u^{\varepsilon_0})^{1/p} \right|^p dx \,.$$

Lemma 2 [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz,Saa, 87] *On the space* $\{u \in L^1(\mathbb{R}^n) \; : \; u^{1/p} \in W^{1,p}(\mathbb{R}^n)\}$*, the functional* $F[u] := \int |\nabla u^\alpha|^p \, dx$ *is convex for any* $p > 1$*,* $\alpha \in [\frac{1}{p}, 1]$*.*

Existence (end). From $(p-1)\nabla u^{1/(p-1)} = p\, u^{1/(p(p-1))}\nabla u^{1/p}$, we get by Hölder's inequality (with Hölder exponents p and p^*)

$$\|\nabla u^{1/(p-1)}\|_{p-1} \leq p^* \|u\|_1^{1/(p(p-1))}\, \|\nabla u^{1/p}\|_p$$

Remark 3 *The entropy decays exponentially since*
$\frac{d}{dt}\int u \log\left(\frac{u}{\int u\, dx}\right) dx = -\frac{1}{p-1}\int |p^*\, \nabla u^{1/p}|^p\, dx$

For any $w \in W^{1,p}(\mathbb{R}^n)$, $w \neq 0$, *for any* $\mu > 0$,
$$p\int |w|^p \log\left(\frac{|w|}{\|w\|_p}\right)\, dx + \frac{n}{p}\log\left(\frac{p\,\mu\, e}{n\,\mathcal{L}_p}\right)\int |w|^p\, dx \leq \mu \int |\nabla w|^p\, dx.$$

applied with $w = u^{1/p}$, $\mu = \frac{n\,\mathcal{L}_p}{p\,e}$ *gives*

$$\frac{d}{dt}\int u \log\left(\frac{u}{\int u\, dx}\right) dx \leq -\frac{(p*)^{p+1} e}{n\,\mathcal{L}_p}\int u \log\left(\frac{u}{\int u\, dx}\right) dx\ .$$

Uniqueness. Consider two solutions u_1 and u_2 of (1).

$$\frac{d}{dt}\int u_1 \log\left(\frac{u_1}{u_2}\right) dx$$
$$= \int \left(1 + \log\left(\frac{u_1}{u_2}\right)\right) (u_1)_t \, dx - \int \left(\frac{u_1}{u_2}\right) (u_2)_t \, dx$$
$$= -(p-1)^{-(p-1)} \int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right] \cdot \left[\left|\frac{\nabla u_1}{u_1}\right|^{p-2} \frac{\nabla u_1}{u_1} - \left|\frac{\nabla u_2}{u_2}\right|^{p-2} \frac{\nabla u_2}{u_2}\right] dx \,.$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4\,\|f\|_1} \|u_1(\cdot,t) - u_2(\cdot,t)\|_1^2 \le \int u_1(\cdot,t) \log\left(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\right) dx \le \int f \log\left(\frac{f}{f}\right) dx = 0$$

by the Csiszár-Kullback inequality.

Part II — Optimal constants, Optimal rates A variational approach, with applications to nonlinear diffusions

[Manuel Del Pino, J.D.]

Intermediate asymptotics for:

$$u_t = \Delta_p u^m$$

A brief review of the literature

1. Comparison techniques: [Friedman-Kamin, 1980], [Kamin-Vazquez,1988],...

2. Entropy-Entropy production methods:
• *probability theory:* [Bakry], [Emery], [Ledoux], [Coulhon],...
• *linear diffusions:* [Toscani], [Arnold-Markowich-Toscani-Unterreiter]
• *nonlinear diffusions:* [Carrillo-Toscani], [Del Pino, JD], [Otto], [Juengel-Markowich-Toscani], [Carrillo-Juengel-Markowich-Toscani-Unterreiter]
+ *applications* (...)

3. Variational approach: [Gross], [Aubin], [Talenti], [Weissler], [Carlen], [Carlen-Loss], [Beckner], [Del Pino, JD]

[4. Mass transportation methods] [Cordero-Eruasquin, Gangbo, Houdré], [Cordero-Eruasquin, Nazaret, Villani]

Ground states

A 3 steps strategy for the study of *nonlinear (local) scalar fields equations.* The original goal is to identify all the **minimizers** of an energy functional:

1. Get *a priori* **qualitative properties**: positivity, decay at infinity, regularity, Euler-Lagrange equations [1960→] Standard results are easy

2. Characterize the symmetry of the **solutions**
[Serrin], [Gidas-Ni-Nirenberg], [Serrin], [~1979→]
using moving planes techniques: [JD-Felmer, 1999],
[Damascelli-Pacella-Ramaswamy, 2000→]
or elaborate symmetrization methods: [Brock, 2001]

3. Prove the **uniqueness** of the solutions of an ODE (radial solutions): [Coffman, 1972], [Peletier-Serrin, 1986], (...) [Erbe-Tang, 1997], [Pucci-Serrin, 1998], [Serrin-Tang, 2000]

"Ground states" is now used in the literature to qualify postive solutions dacaying to zero at ∞.

Intermediate asymptotics (case $m = 1$)

$$u_t = \Delta_p u \quad \text{in } \mathbb{R}^n$$

Theorem 1 $n \geq 2$, $\frac{2n+1}{n+1} \leq p < n$, $q = 2 - \frac{1}{p-1}$
initial data: $u_0 \geq 0$ *in* $L^1 \cap L^\infty$, $|x|^{p/(p-1)} u_0 \in L^1$
$u_0^q \in L^1$ *in case* $p < 2$
$\forall\, s > 1$ $\exists K > 0$ $\forall\, t > 0$

$$\|u(t,\cdot) - u_\infty(t,\cdot)\|_s \leq K\, R^{-(\frac{\alpha}{2}\frac{q}{s} + n(1-\frac{1}{s}))} \quad \textit{if } p \geq 2\,,\ s \geq q$$

$$\|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_s \leq K\, R^{-(\frac{\alpha}{2qs} + n(q-\frac{1}{s}))} \quad \textit{if } p \leq 2\,,\ s \geq \frac{1}{q}$$

$$\alpha = \left(1 - \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}\right) \frac{p}{p-1}$$

$$u_\infty(t,x) = R^{-n}\, v_\infty(\log R, R^{-1}x)$$

$R = R(t) = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (n+1)p - 2n$,
$v_\infty(x) = (C - \frac{p-2}{p}\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ if p$\neq$ 2
$v_\infty(x) = C\, e^{-|x|^2/2}$ if $p = 2$.

Existence results, uniform convergence for large time: [diBenedetto, 1993].

Time-dependent rescaling: $u(t,x) = R^{-n}\, v(\log R, \frac{x}{R})$

$$u_t = \Delta_p u \Longleftrightarrow v_t = \Delta_p v + \nabla \cdot (x\, v) \qquad (3)$$

v_∞ is a stationary, nonnegative radial and nonincreasing solution.

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \Big[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)\Big]\, dx$$

$\sigma(s) = \frac{s^q - 1}{q-1}$ if $q \neq 1$
$\sigma(s) = s \log s$ if $q = 1$ $(p = 2)$
$\frac{d\Sigma}{dt} = -q\,(I_1 + I_2 + I_3 + I_4)$ where

$$I_1 = \int v^{-\frac{1}{p-1}}\,|\nabla v|^p\, dx, \qquad I_2 = \int |x|^{\frac{p}{p-1}} v\, dx$$
$$I_3 = -\frac{d}{q}\int v^q\, dx, \qquad I_4 = \int |\nabla v|^{p-2}\nabla v \cdot |x|^{\frac{1}{p-1}-1} x\, dx$$

Heat equation $(p = 2)$: [Toscani, 1997]. From now on: $p \neq 2$

Lemma 4 *Let* $\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$

$$\frac{1}{q}\frac{d\Sigma}{dt} \leq -(1 - \kappa_p)(I_1 + I_2) - I_3$$

Optimal constants for Gagliardo-Nirenberg ineq. [Del Pino, J.D.]

Theorem 5 $1<p<n$, $1<a\le\frac{p(n-1)}{n-p}$, $b=p\,\frac{a-1}{p-1}$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \quad \textit{if } a>p$$
$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \quad \textit{if } a<p$$
$$\textit{Equality if } w(x)=A\,(1+B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$$

$a>p$: $\theta=\frac{(q-p)\,n}{(q-1)\,(n\,p-(n-p)\,q)}$

$a<p$: $\theta=\frac{(p-q)\,n}{q\,(n\,(p-q)+p\,(q-1))}$

Proof based on [Serrin, Tang]

$b = \frac{p\,(p-1)}{p^2-p-1}$, $a = b\,q$, $v = w^b$. For $p \neq 2$, let

$$\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p \, dx - \frac{1}{q} \left(\frac{n}{1-\kappa_p} + \frac{p}{p-2} \right) \int v^q \, dx$$

$\Sigma \leq \frac{q}{1-\kappa_p} \frac{p-1}{p} \left[(1-\kappa_p)(I_1 + I_2) + I_3 \right]$
The inequality is autonomous.

Corollary 2 *$n \geq 2$, $(2n+1)/(n+1) \leq p < n$. $\forall\, v$ s.t. $\|v\|_{L^1} = \|v_\infty\|_{L^1}$*

$$\mathcal{F}[v] \geq \mathcal{F}[v_\infty]$$

Corollary 3 *With $\alpha = (1-\kappa_p)\,\frac{p}{p-1}$, for any $t > 0$, $\Sigma(t) \leq e^{-\alpha\, t}\,\Sigma(0)$.*

A variant of the Csiszár-Kullback inequality [Caceres-Carrillo-JD]

Lemma 6 *Let f and g be two nonnegative functions in $L^q(\Omega)$ for a given domain Ω in $\mathbb{R}^n$. Assume that $q \in (1,2]$. Then*

$$\int_\Omega [\sigma(\frac{f}{g}) - \sigma'(1)(\frac{f}{g} - 1)]\, g^q \, dx \geq \frac{q}{2} \max(\|f\|^{q-2}_{L^q(\Omega)}, \|g\|^{q-2}_{L^q(\Omega)})\, \|f - g\|^2_{L^q(\Omega)}$$

End of the proof

For $q > 1$ ($p > 2$): $\sigma(s) = \frac{s^q - 1}{q-1}$

For $q < 1$ ($p < 2$)

$\frac{s^q-1}{q-1} - \frac{q}{q-1}(s-1) = (\frac{1}{q} - 1)^{-1}[(s^q)^{\frac{1}{q}} - 1 - \frac{1}{q}((s^q) - 1)]$

Extension to other nonlinear diffusions

$$u_t = \Delta_p u^m \tag{4}$$

Formal in the sense that apparently a complete existence theory for such an equation is not yet available, except in the special cases $p = 2$, $m = 1$ or $m = 1/(p-1)$. Let

$$q = 1 + m - (p-1)^{-1}$$

Whether q is bigger or smaller than 1 determines two different regimes like for $m = 1$ (depending if p is bigger or smaller than 2).

(H) the solution corresponding to a given nonnegative initial data u_0 in $L^1 \cap L^\infty$, such that $u_0^q \in L^1$ (in case $m < \frac{1}{p-1}$) and $\int |x|^{\frac{p}{p-1}} u_0 \, dx < +\infty$, is well defined for any $t > 0$, belongs to $C^0(\mathbb{R}^+; L^1(\mathbb{R}^n, (1+|x|^{\frac{p}{p-1}})\, dx) \cap L^\infty(\mathbb{R}^+ \times \mathbb{R}^n)$, such that u^q and $t \mapsto \int u^{-\frac{1}{p-1}} |\nabla u|^p \, dx$ belong to $C^0(\mathbb{R}^+; L^1(\mathbb{R}^n)$ and $L^1_{\rm loc}(\mathbb{R}^+)$ respectively.

[Del Pino, J.D.] Intermediate asymptotics of $u_t = \Delta_p u^m$

Theorem 7 $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

$$\text{(i)} \quad \|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-\left(\frac{\alpha}{2}+n\left(1-\frac{1}{q}\right)\right)}$$

$$\text{(ii)} \quad \|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$ *(ii)*: $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$

$\alpha = (1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (mn+1)(p-1) - (n-1)$
$u_\infty(t,x) = \frac{1}{R^n}\, v_\infty(\log R, \frac{x}{R})$
$v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ *if* $m \neq \frac{1}{p-1}$
$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.

Use $v_t = \Delta_p v^m + \nabla\cdot(x\, v)$
$w = v^{(mp+q-(m+1))/p}$, $a = b\, q = p\,\frac{m(p-1)+p-2}{mp(p-1)-1}$.

Gagliardo-Nirenberg inequalities

$$1 < p < n, \; 1 < a \le \frac{p(n-1)}{n-p}, \; b = p\,\frac{a-1}{p-1}$$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_a^{1-\theta} \quad \text{if } a > p$$

$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^\theta\,\|w\|_b^{1-\theta} \quad \text{if } a < p$$

Equality if $w(x) = A\,(1 + B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$

$a > p$: $\theta = \frac{(q-p)\,n}{(q-1)\,(n\,p-(n-p)\,q)}$ $\qquad$ $a < p$: $\theta = \frac{(p-q)\,n}{q\,(n\,(p-q)+p\,(q-1))}$

The case $q = 1$ (which corresponds to $m = (p-1)^{-1} \Longleftrightarrow a = p$) is a limiting case, for which we can use the generalization to $W^{1,p}$ of the logarithmic Sobolev inequality.

The optimal L^p-Euclidean logarithmic Sobolev inequality (an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 8 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx\right]$$

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)}\right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)}\right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 79]

$p = 1$: [Ledoux 96], [Beckner, 99]

Part III — Hypercontractivity, Ultracontractivity, Large deviations

[Manuel Del Pino, J.D., Ivan Gentil]

Understanding the regularizing properties of

$$u_t = \Delta_p u^{1/(p-1)}$$

Hypercontractivity

Theorem 9 *Let α, $\beta \in [1,+\infty]$ with $\beta \geq \alpha$. Under the same assumptions as in the existence Theorem, if moreover $f \in L^\alpha(\mathbb{R}^n)$, any solution with initial data f satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta)\, t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}} \quad \forall\, t > 0$$

with

$$A(n,p,\alpha,\beta) = \left(\mathcal{C}_1\,(\beta-\alpha)\right)^{\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}\,,$$

$$\mathcal{C}_1 = n\,\mathcal{L}_p\,e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}\,, \quad \mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}\,.$$

Case $p = 2$, $\mathcal{L}_2 = \frac{2}{\pi\,n\,e}$, [Gross 75]

Ultracontractivity

As a special case of Theorem 9, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

Corollary 10 *Consider a solution u with a nonnegative initial data $f \in L^1(\mathbb{R}^n)$. Then for any $t > 0$*

$$\|u(\cdot,t)\|_\infty \le \|f\|_1 \left(\frac{\mathcal{C}_1}{t}\right)^{\frac{n}{p}} .$$

Case $p = 2$, [Varopoulos 85]

Main tool: optimal L^p-Euclidean log-Sobolev inequality

Theorem 11 *Let $p \in (1, +\infty)$. Then for any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$ we have,*

$$\int |w|^p \log |w|^p \, dx \le \frac{n}{p} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right] ,$$

with $\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e} \right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n \frac{p-1}{p}+1)} \right]^{\frac{p}{n}}$. The inequality is optimal and it is an equality if

$$w(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)} \right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}} \quad \forall\, x \in \mathbb{R}^n$$

for any $p > 1$, $\sigma > 0$ and $\bar{x} \in \mathbb{R}^n$. For $p \in (1, n)$ the equality holds only if w takes the above form.

For our purpose, it is more convenient to use this inequality in a non homogeneous form, which is based on the fact that

$$\inf_{\mu>0}\left[\frac{n}{p}\log\left(\frac{n}{p\mu}\right)+\mu\,\frac{\|\nabla w\|_p^p}{\|w\|_p^p}\right]=n\,\log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right)+\frac{n}{p}\,.$$

Corollary 12 *For any* $w\in W^{1,p}(\mathbb{R}^n)$, $w\neq 0$, *for any* $\mu>0$,

$$p\int|w|^p\log\left(\frac{|w|}{\|w\|_p}\right)\,dx+\frac{n}{p}\log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right)\int|w|^p\,dx\le\mu\int|\nabla w|^p\,dx\,.$$

Proof

Take a nonnegative function $u \in L^q(\mathbb{R}^n)$ with $u^q \log u$ in $L^1(\mathbb{R}^n)$. It is straightforward that

$$\frac{d}{dq} \int u^q \, dx = \int u^q \log u \; dx \; . \qquad (5)$$

Consider now a solution $u_t = \Delta_p u^{1/(p-1)}$. For a given $q \in [1, +\infty)$,

$$\frac{d}{dt} \int u^q \, dx = -\frac{q(q-1)}{(p-1)^{p-1}} \int u^{q-p} |\nabla u|^p \, dx \; . \qquad (6)$$

Assume that q depends on t and let $F(t) = \|u(\cdot, t)\|_{q(t)}$. Let $' = \frac{d}{dt}$. A combination of (5) and (6) gives

$$\frac{F'}{F} = \frac{q'}{q^2} \left[\int \frac{u^q}{F^q} \log\left(\frac{u^q}{F^q}\right) dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}} \frac{1}{F^q} \int u^{q-p} |\nabla u|^p \, dx \right] .$$

Since $\int u^{q-p}|\nabla u|^p\,dx = (\frac{p}{q})^p \int |\nabla u^{q/p}|^p\,dx$, Corollary 12 applied with $w = u^{q/p}$,

$$\mu = \frac{(q-1)\,p^p}{q'\,q^{p-2}\,(p-1)^{p-1}}$$

gives for any $t \geq 0$

$$F(t) \leq F(0)\,e^{A(t)} \qquad \text{with } A(t) = \frac{n}{p}\int_0^t \frac{q'}{q^2}\log\left(\mathcal{K}_p\,\frac{q^{p-2}\,q'}{q-1}\right)ds$$

$$\text{and } \mathcal{K}_p = \frac{n\,\mathcal{L}_p}{e}\,\frac{(p-1)^{p-1}}{p^{p+1}}\,.$$

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$q''\,q = 2\,{q'}^2 \Longleftrightarrow q(t) = \frac{1}{a\,t+b}\,.$$

Take $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha-\beta}{\alpha\beta}$ and $b = \frac{1}{\alpha}$.

$$A(t) = -\frac{n}{p} \int_0^t a \log \left(\frac{a\,\mathcal{K}_p}{(a\,s+b)^{p-1}(a\,s+b-1)} \right) ds$$

Let $f(u) = (p-1)\,u \log u - (1-u)\log(1-u) - p\,u$. Then

$$\begin{aligned} A(t) &= -\tfrac{n}{p}\,a \int_0^t \left[\log(-a\,\mathcal{K}_p) - f'(as+b)\right] ds \\ &= \tfrac{n}{p}\,\tfrac{\beta-\alpha}{\alpha\,\beta} \log\left(\tfrac{\beta-\alpha}{\alpha\,\beta}\,\tfrac{\mathcal{K}_p}{t}\right) + \tfrac{n}{p}\left[f(\tfrac{1}{\beta}) - f(\tfrac{1}{\alpha})\right]\,. \end{aligned}$$

This ends the proof of Theorem 9.

With a minor adaptation of the above proof, one can state a result similar to the one of Theorem 9 in the case α, $\beta \in (0,1]$ with $\beta \le \alpha$ and at a formal level in the case $\beta \le \alpha < 0$ (in both cases, $a > 0$). Since the sign of q' is changed, the inequality is reversed, compared to Theorem 9.

Theorem 13 *Let α, $\beta \in (0,1]$ with $\beta \leq \alpha$. Any solution u of $u_t = \Delta_p u^{1/(p-1)}$ with initial data f such that f^α belongs to $L^1(\mathbb{R}^n)$ satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \geq \|f\|_\alpha \, A(n,p,\alpha,\beta)\, t^{\frac{n}{p}\,\frac{\alpha-\beta}{\alpha\beta}} \quad \forall\, t > 0$$

with

$$A(n,p,\alpha,\beta) = \left(\mathcal{C}_1\,(\alpha-\beta)\right)^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}\, \mathcal{C}_2^{\frac{n}{p}}\,,$$

$$\mathcal{C}_1 = n\,\mathcal{L}_p\, e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}\,, \quad \mathcal{C}_2 = \frac{(1-\beta)^{\frac{1-\beta}{\beta}}}{(1-\alpha)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}\,.$$

Consider a solution of

$$\begin{cases} v_t + \frac{1}{p}\,|\nabla v|^p = \frac{1}{p-1}\,p^{\frac{2-p}{p-1}}\,\varepsilon^{p^*}\Delta_p v & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, t=0) = g \end{cases} \tag{7}$$

Lemma 14 *Let $\varepsilon > 0$. Then v is a C^2 solution of (7) iff*

$$u = e^{-\frac{1}{\lambda \varepsilon^{p^*}}\,v} \quad \textit{with } \lambda = \frac{p^{\frac{1}{p-1}}}{p-1}$$

is a C^2 positive solution of

$$u_t = \varepsilon^p\,\Delta_p(u^{1/(p-1)})$$

with initial data $f = e^{-\frac{1}{\lambda \varepsilon^{p^*}}\,g}$.

Let $P_t^p f := u \Longleftrightarrow u_t = \Delta_p(u^{1/(p-1)}, u(\cdot,0) = f$.
In the limit case $\varepsilon = 0$,

$$Q_t^p g(x) := v(x,t) = \inf_{y \in \mathbf{R}^n} \left\{ g(y) + \frac{t}{p^*} \left| \frac{x-y}{t} \right|^{p^*} \right\}$$

is the Hopf-Lax solution of the Hamilton-Jacobi equation:

$$v_t + \frac{1}{p} |\nabla v|^p = 0$$

Theorem 15 *For any C^2 function g,*

$$Q_t^p g(x) = \lim_{\varepsilon \to 0} \left[-\lambda\, \varepsilon^{p^*} \log \left(P^p_{\varepsilon^p t} \left(e^{-\frac{g}{\lambda\, \varepsilon^{p^*}}} \right) \right) \right] \quad \forall\, t > 0\ .$$

The family $(P^p_{\varepsilon^p t})_{\varepsilon>0}$ satisfies a *large deviation principle* of order ε^{p^*} and rate function $\frac{1}{p^*\, t^{p^*-1}} |\, x - \cdot\, |^{p^*}$.

This provides a new proof of the main result of [Gentil 02].

Corollary 16 *Let* $\lambda = \frac{p^{\frac{1}{p-1}}}{p-1}$. *For any* α, β *with* $0 \le \alpha \le \beta$, *we may write*

$$\|e^{Q_t^p g}\|_\beta \le \|e^g\|_\alpha \, B(n,p,\alpha,\beta)\, t^{\frac{n}{p}\,\frac{\alpha-\beta}{\alpha\beta}} \quad \forall\, t > 0\,,$$

with

$$B(n,p,\alpha,\beta) = \Big((\beta-\alpha)\,\lambda^{p-1}\mathcal{C}_1\Big)^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} \left(\frac{\alpha^{\frac{p-1}{\alpha}+\frac{1}{\beta}}}{\beta^{\frac{p-1}{\beta}+\frac{1}{\alpha}}}\right)^{\frac{n}{p}} .$$

Proof. Replace α, β and t by δ, γ and τ in Theorem 13

$$\|P_\tau^p f\|_\gamma \ge \|f\|_\delta \,(\frac{\mathcal{C}_1}{\tau})^{\frac{n}{p}\,\frac{\gamma-\delta}{\gamma\delta}} \left\{(\delta-\gamma)^{\frac{\gamma-\delta}{\gamma\delta}} \frac{(1-\gamma)^{\frac{1-\gamma}{\gamma}}}{(1-\delta)^{\frac{1-\delta}{\delta}}} \frac{(\gamma)^{\frac{1-p}{\gamma}-\frac{1}{\delta}+1}}{(\delta)^{\frac{1-p}{\delta}-\frac{1}{\gamma}+1}}\right\}^{\frac{n}{p}}$$

Take now $f = e^{-\frac{h}{\lambda \varepsilon^{p^*}}}$, $\tau = \varepsilon^p t$, $\delta = \lambda \varepsilon^{p^*} \alpha$ and $\gamma = \lambda \varepsilon^{p^*} \beta$ and raise the above expression to the power $\lambda \varepsilon^{p^*}$. Take the limit $\varepsilon \to 0$ we obtain,

$$\|e^{-h}\|_\beta \leq \|e^{-Q_t^p h}\|_\alpha \, B(n,p,\alpha,\beta) \, t^{\frac{n}{p} \frac{\alpha-\beta}{\alpha\beta}} \quad \forall \, t > 0 \, .$$

The result then holds by taking $h = -Q_t^p(g)$ and by using the following inequality $-Q_t^p(-Q_t^p(g)) \leq g$.

Conclusion: The three following identities have been established:
(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \le \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \le \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p g}\|_\beta \le \|e^g\|_\alpha \, B(n,p,\alpha,\beta) \, t^{-\frac{n}{p} \frac{\beta-\alpha}{\alpha\beta}}$$

The equivalence (i) $\Longleftrightarrow$ (iii) has been established in [Gentil 02]. What we have seen is that (i) $\Longrightarrow$ (ii) and that (ii) $\Longrightarrow$ (iii). It is not difficult to check that (ii) $\Longrightarrow$ (i), so that the constants in (ii) are optimal.

Part IV — Optimal transport

[Cordero-Eruasquin, Gangbo, Houdré, Nazaret, Villani]

- Sobolev inequality: $\|f|_{L^{2^*}} \leq S\,\|\nabla f\|_{L^2}$
- (Standard) logarithmic Sobolev inequality
- Logarithmic Sobolev inequality in $W^{1,p}(\mathbb{R}^N)$

Sobolev inequalities

$$\|f\|_{L^{2^*}} \leq S\,\|\nabla f\|_{L^2}$$

$N \geq 3$. Optimal function: $f(x) = (\sigma + |x|^2)^{-(N-2)/2}$ A proof based on mass transportation:

$$\inf\left\{\frac{1}{2\lambda^2}\int_{\mathbf{R}^N} |\nabla f|^2\,dx \;:\; \int_{\mathbf{R}^N} |f|^{2^*}\,dx = 1\right\}$$

$$= \tfrac{n(n-2)}{2(n-1)}\,\sup\left\{\textstyle\int_{\mathbf{R}^N} |g|^{2^*(1-\frac{1}{n})}dy - \frac{\lambda^2}{2}\int_{\mathbf{R}^N} |y|^2\,|g|^{2^*}dy \;:\; \int_{\mathbf{R}^N} |g|^{2^*}dy = 1\right\}$$

Mass transportation: basic results

μ and ν two Borel probability measures on $\mathbb{R}^N$. $T : \mathbb{R}^N \to \mathbb{R}^N$
$T\#\mu = \nu \Longleftrightarrow \nu(A) = \mu(T^{-1}(A))$ for any Borel measurable set A.

Theorem 17 (Brenier, McCann) *$\exists\, T = \nabla\phi$ such that $T\#\mu = \nu$ and ϕ is convex.*

$\mu = F(x)\,dx$, $\nu = G(x)\,dx$, $\int_{\mathbf{R}^N} F(x)\,dx = \int_{\mathbf{R}^N} G(y)\,dy = 1$

$$\forall\, b \in C(\mathbb{R}^N, \mathbb{R}^+) \quad \int_{\mathbf{R}^N} b(y)G(y)\,dy = \int_{\mathbf{R}^N} b(\nabla\phi(x))F(x)\,dx$$

Under technical assumptions: $\phi \in C^2$, supp(F) or supp(G) is convex... [Caffarelli] ϕ solves the Monge-Ampère equation

$$G(\nabla\phi)\,\det \mathrm{Hess}(\phi) = F$$

A proof of the Sobolev inequality

$$G(\nabla\phi)^{-\frac{1}{n}} = (\det \mathrm{Hess}(\phi))^{\frac{1}{n}} F^{-\frac{1}{n}} \leq \frac{1}{n} \Delta\phi\, F^{-\frac{1}{n}}$$

$$\int G(y)^{1-\frac{1}{n}} dy \leq \frac{1}{n} \int G(\nabla\phi(x))^{1-\frac{1}{n}} (\det \mathrm{Hess}(\phi))^{\frac{1}{n}} \Delta\phi \, dx$$

$$= \frac{1}{n} \int F^{1-\frac{1}{n}} \Delta\phi\, dx = -\frac{1}{n} \int \nabla(F^{1-\frac{1}{n}}) \cdot \nabla\phi\, dx$$

by the arithmetic-geometric inequality. $F = |f|^{2^*}$, $G = |g|^{2^*}$

$$\int |g|^{2^*(1-\frac{1}{n})} dy \leq -\frac{2(n-1)}{n(n-2)} \int (f^{\frac{n}{n-2}}) \nabla f \cdot \nabla\phi\, dx$$

$$\frac{n(n-2)}{2(n-1)} \int |g|^{2^*(1-\frac{1}{n})} dy \leq \frac{2}{\lambda^2} \int |\nabla f|^2 dx + \frac{\lambda^2}{2} \int |f|^{2^*} |\nabla\phi|^2 dx$$

by Young's inequality. Use: $\int F\, |\nabla\phi|^2\, dx = \int G |y|^2 dy$

A proof of the standard logarithmic Sobolev inequality

$G(y) = e^{-|y|^2/2}$, $F(x) = f(x)\, e^{-|x|^2/2}$, $\nabla\phi \# F\, dx = G\, dy$.

$$e^{-|\nabla\phi|^2/2} \det \mathrm{Hess}(\phi) = f(x)\, e^{-|x|^2/2}$$

$\theta(x) = \phi(x) - \frac{1}{2}|x|^2$

$$f(x)\, e^{-|x|^2/2} = \det(\mathrm{Id} + \mathrm{Hess}(\theta))\, e^{-|x+\nabla\theta(x)|^2/2}$$

$$\begin{aligned} \log f - |x|^2/2 &= -|x+\nabla\theta(x)|^2/2 + \log\Big[\det(\mathrm{Id} + \mathrm{Hess}(\theta))\Big] \\ &\le -|x+\nabla\theta(x)|^2/2 + \Delta\theta \end{aligned}$$

(use $\log(1+t) \le t$). Let $d\mu(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$.

$$\log f \le -\frac{1}{2}|\nabla\theta|^2 - x \cdot \nabla\theta + \Delta\theta$$

$$\int f \log f\, d\mu \le -\frac{1}{2}\int \left|\sqrt{f}\,\nabla\theta + \frac{\nabla f}{\sqrt{f}}\right|^2 d\mu + \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu \le \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu$$

Logarithmic Sobolev inequality in $W^{1,p}(\mathbb{R}^N)$

$G(y) = c_{p,n}\, e^{-\frac{p}{p-1}|y|^{p/(p-1)}} =: f_\infty(y)$, $F(x) = f(x)\, c_{p,n}\, e^{-\frac{p}{p-1}|x|^{p/(p-1)}}$
$\nabla\phi \# F\, dx = G\, dy$, $d\mu(x) = f_\infty^p(x)\, dx$

$$f(x)\, e^{-\frac{p}{p-1}|x|^{p/(p-1)}} = \det\left(\text{Hess}(\phi)\right) e^{-\frac{p}{p-1}|x+\nabla\theta(x)|^{p/(p-1)}}$$

$$f^p(x) = f_\infty^p(\nabla\phi)\det\left(\text{Id} + \text{Hess}(\phi)\right)$$

$$\int f^p \log f^p\, d\mu = \int f^p \log f_\infty^p\, d\mu + \int (\Delta\phi - n) f^p d\mu$$

$$\int \Delta\phi\, f^p d\mu = -p\int f^{p-1}\nabla f\cdot\nabla\phi\, d\mu \le \frac{\lambda^{-q}}{q}\int |f|^p\, |\nabla\phi|^{p/(p-1)} + \frac{\lambda^p}{p}\int |\nabla f|^p d\mu$$

using Young's inequality: $X = f^{p-1}\nabla\phi$, $Y = \nabla f$

$$\int X\cdot Y\, d\mu \le \frac{\lambda^{-q}}{q}\|X\|_q^q + \frac{\lambda^p}{p}\|Y\|_p^p$$