Nonlinear diffusions, entropy methods and stability

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Outline

Proving inequalities
by the carré du champ method
using a fast diffusion flow
or a porous medium flow
Reduction of nonlinear flows
to linear flows

Introducing the flow

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

Check: if $m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$, $v = u^\beta$, then $\rho = v^p$, solves $\frac{\partial \rho}{\partial t} = \Delta \rho^m$

$$\frac{d}{dt}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} = 0\,, \quad \frac{d}{dt}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 2\,(p-2) \int_{\mathbb{S}^d} u^{-\,p\,(1-m)}\,|\nabla u|^2\,d\mu_d\,,$$

$$\frac{d}{dt}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = -\,2 \int_{\mathbb{S}^d} \left(\beta\,v^{\beta-1}\,\frac{\partial v}{\partial t} \right) \left(\Delta v^\beta \right) d\mu_d = -\,2\,\beta^2\,\mathscr{K}[v]$$

Lemma

Assume that $p \in (1, 2^*)$ and $m \in [m_-(d,p), m_+(d,p)]$. Then

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt} \left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u] \right) \le -\,\gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2}\,d\mu_d \le 0$$

The *carré du champ* strategy

In the linear case ($m = 1$), the method goes back to [Bakry, Emery, 1985], but it applies also with $m \neq 1$

$$\frac{d}{dt}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u]\right) \le 0$$

$\lim_{t\to+\infty}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[u]\right) = 0$ proves the

- ***Gagliardo-Nirenberg-Sobolev inequality***

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\mathcal{E}_p[u] := \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $p \in [1,2) \cup (2, 2^*)$
with $2^* := \frac{2\,d}{d-2}$ if $d \ge 3$ and $2^* = +\infty$ if $d = 1$ or 2

- Limit $p \to 2$: the ***logarithmic Sobolev inequality***

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\mu_d \ge \frac{d}{2}\int_{\mathbb{S}^d} u^2\,\log\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu_d \quad \forall\,u \in \mathrm{H}^1(\mathbb{S}^d, d\mu_d)$$

Algebraic preliminaries

$$\mathrm{L}v := \mathrm{H}v - \frac{1}{d}\,(\Delta v)\,g_d \quad \text{and} \quad \mathrm{M}v := \frac{\nabla v \otimes \nabla v}{v} - \frac{1}{d}\,\frac{|\nabla v|^2}{v}\,g_d$$

With $a:b = a^{ij}\,b_{ij}$ and $\|a\|^2 := a:a$, we have

$$\|\mathrm{L}v\|^2 = \|\mathrm{H}v\|^2 - \tfrac{1}{d}\,(\Delta v)^2\,, \qquad \|\mathrm{M}v\|^2 = \left\|\tfrac{\nabla v\otimes\nabla v}{v}\right\|^2 - \tfrac{1}{d}\,\tfrac{|\nabla v|^4}{v^2} = \tfrac{d-1}{d}\,\tfrac{|\nabla v|^4}{v^2}$$

- A first identity

$$\int_{\mathbb{S}^d} \Delta v\,\frac{|\nabla v|^2}{v}\,d\mu_d = \frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathrm{M}v\|^2\,d\mu_d - 2\int_{\mathbb{S}^d}\mathrm{L}v : \frac{\nabla v\otimes\nabla v}{v}\,d\mu_d\right)$$

- Second identity (Bochner-Lichnerowicz-Weitzenböck formula)

$$\int_{\mathbb{S}^d}(\Delta v)^2\,d\mu_d = \frac{d}{d-1}\int_{\mathbb{S}^d}\|\mathrm{L}v\|^2\,d\mu_d + d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu_d$$

Constructing the estimate

With $b=(\kappa+\beta-1)\,\frac{d-1}{d+2}$ and $c=\frac{d}{d+2}\,(\kappa+\beta-1)+\kappa\,(\beta-1)$

$$\mathscr{K}[v]:=\int_{\mathbb{S}^d}\left(\Delta v+\kappa\,\frac{|\nabla v|^2}{v}\right)\left(\Delta v+(\beta-1)\,\frac{|\nabla v|^2}{v}\right)d\mu_d$$
$$=\frac{d}{d-1}\,\|\mathrm{L}v-\ b\,\mathrm{M}v\|^2+\left(c-b^2\right)\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu_d+d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu_d$$

Let $\kappa=\beta\,(p-2)+1$. The condition $\gamma:=c-b^2\ge 0$ amounts to

$$\gamma=\tfrac{d}{d+2}\,\beta\,(p-1)+\big(1+\beta\,(p-2)\big)\,(\beta-1)-\left(\tfrac{d-1}{d+2}\,\beta\,(p-1)\right)^2$$

Lemma

$$\mathscr{K}[v]\ge\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu_d+d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu_d$$

Hence $\mathscr{K}[v]\ge d\int_{\mathbb{S}^d}|\nabla v|^2\,d\mu_d$ if $\gamma\ge 0$, a condition on β, *i.e.*, on m

Admissible parameters

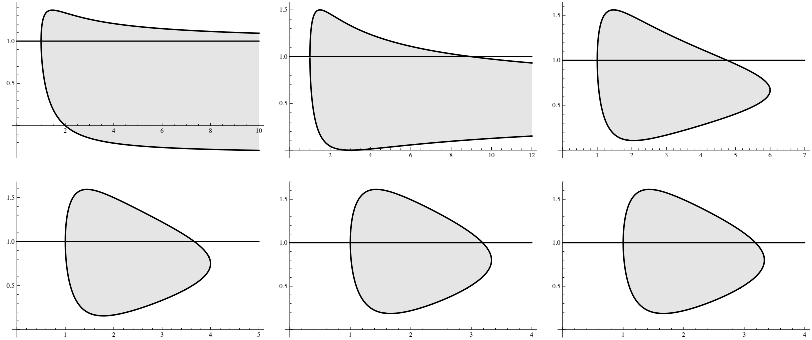


Figure: $d = 1, 2, 3$ (first line) and $d = 4, 5$ and 10 (second line): the curves $p \mapsto m_\pm(p)$ determine the admissible parameters (p, m) [JD, Esteban, Kowalczyk,Loss] [JD, Esteban, 2019]

$$m_\pm(d,p) := \frac{1}{(d+2)\,p}\left(d\,p+2\pm\sqrt{d\,(p-1)\,(2\,d-(d-2)\,p)}\right)$$

From inequalities to *improved* inequalities

Summary

From $\frac{1}{2\,\beta^2}\,\frac{d}{dt}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[u]\right)\le-\gamma\int_{\mathbb{S}^d}\frac{|\nabla v|^4}{v^2}\,d\mu_d\le0$ and $\lim_{t\to+\infty}\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-d\,\mathcal{E}_p[u]\right)=0$, we deduce the inequality

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge d\,\mathcal{E}_p[u]$$

[Bakry-Emery, 1984], [Bidaut-Véron, Véron, 1991], [Beckner,1993]

... but we can do better

[Demange, 2008], [JD, Esteban, Kowalczyk, Loss]

Logarithmic Sobolev and Gagliardo-Nirenberg inequalities on the sphere

A joint work with G. Brigati and N. Simonov
Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results
arXiv:2211.13180,
to appear in Annales Inst. Henri Poincaré C, Analyse non linéaire

▷ *Carré du champ methods combined with spectral information*

(Improved) logarithmic Sobolev inequality: stability (1)

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu_d \ge \frac{d}{2}\int_{\mathbb{S}^d} F^2\,\log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu_d \quad \forall\,F\in\mathrm{H}^1(\mathbb{S}^d,d\mu) \tag{LSI}$$

$d\mu_d$: uniform probability measure; equality case: constant functions
Optimal constant: test functions $F_\varepsilon(x)=1+\varepsilon\,x\cdot\nu$, $\nu\in\mathbb{S}^d$, $\varepsilon\to0$
$\triangleright$ *improved inequality* under an appropriate *orthogonality condition*

Theorem

Let $d\ge1$. For any $F\in\mathrm{H}^1(\mathbb{S}^d,d\mu)$ such that $\int_{\mathbb{S}^d} x\,F\,d\mu_d=0$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu_d - \frac{d}{2}\int_{\mathbb{S}^d} F^2\,\log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu_d \ge \frac{2}{d+2}\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu_d$$

Improved ineq. $\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu_d \ge \left(\frac{d}{2}+1\right)\int_{\mathbb{S}^d} F^2\,\log\left(F^2/\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)d\mu_d$

Earlier/weaker results in [JD, Esteban, Loss, 2015]

Logarithmic Sobolev inequality: stability (2)

What if $\int_{\mathbb{S}^d} x\,F\,d\mu_d \neq 0$? Take $F_\varepsilon(x) = 1 + \varepsilon\,x\cdot\nu$ and let $\varepsilon \to 0$

$$\|\nabla F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{2}\int_{\mathbb{S}^d} F_\varepsilon^2\,\log\left(\frac{F_\varepsilon^2}{\|F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu_d = O(\varepsilon^4) = O\left(\|\nabla F_\varepsilon\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Such a behaviour is in fact optimal: *carré du champ* method

Proposition

Let $d \geq 1$, $\gamma = 1/3$ if $d = 1$ and $\gamma = (4\,d-1)\,(d-1)^2/(d+2)^2$ if $d \geq 2$. Then, for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with $\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$ we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu_d - \frac{d}{2}\int_{\mathbb{S}^d} F^2\,\log F^2\,d\mu_d \geq \frac{1}{2}\,\frac{\gamma\,\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\gamma\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d}$$

In other words, if $\|\nabla F\|_{\mathrm{L}^2(\mathbb{S}^d)}$ is small

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu_d - \tfrac{d}{2}\int_{\mathbb{S}^d} F^2\,\log F^2\,d\mu_d \geq \tfrac{\gamma}{2\,d}\,\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)} + o\left(\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Logarithmic Sobolev inequality: stability (3)

Let $\Pi_1 F$ denote the orthogonal projection of a function $F \in \mathrm{L}^2(\mathbb{S}^d)$ on the spherical harmonics corresponding to the first eigenvalue on $\mathbb{S}^d$

$$\Pi_1 F(x) = \frac{x}{d+1} \cdot \int_{\mathbb{S}^d} y\, F(y)\, d\mu(y) \quad \forall\, x \in \mathbb{S}^d$$

▷ a global (and detailed) stability result

Theorem

Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu_d - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu_d$$
$$\geq \mathscr{S}_d \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{2}\, \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for some explicit stability constant $\mathscr{S}_d > 0$

Gagliardo-Nirenberg inequalities: a result by R. Frank

[Frank, 2022]: if $p \in (2, 2^*)$, there is $\mathsf{c}(d,p) > 0$ such that

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \geq \mathsf{c}(d,p)\,\frac{\left(\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F - \overline{F}\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

where $\overline{F} := \int_{\mathbb{S}^d} F\,d\mu_d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \geq \mathsf{c}(d,p)\,\frac{\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2}\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

- a compactness method,
- the exponent 4 in the r.h.s. is optimal
- the (generalized) entropy is

$$\mathcal{E}_p[u] := \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Gagliardo-Nirenberg inequalities: stability

As in the case of the logarithmic Sobolev inequality, an improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined

Theorem

Let $d \geq 1$ and $p \in (1,2) \cup (2, 2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d - d \, \mathcal{E}_p[F] \geq \mathscr{S}_{d,p} \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1) \, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for some explicit stability constant $\mathscr{S}_{d,p} > 0$

A first stability result based on an improved inequality under an orthogonality constraint: a spectral analysis

Improved interpolation inequalities under orthogonality

Decomposition of $\mathrm{L}^2(\mathbb{S}^d, d\mu) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell$ into spherical harmonics
Let Π_k be the orthogonal projection onto $\bigoplus_{\ell=1}^{k} \mathcal{H}_\ell$

Theorem

Assume that $d \geq 1$, $p \in (1, 2^*)$ and $k \in \mathbb{N} \setminus \{0\}$ be an integer. For some $\mathscr{C}_{d,p,k} \in (0,1)$ with $\mathscr{C}_{d,p,k} \leq \mathscr{C}_{d,p,1} = \frac{2\,d - p\,(d-2)}{2\,(d+p)}$

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d - d\,\mathcal{E}_p[F] \geq \mathscr{C}_{d,p,k} \int_{\mathbb{S}^d} \left|\nabla(\mathrm{Id} - \Pi_k)\,F\right|^2 d\mu_d$$

- $\mathcal{H}_1$ is generated by the coordinate functions x_i, $i = 1, 2, \ldots d+1$

▷ ***Funk-Hecke formula*** as in [Lieb, 1983] and [Beckner,1993]
▷ Use convexity estimates and monotonicity properties of the coefficients

The convex improvement based on the carré du champ method

Improved inequalities: flow estimates

With $\|u\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, consider the *entropy* and the *Fisher information*

$$\mathsf{e} := \frac{1}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \text{and} \quad \mathsf{i} := \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

Lemma

With $\delta := \frac{2-(4-p)\,\beta}{2\,\beta\,(p-2)}$ *if* $p > 2\,,\quad \delta := 1$ *if* $p \in [1,2]$

$$(\mathsf{i} - d\,\mathsf{e})' \le -\gamma \int_{\mathbb{S}^d} \frac{|\nabla v|^4}{v^2}\,d\mu_d \le \frac{\gamma\,\mathsf{i}\,\mathsf{e}'}{\big(1-(p-2)\,\mathsf{e}\big)^\delta}$$

If $F \in \mathrm{H}^1(\mathbb{S}^d)$ is such that $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, then

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge d\,\varphi\left(\mathcal{E}_p[F]\right)$$

Some global stability estimates

[JD, Esteban, Kowalczyk, Loss], [JD, Esteban 2020]
[Brigati, JD, Simonov]

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \geq d\,\varphi\left(\mathcal{E}_p[F]\right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d) \text{ s.t. } \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} = 1$$

Since $\varphi(0) = 0$, $\varphi'(0) = 1$, $\varphi'' > 0$, we know that $\varphi : [0, s_\star) \to \mathbb{R}^+$ is invertible and $\psi : \mathbb{R}^+ \to [0, s_\star)$, $s \mapsto \psi(s) := s - \varphi^{-1}(s)$, is convex increasing: $\psi'' > 0$, with $\psi(0) = \psi'(0) = 0$, $\lim_{t\to+\infty} \big(t - \psi(t)\big) = s_\star$

Proposition

If $d \geq 1$ and $p \in (1, 2^\#)$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \geq d\, \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\, \psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d)$$

▷ If $p = 2$, notice that $\psi(t) = t - \frac{1}{\gamma}\,\log(1 + \gamma\, t)$

Stability: the general result

It remains to combine the *improved entropy – entropy production inequality* (carré du champ method) and the *improved interpolation inequalities under orthogonality constraints*

The “far away” regime and the “neighborhood” of $\mathcal{M}$

▷ If $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \,/\, \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \ge \vartheta_0 > 0$, by the convexity of ψ

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - d\,\mathcal{E}_p[F] \ge d\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\,\psi\left(\frac{1}{d}\,\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}\right) \ge \frac{d}{\vartheta_0}\,\psi\left(\frac{\vartheta_0}{d}\right)\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}$$

▷ From now on, we assume that $\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta_0\,\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$, take $\|F\|_{\mathrm{L}^p(\mathbb{S}^d)} = 1$, learn that

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} < \vartheta := \frac{d\,\vartheta_0}{d-(p-2)\,\vartheta_0} > 0$$

from the standard interpolation inequality and deduce from the Poincaré inequality that

$$\frac{d-\vartheta}{d} < \left(\int_{\mathbb{S}^d} F\,d\mu_d\right)^2 \le 1$$

Partial decomposition on spherical harmonics

$\mathscr{M} = \Pi_0 F$ and $\Pi_1 F = \varepsilon\,\mathscr{Y}$ where $\mathscr{Y}(x) = \sqrt{\frac{d+1}{d}}\,x\cdot\nu$, $\nu\in\mathbb{S}^d$

$$F = \mathscr{M}\,\big(1+\varepsilon\,\mathscr{Y}+\eta\,G\big)$$

Apply $c_{p,d}^{(-)}\,\varepsilon^6 \le \|1+\varepsilon\,\mathscr{Y}\|_{\mathrm{L}^p(\mathbb{S}^d)}^p - \big(1+a_{p,d}\,\varepsilon^2+b_{p,d}\,\varepsilon^4\big) \le c_{p,d}^{(+)}\,\varepsilon^6$ (with explicit constants) to $u=1+\varepsilon\,\mathscr{Y}$ and $r=\eta\,G$ the estimate

$$\begin{aligned}
&\|u+r\|_{\mathrm{L}^p(\mathbb{S}^d)}^2 - \|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^2\\
&\quad\le \tfrac{2}{p}\,\|u\|_{\mathrm{L}^p(\mathbb{S}^d)}^{2-p}\,\Big(p\int_{\mathbb{S}^d} u^{p-1}\,r\,d\mu_d + \tfrac{p}{2}\,(p-1)\int_{\mathbb{S}^d} u^{p-2}\,r^2\,d\mu_d\\
&\qquad\qquad + \textstyle\sum_{2<k<p} C_k^p \int_{\mathbb{S}^d} u^{p-k}\,|r|^k\,d\mu_d + K_p \int_{\mathbb{S}^d} |r|^p\,d\mu_d\Big)
\end{aligned}$$

Estimate $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-1}\,G\,d\mu_d$, $\int_{\mathbb{S}^d}(1+\varepsilon\,\mathscr{Y})^{p-k}\,|G|^k\,d\mu_d$, etc. to obtain (under the condition that $\varepsilon^2+\eta^2\sim\vartheta$)

$$\begin{aligned}
\int_{\mathbb{S}^d}|\nabla F|^2\,d\mu_d - d\,\mathcal{E}_p[F] &\ge \mathscr{M}^2\,\Big(A\,\varepsilon^4 - B\,\varepsilon^2\,\eta + C\,\eta^2 - \mathcal{R}_{p,d}\,\big(\vartheta^p+\vartheta^{5/2}\big)\Big)\\
&\ge \mathscr{C}\,\left(\frac{\varepsilon^4}{\varepsilon^2+\eta^2+1}+\eta^2\right)
\end{aligned}$$

Gaussian interpolation inequalities

Joint work with G. Brigati and N. Simonov
Gaussian interpolation inequalities
arXiv:2302.03926

▷ *The large dimensional limit of the sphere*

Large dimensional limit

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \geq \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

Theorem

Let $v \in \mathrm{H}^1(\mathbb{R}^n,dx)$ with compact support, $d \geq n$ and

$$u_d(\omega) = v\Big(\omega_1/\sqrt{d},\omega_2/\sqrt{d},\ldots,\omega_n/\sqrt{d}\Big)$$

where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\,\pi)^{-n/2}\,e^{-\frac12\,|y|^2}\,dy$,

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

Gaussian interpolation inequalities on $\mathbb{R}^n$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right) \tag{1}$$

- $1 \leq p < 2$ [Beckner, 1989], [Bakry, Emery, 1984]
- Poincaré inequality corresponding: $p = 1$
- *Gaussian logarithmic Sobolev inequality* $p \to 2$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \geq \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

$$d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$$

Admissible parameters on $\mathbb{S}^d$

Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left(\Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

$$m_\pm(d,p) := \tfrac{1}{(d+2)\,p} \left(d\,p + 2 \pm \sqrt{d\,(p-1)\,\big(2\,d-(d-2)\,p\big)} \right)$$

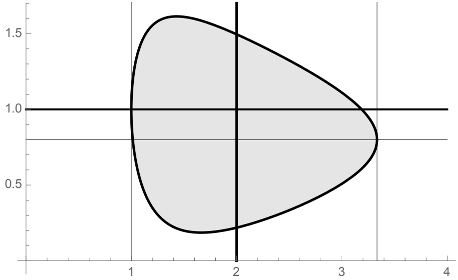


Figure: Case $d=5$: admissible parameters $1 \le p \le 2^* = 10/3$ and m (horizontal axis: p, vertical axis: m)

Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)}\left(\mathcal{L} v + (m\,p-1)\,\frac{|\nabla v|^2}{v}\right) \quad \text{on} \quad \mathbb{R}^n$$

Ornstein-Uhlenbeck operator: $\mathcal{L} = \Delta - x\cdot\nabla$

$$m_\pm(p) := \lim_{d\to+\infty} m_\pm(d,p) = 1 \pm \frac{1}{p}\sqrt{(p-1)\,(2-p)}$$

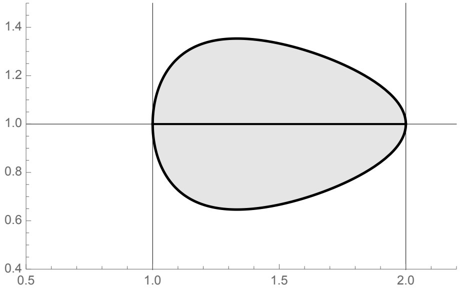


Figure: The admissible parameters $1 \le p \le 2$ and m are independent of n

A stability result for Gaussian interpolation inequalities

Theorem

For all $n \geq 1$, and all $p \in (1,2)$, there is an explicit constant $c_{n,p} > 0$ such that, for all $v \in \mathrm{H}^1(d\gamma)$,

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{p-2}\left(\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right)$$
$$\geq c_{n,p}\left(\|\nabla(\mathrm{Id}-\Pi_1)v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \frac{\|\nabla \Pi_1 v\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)$$

More results on logarithmic Sobolev inequalities

Joint work with G. Brigati and N. Simonov
Stability for the logarithmic Sobolev inequality
arXiv:2303.12926

▷ *Entropy methods, with constraints*

Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon,1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R},d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w\in\mathscr{M}} \mathrm{d}(u_\varepsilon,w)^\alpha \le \frac{1}{2}\,\varepsilon^4 + O\big(\varepsilon^6\big)\,.$$

$\mathscr{M} := \big\{ w_{a,c} \,:\, (a,c) \in \mathbb{R}^d \times \mathbb{R} \big\}$ where $w_{a,c}(x) = c\, e^{-a\cdot x}$

Proposition

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ and $\|x\,u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 \log|u|^2\, d\gamma \ge \frac{1}{2\,d}\left(\int_{\mathbb{R}^d} |u|^2 \log|u|^2\, d\gamma\right)^2$$

and, with $\psi(s) := s - \frac{d}{4}\,\log\left(1 + \frac{4}{d}\,s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)} - \frac{1}{2}\int_{\mathbb{R}^d} |u|^2 \log|u|^2\, d\gamma \ge \psi\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}\right)$$

Stability under log-concavity

Theorem

For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that

$$\int_{\mathbb{R}^d} (1, x)\, |u|^2\, d\gamma = (1, 0) \quad \textit{and} \quad \int_{\mathbb{R}^d} |x|^2\, |u|^2\, d\gamma \le \mathsf{K}$$

we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\gamma \ge 0$$

$$\mathscr{C}_\star = 1 + \frac{1}{1728\,\mathsf{K}} \approx 1 + \frac{0.0005787}{\mathsf{K}}$$

Theorem

Let $d \geq 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on ε such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with

$$\int_{\mathbb{R}^d} (1, x)\, |u|^2\, d\gamma = (1, 0)\,,\quad \int_{\mathbb{R}^d} |x|^2\, |u|^2\, d\gamma \leq d\,,\quad \int_{\mathbb{R}^d} |u|^2\, e^{\,\varepsilon\, |x|^2}\, d\gamma < \infty$$

for some $\varepsilon > 0$, then we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\gamma)} \geq \frac{\mathscr{C}}{2} \int_{\mathbb{R}^d} |u|^2\, \log |u|^2\, d\gamma$$

Compact support: [Lee, Vázquez, '03]; [Chen, Chewi, Niles-Weed, '21]

Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

A joint work with JD, M.J. Esteban, A. Figalli, R. Frank, M. Loss
Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence
arXiv: 2209.08651

A stability results for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g(x) = c\left(a + |x - b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

Theorem

There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on d such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

[JD, Esteban, Figalli, Frank, Loss]

- No compactness argument
- The (estimate of the) constant β is explicit
- The decay rate β/d is optimal as $d \to +\infty$

A stability results for the logarithmic Sobolev inequality

Use the inverse stereographic projection to rewrite the result on $\mathbb{S}^d$

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{1}{4}\,d\,(d-2)\left(\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$
$$\geq \frac{\beta}{d}\,\inf_{g\in\mathcal{M}}\left(\|\nabla F - \nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{1}{4}\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Corollary

With $\beta > 0$ as above

$$\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \pi\int_{\mathbb{R}^d} F^2\,\ln\left(\frac{|F|^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)d\gamma$$
$$\geq \frac{\beta\,\pi}{2}\,\inf_{a\in\mathbb{R}^d,\,c\in\mathbb{R}}\int_{\mathbb{R}^d}\left|F - c\,e^{a\cdot x}\right|^2 d\gamma$$

Stability for Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

Rényi entropy powers, inequalities and flow, a formal approach

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

▷ *How do we relate Gagliardo-Nirenberg-Sobolev inequalities on* $\mathbb{R}^d$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\geq\mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\tag{GNS}$$

and the fast diffusion equation

$$\frac{\partial u}{\partial t}=\Delta u^m\tag{FDE}$$

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \geq m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u(t,x)\,dx = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+\left(\mathbb{R}^d, (1+|x|^2)\,dx\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |x|^2\,u(t,x)\,dx = 2\,d \int_{\mathbb{R}^d} u^m(t,x)\,dx$$

(iii) *Entropy estimate.* With $m \geq m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+\left(\mathbb{R}^d, (1+|x|^2)\,dx\right)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} u^m(t,x)\,dx = \frac{m^2}{1-m} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2} \int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy growth rate as a consequence of (GNS)

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\ge \mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

$$p=\frac{1}{2\,m-1}\iff m=\frac{p+1}{2\,p}\in[m_1,1)$$

$u=f^{2\,p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathcal{M}=\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{2\,p}\,,\quad \mathsf{E}[u]=\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1}\,,\quad \mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$, then $\mathsf{E}'=m\,\mathsf{I}$

$$\mathsf{E}'\ge\frac{p-1}{2\,p}\,(p+1)^2\,\mathcal{C}_{\mathrm{GNS}}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}^{\frac{2}{\theta}}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{-\frac{2\,(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Self-similar solutions

$$\int_{\mathbb{R}^d} u^m(t,x)\,dx \geq \left(\int_{\mathbb{R}^d} u_0^m\,dx + \frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}} \quad \forall\, t \geq 0$$

Equality case is achieved if and only if, up to a normalisation and a a translation

$$u(t,x) = \frac{c_1}{R(t)^d}\,\mathcal{B}\left(\frac{c_2\,x}{R(t)}\right)$$

where $\mathcal{B}$ is the ***Barenblatt self-similar solution***

$$\mathcal{B}(x) := \left(1+|x|^2\right)^{\frac{1}{m-1}}$$

Notice that $\mathcal{B} = \varphi^{2p}$ means that

$$\varphi(x) = \left(1+|x|^2\right)^{-\frac{1}{p-2}}$$

is an ***Aubin-Talenti profile***

Pressure variable and decay of the Fisher information

The derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta\mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$
$$= -2\int_{\mathbb{R}^d} u^m\,\Big(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta\mathsf{P})^2\Big)\,dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares: with $m_1 = \frac{d-1}{d}$

$$-\frac{\mathsf{I}}{2\,\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^\theta\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$
$$=\int_{\mathbb{R}^d}u^m\,\left\|\mathrm{D}^2\mathsf{P}-\frac{1}{d}\,\Delta\mathsf{P}\,\mathrm{Id}\right\|^2 dx+(m-m_1)\int_{\mathbb{R}^d}u^m\,\left|\Delta\mathsf{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2 dx$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^\theta\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathcal{M}^{\frac{2\,\theta}{p}}}=\frac{\mathsf{I}[\mathcal{B}]^\theta\,\mathsf{E}[\mathcal{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathcal{B}\|_{\mathrm{L}^1(\mathbb{R}^d)}^{\frac{2\,\theta}{p}}}=(p+1)^{2\,\theta}\,\mathcal{C}_{\mathrm{GNS}}^{2\,\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^\theta\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\ge(p+1)^{2\,\theta}\,\left(\mathcal{C}_{\mathrm{GNS}}\right)^{2\,\theta}\,\mathcal{M}^{\frac{2\,\theta}{p}}$$

Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

in collaboration with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method
arXiv:2007.03674, to appear in *Memoirs of the AMS*

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion eq.
DCDS, 43 (3 & 4): 1070-1089, 2023

Entropy – entropy production inequality

Fast diffusion equation (written in self-similar variables)

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \Big(v \left(\nabla v^{m-1} - 2\,x \right) \Big) = 0 \qquad (r\,\mathsf{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1} \left(v - \mathcal{B} \right) \right) \, dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 \, dx$$

satisfy an *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

[del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-4\,t}$$

The *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

is equivalent to the *Gagliardo-Nirenberg-Sobolev inequalities*

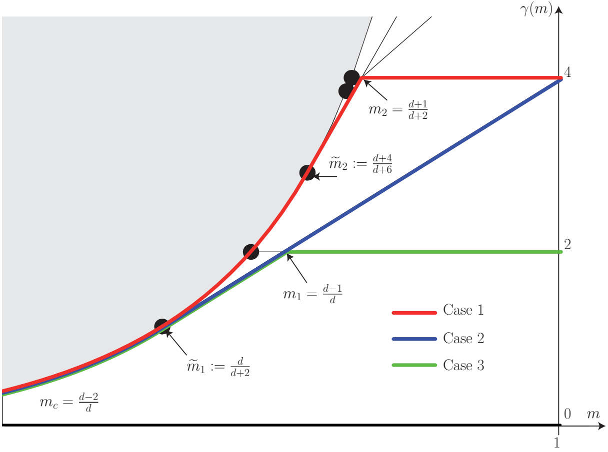
$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with equality if and only if $|f|^{2\,p}$ is the *Barenblatt profile* such that

$$|f(x)|^{2p} = \mathcal{B}(x) = \left(1+|x|^2\right)^{\frac{1}{m-1}}$$

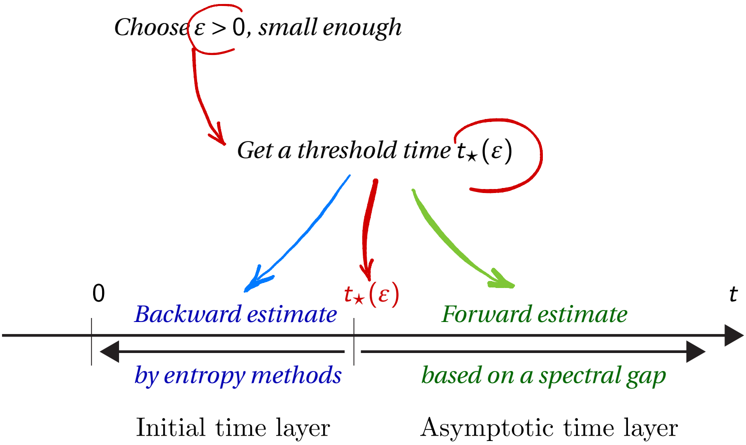
$v = f^{2\,p}$ so that $v^m = f^{p+1}$ and $v\left|\nabla v^{m-1}\right|^2 = (p-1)^2\,|\nabla f|^2$

Spectral gap and Taylor expansion around $\mathcal{B}$



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Strategy of the method



A constructive stability result (subcritical case)

Stability in the entropy - entropy production estimate
$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

if $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and $\mathsf{A}[v] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} v\,dx < \infty$

Theorem

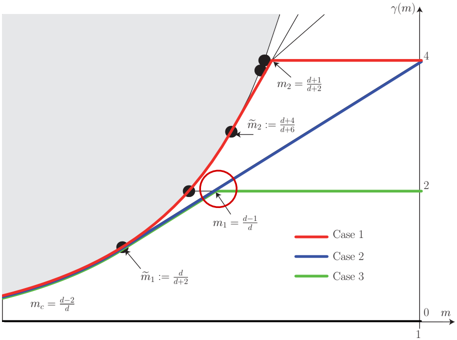
Let $d \geq 1$ and $p \in (1, p^*)$. There is an explicit $\mathcal{C} = \mathcal{C}[f] > 0$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d, (1+|x|^2)\,dx\big)$ s.t. $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}\big[f^{2p}\big] < \infty$

$$(p-1)^2\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + 4\,\frac{d-p\,(d-2)}{p+1}\,\|f\|^{p+1}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} - \mathcal{K}_{\mathrm{GNS}}\,\|f\|^{2\,p\,\gamma}_{\mathrm{L}^{2\,p}(\mathbb{R}^d)}$$
$$\geq \mathcal{C}[f]\,\inf_{\varphi\in\mathfrak{M}} \int_{\mathbb{R}^d} \big|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\big|^2\,dx$$

Extending the subcritical result to the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathcal{B}_\lambda(x) = \lambda^{-d/2}\,\mathcal{B}\left(x/\sqrt{\lambda}\right)$ in order to match $\int_{\mathbb{R}^d} |x|^2\,v\,dx$ where the function v solves $(r\,\mathrm{FDE})$ or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t), \frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d} |x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0) = 0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\;\; |x|\, f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Theorem

Let $d \ge 3$ and $A > 0$. For any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \;\; \text{and} \;\; \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \le A$$

we have

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \frac{\mathcal{C}_\star(A)}{4 + \mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\,\left(1 + A^{1/(2\,d)}\right)^{-1}$ *and* $\mathcal{C}_\star(0) > 0$ *depends only on* d

Symmetry in the "critical" case of the Caffarelli-Kohn-Nirenberg inequalities

in collaboration with M.J. Esteban and M. Loss

and M.J. Esteban, M. Loss and M. Muratori

▷A formal proof based on a fast diffusion flow

Critical Caffarelli-Kohn-Nirenberg inequalities

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

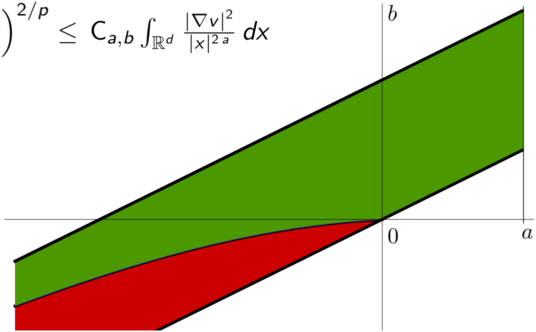
▷ *An optimal function among radial functions:*

$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

▷ *Is $v_\star$ optimal without symmetry assumption ?*

Symmetry *versus* symmetry breaking: the sharp result in the critical case

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



Theorem

Let $d \ge 2$ and $p < 2^*$. $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ (symmetry) if and only if either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \ge b_{\rm FS}(a)$

[JD, Esteban, Loss, 2016]

Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,,\quad \alpha = 1 + \frac{\beta-\gamma}{2}\quad\text{and}\quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathsf{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,,\quad \alpha > 0\,,\quad n > d\quad\text{and}\quad p\in(1,p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1+|x|^2\right)^{-1/(p-1)}\quad\forall\,x\in\mathbb{R}^d$$

The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu_d\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathsf{D}_\alpha \mathsf{P}|^2 \, d\mu_d$$

where $\sigma = \frac{2}{d} \, \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

Looking for an optimal function in (CKN) is equivalent to minimize $\mathcal{G}$ under a mass constraint

With $\mathsf{L}_\alpha = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathsf{L}_\alpha\,u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d} u^m\,d\mu_d\right)^{1-\sigma}\\
&\geq (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\left|\mathsf{L}_\alpha\mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu_d}{\int_{\mathbb{R}^d} u^m\,d\mu_d}\right|^2 d\mu_d\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}' - \frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right) u^m\,d\mu_d\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu_d =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \leq \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result. *A quantifier elimination problem [Tarski, 1951] ?*

(3/3: elliptic regularity, boundary terms)

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings
- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathsf{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathsf{L}_\alpha u^m\,d\mu_d\geq\mathcal{H}[u]\geq 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Stability in Caffarelli-Kohn-Nirenberg inequalities ?

in collaboration with M. Bonforte, B. Nazaret and N. Simonov

Constructive stability results in interpolation inequalities and explicit improvements of decay rates of fast diffusion eq.
DCDS, 43 (3 & 4): 1070-1089, 2023

Subcritical Caffarelli-Kohn-Nirenberg inequalities

On $\mathbb{R}^d$ with $d \geq 1$, let us consider the *Caffarelli-Kohn-Nirenberg interpolation inequalities*

$$\|f\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \leq \mathcal{C}_{\beta,\gamma,p}\, \|\nabla f\|^{\theta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\, \|f\|^{1-\theta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)}$$

$$\gamma-2 < \beta < \frac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2}\,,$$

with $\theta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$

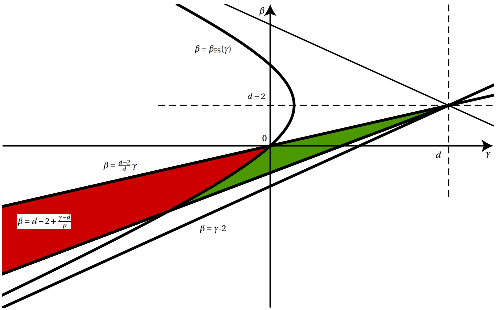
and $\|f\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$

Symmetry: equality is achieved by the *Aubin-Talenti type functions*

$$\mathsf{g}(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-\frac{1}{p-1}}$$

[JD, Esteban, Loss, Muratori, 2017] Symmetry holds if and only if

$$\gamma < d\,, \quad \text{and} \quad \gamma - 2 < \beta < \frac{d-2}{d}\,\gamma \quad \text{and} \quad \beta \leq \beta_{\mathrm{FS}}(\gamma)$$



$d = 4$ and $p = 6/5$: (γ, β) admissible region

V

An improved decay rate along the flow

In self-similar variables, with $m=(p+1)/(2\,p)$

$$|x|^{-\gamma}\,\frac{\partial v}{\partial t}+\nabla\cdot\left(|x|^{-\beta}\,v\,\nabla v^{m-1}\right)=\sigma\,\nabla\cdot\left(x\,|x|^{-\gamma}\,v\right)$$

$$\mathcal{F}[v]=\frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(v^{\frac{p+1}{2\,p}}-\mathrm{g}^{p+1}-\frac{p+1}{2\,p}\mathrm{g}^{1-p}\left(v-\mathrm{g}^{2p}\right)\right)|x|^{-\gamma}\,dx$$

Theorem

In the symmetry region, if $v\ge0$ is a solution with a initial datum v_0 s.t.

$$A[v_0]:=\sup_{R>0}R^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)}\int_{|x|>R}v_0(x)\,|x|^{-\gamma}\,dx<\infty$$

then there are some $\zeta>0$ and some $T>0$ such that

$$\mathcal{F}[v(t,.)]\le\mathcal{F}[v_0]\,e^{-\,(4\,\alpha^2+\zeta)\,t}\quad\forall\,t\ge2\,T$$

[Bonforte, JD, Nazaret, Simonov, 2022]

Entropy methods and stability: some basic references

- Model inequalities: [Gagliardo, 1958], [Nirenberg, 1958]
Carré du champ: [Bakry, Emery, 1985]
- Motivated by asymptotic rates of convergence in kinetic equations:
▷ linear diffusions: [Toscani, 1998], [Arnold, Markowich, Toscani, Unterreiter, 2001]
▷ Nonlinear diffusion for the carré du champ [Carrillo, Toscani], [Carrillo, Vázquez], [Carrillo, Jüngel, Markowich, Toscani, Unterreiter]
▷ Sharp global decay rates, nonlinear diffusions: [del Pino, JD, 2001] (variational methods), [Carrillo, Jüngel, Markowich, Toscani, Unterreiter] (carré du champ), [Jüngel], [Demange] (manifolds)
- Refinements and stability [Arnold, Dolbeault], [Blanchet, Bonforte, JD, Grillo, Vázquez], [JD, Toscani], [JD, Esteban, Loss]
- Detailed stability results by entropy methods
on $\mathbb{R}^d$: [Bonforte, JD, Nazaret, Simonov]
on $\mathbb{S}^d$: [Brigati, JD, Simonov]

▷ Side results: hypocoercivity; symmetry in CKN inequalities
▷ Angle of attack: *entropy methods and diffusion flows as a tool*

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Thank you for your attention !