Asymptotiques intermédiaires et comportement en temps grand des solutions d'équations de diffusion non linéaires

Jean DOLBEAULT

Ceremade, Université Paris IX-Dauphine,
Place de Lattre de Tassigny,
75775 Paris Cédex 16, France

E-mail: dolbeaul@ceremade.dauphine.fr

http://www.ceremade.dauphine.fr/~dolbeaul/

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I — Entropy methods for linear diffusions

The logarithmic Sobolev inequality

Convex Sobolev inequalities

- *logarithmic Sobolev inequality:* [Gross], [Weissler], [Coulhon],...
- *probability theory:* [Bakry], [Emery], [Ledoux], [Coulhon],...
- *linear diffusions:* [Toscani], [Arnold, Markowich, Toscani, Unterreiter], [Otto, Kinderlehrer, Jordan], [Arnold, J.D.]

I-A. Intermediate asymptotics: heat equation

Heat equation: $$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^n\,,\ t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0 \, dx = 1 \end{cases} \qquad (1)$$

As $t \to +\infty$, $u(x,t) \sim \mathcal{U}(x,t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{n/2}}$.

Optimal rate of convergence of $\|u(\cdot,t) - \mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^n)}$?

The time dependent rescaling

$$u(x,t) = \frac{1}{R^n(t)}\, v\left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0)\right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{n/2}\, e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R} \quad R(0) = 1 \quad \tau(0) = 0$:

$$R(t) = \sqrt{1 + 2\,t}\,, \quad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi\, v) & \xi \in \mathbb{R}^n\,,\ \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^n} u_0\, dx = 1 \end{cases} \tag{2}$$

Entropy (relative to the stationary solution v_∞):

$$\Sigma[v] := \int_{\mathbf{R}^n} v \log\left(\frac{v}{v_\infty}\right) dx = \int_{\mathbf{R}^n} \left(v \log v + \frac{1}{2}|x|^2 v\right) dx + Const$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau}\Sigma[v(\cdot,\tau)] = -\int_{\mathbf{R}^n} v \left|\nabla \log\left(\frac{v}{v_\infty}\right)\right|^2 dx =: -I[v(\cdot,\tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\int_{\mathbf{R}^n} v \log v \, dx + n\left(1 + \frac{1}{2}\log(2\pi)\right) \leq \frac{1}{2}\int_{\mathbf{R}^n} \frac{|\nabla v|^2}{v} dx$$

Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-n/2} e^{-|\xi|^2/2}$

$$\Longrightarrow \Sigma[v(\cdot,\tau)] \leq \tfrac{1}{2} I[v(\cdot,\tau)]$$

$$\Sigma[v(\cdot,\tau)] \leq e^{-2\tau}\Sigma[u_0] = e^{-2\tau}\int_{\mathbf{R}^n} u_0 \log\left(\frac{u_0}{v_\infty}\right) dx$$

- Csiszár-Kullback inequality: Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbf{R}^n} v\,dx = \int_{\mathbf{R}^n} \bar{v}\,dx =: M > 0$

$$\int_{\mathbf{R}^n} v \log\left(\frac{v}{\bar{v}}\right)\,dx \geq \frac{1}{4\,M}\|v - \bar{v}\|^2_{L^1(\mathbf{R}^n)}$$

$$\Longrightarrow \|v - v_\infty\|^2_{L^1(\mathbf{R}^n)} \leq 4M\boldsymbol{\Sigma}[u_0]\,e^{-2\tau}$$

$\tau(t) = \log\sqrt{1+2t}$

$$\|u(\cdot,t) - u_\infty(\cdot,t)\|^2_{L^1(\mathbf{R}^n)} \leq \frac{4}{1+2t}\boldsymbol{\Sigma}[u_0]$$

$$u_\infty(x,t) = \frac{1}{R^n(t)}\,v_\infty\left(\frac{x}{R(t)}, \tau(t)\right)$$

Proof of the Csiszár-Kullback inequality: Taylor development at second order.

Euclidean logarithmic Sobolev inequality: other formulations

1) independent from the dimension [Gross, 75]

$$\int_{\mathbb{R}^n} w \log w \, d\mu(x) \leq \frac{1}{2} \int_{\mathbb{R}^n} w \, |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M\, v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x)\, dx$.

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbb{R}^n} w^2 \log w^2 \, dx \leq \frac{n}{2} \log \left(\frac{2}{\pi\, n\, e} \int_{\mathbb{R}^n} |\nabla w|^2 \, dx \right)$$

for any $w \in H^1(\mathbb{R}^n)$ such that $\int w^2 \, dx = 1$

Proof: take $w = \sqrt{\frac{v}{M\, v_\infty}}$ and optimize for $w_\lambda(x) = \lambda^{n/2} w(\lambda\, x)$

$$\int_{\mathbb{R}^n} |\nabla w_\lambda|^2\, dx - \int_{\mathbb{R}^n} w_\lambda^2 \log w_\lambda^2\, dx$$
$$= \lambda^2 \int_{\mathbb{R}^n} |\nabla w|^2\, dx - \int_{\mathbb{R}^n} w^2 \log w^2\, dx - n \log \lambda \int_{\mathbb{R}^n} w^2\, dx$$

□

Entropy-entropy production method: a proof of the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau}\left(I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)]\right) = -C \sum_{i,j=1}^{n} \int_{\mathbb{R}^n} \left| w_{ij} + a\,\frac{w_i w_j}{w} + b\, w\, \delta_{ij} \right|^2 dx < 0$$

for some $C > 0$, a, $b \in \mathbb{R}$. Here $w = \sqrt{v}$.

$I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \searrow I[v_\infty] - 2\Sigma[v_\infty] = 0$

$\Longrightarrow \quad \forall\, u_0\,, \quad I[u_0] - 2\Sigma[u_0] \geq I[v(\cdot,\tau)] - 2\Sigma[v(\cdot,\tau)] \geq 0$ for $\tau > 0$

I-B. Entropy-entropy production method: improved convex Sobolev inequalities

Goal: large time behavior of parabolic equations:

$$\begin{cases} v_t = \mathrm{div}_x[D(x)\,(\nabla_x v + v\nabla_x A(x))] = \mathrm{div}[D(x)\,e^{-A}\nabla(ve^{A})] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R}^n \\ v(x,t=0) = v_0(x) \in L^1_+(\mathbb{R}^n) \end{cases} \tag{3}$$

$A(x)\ldots$ given 'potential'
$v_\infty(x) = e^{-A(x)} \in L^1 \ldots$ (unique) steady state
mass conservation: $\int_{\mathbb{R}^d} v(t)\,dx = \int_{\mathbb{R}^d} v_\infty\,dx = 1$

questions: exponential rate ? connection to logarithmic Sobolev inequalities ?

Entropy-entropy production method

[Bakry, Emery, 84]

[Toscani '96], [Arnold, Markowich, Toscani, Unterreiter, 01]

Relative entropy of $v(x)$ w.r.t. $v_\infty(x)$:

$$\Sigma[v|v_\infty] := \int_{\mathbf{R}^d} \psi\left(\frac{v}{v_\infty}\right) v_\infty \, dx \geq 0$$

with

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \end{aligned}$$

Admissibility condtion: $(\psi''')^2 \leq \frac{1}{2}\psi''\psi^{IV}$

Examples:

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(v|v_\infty) = \int v \ln\left(\frac{v}{v_\infty}\right) dx \ldots$ physical entropy

$\psi_p = w^p - p(w-1) - 1$, $1 < p \leq 2$, $\quad \Sigma_2(v|v_\infty) = \int_{\mathbf{R}^d} (v - v_\infty)^2 v_\infty^{-1} dx$

Exponential decay of entropy production

$$I(v(t)|v_\infty) := \frac{d}{dt}\Sigma[v(t)|v_\infty] = -\int \psi''\left(\frac{v}{v_\infty}\right) |\underbrace{\nabla\left(\frac{v}{v_\infty}\right)}_{=:u}|^2 v_\infty \, dx \leq 0$$

Assume: $D \equiv 1, \quad \underbrace{\frac{\partial^2 A}{\partial x^2}}_{\text{Hessian}} \geq \lambda_1 Id, \ \lambda_1 > 0 \qquad (A(x) \ldots \text{unif. convex})$

Entropy production rate:

$$\begin{aligned} I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u \, v_\infty \, dx + \underbrace{2\int \mathrm{Tr}\,(XY)\, v_\infty \, dx}_{\geq 0} \\ &\geq -2\lambda_1 I \end{aligned}$$

Positivity of $\mathrm{Tr}\,(XY)$?

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} \geq 0$$

$$Y = \begin{pmatrix} \sum_{ij}(\frac{\partial u_i}{\partial x_j})^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0$$

$$\Rightarrow \quad |I(t)| \leq e^{-2\lambda_1 t}\,|I(t=0)| \qquad t > 0$$

$$\forall v_0 \text{ with } |I(v_0|v_\infty)| < \infty$$

EXPONENTIAL DECAY OF RELATIVE ENTROPY

Known: $I' \geq -2\lambda_1 \underbrace{I}_{=\Sigma'}$

$\int_t^\infty \ldots dt$ $\Rightarrow \Sigma' = I \leq -2\lambda_1 \Sigma$ (4)

Theorem 1 [Bakry, Emery], [Arnold, Markowich, Toscani, Unterreiter]

$\frac{\partial^2 A}{\partial x^2} \geq \lambda_1 Id$ (*"Bakry–Emery condition"*), $\Sigma[v_0|v_\infty]<\infty$

$$\Rightarrow \Sigma[v(t)|v_\infty] \leq \Sigma[v_0|v_\infty]\, e^{-2\lambda_1 t}, \qquad t > 0$$

$\|v(t) - v_\infty\|_{L^1}^2 \leq C\, \Sigma[v(t)|v_\infty] \ldots$ Csiszár-Kullback

Convex Sobolev inequalities

Entropy–entropy production estimate (4) for $A(x) = -\ln v_\infty$ (uniformly convex):

$$\Sigma[v|v_\infty] \leq \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

Example 1: logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln \left(\frac{v}{v_\infty} \right) dx \leq \frac{1}{2\lambda_1} \int v \left| \nabla \ln \left(\frac{v}{v_\infty} \right) \right|^2 dx$$

$$\forall v,\ v_\infty \in L^1_+(\mathbb{R}^n),\ \int v\, dx = \int v_\infty\, dx = 1$$

logarithmic Sobolev inequality – "entropy version"

Logarithmic Sobolev inequality–dv_∞ measure version [Gross '75]

$$f^2 = \frac{v}{v_\infty} \Rightarrow \int f^2 \ln f^2 dv_\infty \le \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

$$\forall f \in L^2(\mathbb{R}^n, dv_\infty),\ \int f^2 dv_\infty = 1$$

Example 2: non-logarithmic entropies:

$\psi_p(w) = w^p - p(w-1) - 1, \qquad 1 < p \le 2$

$$(B_p)\ \ \frac{p}{p-1}\left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}}\, dv_\infty\right)^p\right] \le \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

from (4) with $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} dv_\infty}$ $\qquad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, v_\infty dx)$

Poincaré-type inequality [Beckner '89], $(B_p) \Rightarrow (B_2), \quad 1 < p \le 2$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production:

$$\begin{aligned} I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right) u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) v_\infty dx}_{\geq 0} \\ &\geq -2\lambda_1 I \end{aligned}$$

[Arnold, J.D.]: Observe that $\psi_p(w) = w^p - p(w-1) - 1$, $1 < p < 2$:

$$X = \begin{pmatrix} \psi''\left(\frac{v}{v_\infty}\right) & \psi'''\left(\frac{v}{v_\infty}\right) \\ \psi'''\left(\frac{v}{v_\infty}\right) & \frac{1}{2}\psi^{IV}\left(\frac{v}{v_\infty}\right) \end{pmatrix} > 0$$

- Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 Id \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$, $\qquad \kappa = \frac{2-p}{p} < 1$

$$\Rightarrow \boxed{k(\Sigma[v|v_\infty]) \leq \frac{1}{2\lambda_1}|\Sigma'|} = \frac{1}{2\lambda_1}\int \psi''\left(\frac{v}{v_\infty}\right)|\nabla \frac{v}{v_\infty}|^2 dv_\infty$$

Refined convex Sobolev inequality with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty \Rightarrow$

Theorem 2

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^{2(p-1)} \left(\int f^2 dv_\infty\right)^{\frac{2-p}{p}}\right]$$
$$\leq \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^n, dv_\infty)$$

Refined Beckner inequality [Arnold, J.D. '00]
$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$

I-C. Applications

Homogeneous kinetic equations [L. Desvillettes, C. Villani, G. Toscani,...]

Drift-diffusion-Poisson equations for semi-conductors [A. Arnold, P. Markowich, G. Toscani], [P. Biler, J.D., P. Markowich]

Streater's model [P. Biler, J.D., M. Esteban, G. Karch]

Heat equation with a source term [[J.D., G. Karch]

The flashing ratchet [J.D., D. Kinderlehrer, M. Kowalczyk]

Models for traffic flow [J.D., Reinhard Illner]

Navier-Stokes in dimension 2 [T. Gallay, Wayne], [C. Villani], [J.D., A. Munnier]

II — Porous media / fast diffusion equation and generalizations

[coll. Manuel del Pino (Universidad de Chile)] $\Longrightarrow$ Relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Other approaches:
1) "entropy – entropy-production method"
2) mass transportation techniques
3) hypercontractivity for appropriate semi-groups

• *nonlinear diffusions:* [Carrillo, Toscani], [Del Pino, J.D.], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vazquez]

Porous media / fast diffusion equation

$$\begin{aligned} u_t &= \Delta u^m \quad \text{in } \mathbb{R}^n \\ u_{|t=0} &= u_0 \geq 0 \\ u_0(1+|x|^2) \in L^1 \,, &\quad u_0^m \in L^1 \end{aligned} \tag{5}$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0 \, dx = 1$, the self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-n/(2-n(1-m))})$ as $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-n/(2-n(1-m))})$$

Rescaling: Take $u(t,x) = R^{-n}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{n(1-m)-1}\ , \quad R(0) = 1\ , \quad \tau = \log R$$

$$\boxed{v_\tau = \Delta v^m + \nabla \cdot (x\, v)\ , \quad v_{|\tau=0} = u_0}$$

[Ralston, Newman, 1984] Lyapunov functional: *Entropy*

$$\boxed{\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right)\, dx - \Sigma_0}$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\ , \quad I[v] = \int v \left| \frac{\nabla v^{m-1}}{v} + x \right|^2\, dx$$

Stationary solution: C s.t. $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m}\,|x|^2\right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$.

$$\Sigma[v] = \int \psi\left(\frac{v^m}{v_\infty^m}\right)\, v_\infty^{m-1}\, dx \quad \textit{with } \psi(t) = \frac{m t^{1/m} - 1}{1-m} + 1$$

Theorem 1 $m \in [\frac{n-1}{n}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\boxed{I[v] \geq 2\,\Sigma[v]}$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^{m-1}}{v} + x \right|^2 dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$

$$\boxed{\tfrac{1}{2}\left(\tfrac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\tfrac{1}{1-m} - n\right) \int |w|^{1+p} dx + K \geq 0}$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$

$m = \frac{n-1}{n}$: Sobolev, $m \to 1$: logarithmic Sobolev

[Del Pino, J.D.], [Carrillo, Toscani], [Otto]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

$$1 < p \le \frac{n}{n-2} \text{ for } n \ge 3$$

$$\|w\|_{2p} \le A\, \|\nabla w\|_2^{\theta}\, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi n}\right)^{\frac{\theta}{2}} \left(\frac{2y-n}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{n}{2})}\right)^{\frac{\theta}{n}}$$

$$\theta = \frac{n(p-1)}{p(n+2-(n-2)p)},\ y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$. Uses [Serrin-Pucci], [Serrin-Tang].

$1 < p = \frac{1}{2m-1} \le \frac{n}{n-2} \iff$ Fast diffusion case: $\frac{n-1}{n} \le m < 1$

$0 < p < 1 \iff$ Porous medium case: $m > 1$

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau} +$ Csiszár-Kullback inequalities

$\Rightarrow$ Intermediate asymptotics [Del Pino, J.D.]

(i) $\frac{n-1}{n} < m < 1$ if $n \geq 3$

$$\limsup_{t \to +\infty} t^{\frac{1-n(1-m)}{2-n(1-m)}} \| u^m - u_\infty^m \|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t \to +\infty} t^{\frac{1+n(m-1)}{2+n(m-1)}} \| \, [u - u_\infty] \, u_\infty^{m-1} \, \|_{L^1} < +\infty$$

Generalization

Intermediate asymptotics for:

$$u_t = \Delta_p u^m$$

Convergence to a stationary solution for:

$$v_t = \Delta_p v^m + \nabla(x\, v)$$

Let $q = 1 + m - (p-1)^{-1}$. Whether q is bigger or smaller than 1 determines two different regimes like for $p = 1$.

$q < 1 \iff$ Fast diffusion case

$q > 1 \iff$ Porous medium case

For $q > 0$, define the *entropy* by

$$\Sigma[v] = \int \Big[\sigma(v) - \sigma(v_\infty) - \sigma'(v_\infty)(v - v_\infty)\Big]\; dx$$

$\sigma(s) = \frac{s^q - 1}{q-1}$ if $q \neq 1$

$\sigma(s) = s \log s$ if $q = 1$ ($p \neq 2$: see below)

Nonhomogeneous version – Gagliardo-Nirenberg ineq.

$b = \frac{p\,(p-1)}{p^2-p-1}$, $a = b\,q$, $v = w^b$. For $p \neq 2$, let

$$\mathcal{F}[v] = \int v^{-\frac{1}{p-1}} |\nabla v|^p \, dx - \frac{1}{q} \left(\frac{n}{1-\kappa_p} + \frac{p}{p-2} \right) \int v^q \, dx$$

$\kappa_p = \frac{1}{p}\,(p-1)^{\frac{p-1}{p}}$. Based on [Serrin, Tang] (uniqueness result)

Corollary 3 $n \geq 2$, $(2n+1)/(n+1) \leq p < n$. $\forall\, v$ *s.t.* $\|v\|_{L^1} = \|v_\infty\|_{L^1}$

$$\mathcal{F}[v] \geq \mathcal{F}[v_\infty]$$

$$\|w\|_b \leq \mathcal{S}\, \|\nabla w\|_p^\theta \, \|w\|_a^{1-\theta} \quad \text{if } a > p$$
$$\|w\|_a \leq \mathcal{S}\, \|\nabla w\|_p^\theta \, \|w\|_b^{1-\theta} \quad \text{if } a < p$$

[Del Pino, J.D.] Intermediate asymptotics of $u_t = \Delta_p u^m$

Theorem 2 $n \geq 2$, $1 < p < n$, $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{p}{p-1}$ *and* $q = 1 + m - \frac{1}{p-1}$

$$\text{(i)} \quad \|u(t,\cdot) - u_\infty(t,\cdot)\|_q \leq K\, R^{-(\frac{\alpha}{2}+n(1-\frac{1}{q}))}$$

$$\text{(ii)} \quad \|u^q(t,\cdot) - u_\infty^q(t,\cdot)\|_{1/q} \leq K\, R^{-\frac{\alpha}{2}}$$

(i): $\frac{1}{p-1} \leq m \leq \frac{p}{p-1}$ *(ii):* $\frac{n-(p-1)}{n(p-1)} \leq m \leq \frac{1}{p-1}$

$\alpha = (1 - \frac{1}{p}(p-1)^{\frac{p-1}{p}})\,\frac{p}{p-1}$, $R = (1+\gamma\, t)^{1/\gamma}$, $\gamma = (mn+1)(p-1) - (n-1)$
$u_\infty(t,x) = \frac{1}{R^n}\, v_\infty(\log R, \frac{x}{R})$
$v_\infty(x) = (C - \frac{p-1}{mp}\,(q-1)\,|x|^{\frac{p}{p-1}})_+^{1/(q-1)}$ *if* $m \neq \frac{1}{p-1}$
$v_\infty(x) = C\, e^{-(p-1)^2|x|^{p/(p-1)}/p}$ *if* $m = (p-1)^{-1}$.

Use $v_t = \Delta_p v^m + \nabla \cdot (x\, v)$
$w = v^{(mp+q-(m+1))/p}$, $a = b\, q = p\,\frac{m(p-1)+p-2}{mp(p-1)-1}$.

Case $q \neq 1$: apply one of the optimal Gagliardo-Nirenberg inequalities.
Case $q = 1$: apply the optimal L^p-Euclidean logarithmic Sobolev inequality.

$$\frac{d\Sigma}{dt} \leq -C\,\Sigma\,.$$

Csiszár-Kullback inequality: an extension [Cáceres-Carrillo-JD]

Lemma 3 *Let f and g be two nonnegative functions in $L^q(\Omega)$ for a given domain Ω in $\mathbb{R}^n$. Assume that $q \in (1,2]$. Then*

$$\int_\Omega [\sigma(\frac{f}{g}) - \sigma'(1)(\frac{f}{g} - 1)]\, g^q\, dx \geq \frac{q}{2}\, \max(\|f\|^{q-2}_{L^q(\Omega)}, \|g\|^{q-2}_{L^q(\Omega)})\, \|f - g\|^2_{L^q(\Omega)}$$

III – Entropy methods for (non)linear diffusions
The logarithmic Sobolev inequality in $W^{1,p}$

[coll. Manuel del Pino (Universidad de Chile), Ivan Gentil (Ceremade)]

Optimal constants for Gagliardo-Nirenberg ineq.

[Del Pino, J.D.]

Theorem 4 $1<p<n$, $1<a\le\frac{p(n-1)}{n-p}$, $b=p\,\frac{a-1}{p-1}$

$$\|w\|_b \le \mathcal{S}\,\|\nabla w\|_p^{\theta}\,\|w\|_a^{1-\theta} \quad \textit{if } a>p$$
$$\|w\|_a \le \mathcal{S}\,\|\nabla w\|_p^{\theta}\,\|w\|_b^{1-\theta} \quad \textit{if } a<p$$

Equality if $w(x)=A\,(1+B\,|x|^{\frac{p}{p-1}})_+^{-\frac{p-1}{a-p}}$

$a>p$: $\theta=\frac{(q-p)\,n}{(q-1)\,(n\,p-(n-p)\,q)}$

$a<p$: $\theta=\frac{(p-q)\,n}{q\,(n\,(p-q)+p\,(q-1))}$

The optimal L^p-Euclidean logarithmic Sobolev inequality (an optimal under scalings form) [Del Pino, J.D., 2001], [Gentil 2002], [Cordero-Erausquin, Gangbo, Houdré, 2002]

Theorem 5 *If* $\|u\|_{L^p} = 1$, *then*

$$\int |u|^p \log |u| \, dx \le \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla u|^p \, dx\right]$$

$$\mathcal{L}_p = \frac{p}{n} \left(\frac{p-1}{e}\right)^{p-1} \pi^{-\frac{p}{2}} \left[\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(n\,\frac{p-1}{p}+1)}\right]^{\frac{p}{n}}$$

Equality: $u(x) = \left(\pi^{\frac{n}{2}} (\frac{\sigma}{p})^{\frac{n}{p^*}} \frac{\Gamma(\frac{n}{p^*}+1)}{\Gamma(\frac{n}{2}+1)}\right)^{-1/p} e^{-\frac{1}{\sigma}|x-\bar{x}|^{p^*}}$

$p = 2$: Gross' logaritmic Sobolev inequality [Gross, 75], [Weissler, 78]

$p = 1$: [Ledoux 96], [Beckner, 99]

For some purposes, it is sometimes more convenient to use this inequality in a non homogeneous form, which is based upon the fact that

$$\inf_{\mu>0}\left[\frac{n}{p}\log\left(\frac{n}{p\mu}\right)+\mu\,\frac{\|\nabla w\|_p^p}{\|w\|_p^p}\right]=n\,\log\left(\frac{\|\nabla w\|_p}{\|w\|_p}\right)+\frac{n}{p}\,.$$

Corollary 6 *For any $w\in W^{1,p}(\mathbb{R}^n)$, $w\neq 0$, for any $\mu>0$,*

$$p\int|w|^p\log\left(\frac{|w|}{\|w\|_p}\right)\,dx+\frac{n}{p}\log\left(\frac{p\,\mu\,e}{n\,\mathcal{L}_p}\right)\int|w|^p\,dx\leq\mu\int|\nabla w|^p\,dx\,.$$

Consequences [M. Del Pino, J.D., I. Gentil]

III-A. Existence and uniqueness:
Cauchy problem for $u_t = \Delta_p(u^{1/(p-1)}) \quad (x,t) \in \mathbb{R}^n \times \mathbb{R}^+$

III-B. Hypercontractivity, Ultracontractivity, Large deviations:
Connections with $u_t + |\nabla v|^p = 0$

II-A. Existence and uniqueness

Consider the Cauchy problem

$$\begin{cases} u_t = \Delta_p(u^{1/(p-1)}) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ u(\cdot, t=0) = f \geq 0 \end{cases} \tag{6}$$

$\Delta_p u^m = \mathrm{div}\left(|\nabla u^m|^{p-2}\nabla u^m\right)$ is 1-homogeneous $\Longleftrightarrow m = 1/(p-1)$.

Notations: $\|u\|_q = (\int_{\mathbb{R}^n} |u|^q\, dx)^{1/q}$, $q \neq 0$. $p^* = p/(p-1)$, $p > 1$.

Theorem 7 *Let $p > 1$, $f \in L^1(\mathbb{R}^n)$ s.t. $|x|^{p^*} f$, $f \log f \in L^1(\mathbb{R}^n)$. Then there exists a unique weak nonnegative solution $u \in C(\mathbb{R}_t^+, L^1)$ of (6) with initial data f, such that $u^{1/p} \in L^1_{\mathrm{loc}}(\mathbb{R}_t^+, W^{1,p}_{\mathrm{loc}})$.*

[Alt-Luckhaus, 83] [Tsutsumi, 88] [Saa, 91] [Chen, 00] [Agueh, 02]

[Bernis, 88], [Ishige, 96]

The *a priori* estimate on the *entropy term* $\int u \log u\, dx$ plays a crucial role in the proof.

(6) is 1-homogenous: we assume that $\int f\,dx = 1$. u is a solution of (6) if and only if v is a solution of

$$\begin{cases} v_\tau = \Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, \tau = 0) = f \end{cases} \tag{7}$$

provided u and v are related by the transformation

$$u(x,t) = \frac{1}{R(t)^n}\, v(\xi, \tau)\,,\ \xi = \frac{x}{R(t)}\,,\ \tau(t) = \log R(t)\,,\ R(t) = (1 + p\,t)^{1/p}$$

[DelPino, J.D., 01]. Let

$$v_\infty(\xi) = \pi^{-\frac{n}{2}} (\frac{p}{\sigma})^{n/p^*} \frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{p^*}+1)}\ \exp(-\frac{p}{\sigma}\,|x|^{p^*})\ , \quad \sigma = (p^*)^2$$

$\forall \mu > 0$, $\mu\, v_\infty$ is a nonnegative solution of the stationary equation

$$\Delta_p v^{1/(p-1)} + \nabla_\xi(\xi\, v) = 0$$

In the original variables, t and x: consider $u_\infty = \frac{1}{R(t)^n} v_\infty(\frac{x}{R(t)}, \log R(t))$.

$$\int u \log\left(\frac{u}{u_\infty}\right) dx = \int u \log u \, dx + (p-1)(R(t))^{-p^*} \int |x|^{p^*} u \, dx + \sigma(t) \int u \, dx$$

Note that:

$$\frac{d}{dt} \int u \log u \, dx = -\frac{1}{p-1} \int \left| p^* \nabla u^{1/p} \right|^p dx \,.$$

Lemma 8 [Benguria, 79], [Benguria, Brezis, Lieb, 81], [Diaz,Saa, 87]
On the space $\{u \in L^1(\mathbb{R}^n) \ : \ u^{1/p} \in W^{1,p}(\mathbb{R}^n)\}$*, the functional* $F[u] := \int |\nabla u^\alpha|^p \, dx$ *is convex for any* $p > 1$*,* $\alpha \in [\frac{1}{p}, 1]$*.*

From $(p-1)\nabla u^{1/(p-1)} = p \, u^{1/(p(p-1))} \nabla u^{1/p}$, we get by Hölder's inequality (with Hölder exponents p and p^*)

$$\|\nabla u^{1/(p-1)}\|_{p-1} \le p^* \|u\|_1^{1/(p(p-1))} \, \|\nabla u^{1/p}\|_p$$

Uniqueness. Consider two solutions u_1 and u_2 of (6).

$$\frac{d}{dt}\int u_1 \log\left(\frac{u_1}{u_2}\right) dx$$
$$= \int \left(1 + \log\left(\frac{u_1}{u_2}\right)\right) (u_1)_t \, dx - \int \left(\frac{u_1}{u_2}\right) (u_2)_t \, dx$$
$$= -(p-1)^{-(p-1)} \int u_1 \left[\frac{\nabla u_1}{u_1} - \frac{\nabla u_2}{u_2}\right] \cdot \left[\left|\frac{\nabla u_1}{u_1}\right|^{p-2} \frac{\nabla u_1}{u_1} - \left|\frac{\nabla u_2}{u_2}\right|^{p-2} \frac{\nabla u_2}{u_2}\right] dx \, .$$

It is then straightforward to check that two solutions with same initial data f have to be equal since

$$\frac{1}{4\,\|f\|_1} \|u_1(\cdot,t) - u_2(\cdot,t)\|_1^2 \le \int u_1(\cdot,t) \log\left(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\right) dx \le \int f \log\left(\frac{f}{f}\right) dx = 0$$

by the Csiszár-Kullback inequality.

III-B. Hypercontractivity, Ultracontractivity, Large deviations

Understanding the regularizing properties of

$$u_t = \Delta_p u^{1/(p-1)}$$

Theorem 9 *Let α, $\beta \in [1, +\infty]$ with $\beta \geq \alpha$. Under the same assumptions as in the existence Theorem, if moreover $f \in L^\alpha(\mathbb{R}^n)$, any solution with initial data f satisfies the estimate*

$$\|u(\cdot,t)\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta)\, t^{-\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}} \quad \forall\, t > 0$$

with $A(n,p,\alpha,\beta) = \left(\mathcal{C}_1\,(\beta-\alpha)\right)^{\frac{n}{p}\,\frac{\beta-\alpha}{\alpha\beta}}\,\mathcal{C}_2^{\frac{n}{p}}$, $\mathcal{C}_1 = n\,\mathcal{L}_p\,e^{p-1}\,\frac{(p-1)^{p-1}}{p^{p+1}}$,

$\mathcal{C}_2 = \frac{(\beta-1)^{\frac{1-\beta}{\beta}}}{(\alpha-1)^{\frac{1-\alpha}{\alpha}}}\,\frac{\beta^{\frac{1-p}{\beta}-\frac{1}{\alpha}+1}}{\alpha^{\frac{1-p}{\alpha}-\frac{1}{\beta}+1}}$. Case $p = 2$, $\mathcal{L}_2 = \frac{2}{\pi\, n\, e}$, [Gross 75]

As a special case of Theorem 9, we obtain an *ultracontractivity* result in the limit case corresponding to $\alpha = 1$ and $\beta = \infty$.

Corollary 10 *Consider a solution u with a nonnegative initial data $f \in L^1(\mathbb{R}^n)$. Then for any $t > 0$*

$$\|u(\cdot,t)\|_\infty \leq \|f\|_1 \left(\frac{\mathcal{C}_1}{t}\right)^{\frac{n}{p}} .$$

Case $p = 2$, [Varopoulos 85]

Proof. Take a nonnegative function $u \in L^q(\mathbb{R}^n)$ with $u^q \log u$ in $L^1(\mathbb{R}^n)$. It is straightforward that

$$\frac{d}{dq}\int u^q\,dx = \int u^q \log u\;dx\;. \tag{8}$$

Consider now a solution $u_t = \Delta_p u^{1/(p-1)}$. For a given $q \in [1, +\infty)$,

$$\frac{d}{dt}\int u^q\,dx = -\frac{q(q-1)}{(p-1)^{p-1}}\int u^{q-p}|\nabla u|^p\,dx\;. \tag{9}$$

Assume that q depends on t and let $F(t) = \|u(\cdot,t)\|_{q(t)}$. Let $' = \frac{d}{dt}$. A combination of (8) and (9) gives

$$\frac{F'}{F} = \frac{q'}{q^2}\left[\int \frac{u^q}{F^q}\log\left(\frac{u^q}{F^q}\right)dx - \frac{q^2(q-1)}{q'(p-1)^{p-1}}\,\frac{1}{F^q}\int u^{q-p}|\nabla u|^p\,dx\right]\;.$$

Since $\int u^{q-p}|\nabla u|^p\,dx = (\frac{p}{q})^p \int |\nabla u^{q/p}|^p\,dx$, Corollary 6 applied with $w = u^{q/p}$,

$$\mu = \frac{(q-1)\,p^p}{q'\,q^{p-2}\,(p-1)^{p-1}}$$

gives for any $t \geq 0$

$$F(t) \leq F(0)\,e^{A(t)} \qquad \text{with } A(t) = \frac{n}{p}\int_0^t \frac{q'}{q^2}\log\left(\mathcal{K}_p\,\frac{q^{p-2}\,q'}{q-1}\right)ds$$

$$\text{and } \mathcal{K}_p = \frac{n\,\mathcal{L}_p}{e}\,\frac{(p-1)^{p-1}}{p^{p+1}}\,.$$

Now let us minimize $A(t)$: the optimal function $t \mapsto q(t)$ solves the ODE

$$q''\,q = 2\,{q'}^2 \Longleftrightarrow q(t) = \frac{1}{a\,t+b}\,.$$

Take $q_0 = \alpha$, $q(t) = \beta$ allows to compute $at = \frac{\alpha-\beta}{\alpha\beta}$ and $b = \frac{1}{\alpha}$. $\square$

Large deviations and Hamilton-Jacobi equations

Consider a solution of

$$\begin{cases} v_t + \frac{1}{p}\,|\nabla v|^p = \frac{1}{p-1}\,p^{\frac{2-p}{p-1}}\,\varepsilon^{p^*}\Delta_p v & (x,t) \in \mathbb{R}^n \times \mathbb{R}^+ \\ v(\cdot, t=0) = g \end{cases} \qquad (10)$$

Lemma 11 *Let $\varepsilon > 0$. Then v is a C^2 solution of (10) iff*

$$u = e^{-\frac{1}{\lambda\varepsilon^{p^*}}\,v} \quad \text{with } \lambda = \frac{p^{\frac{1}{p-1}}}{p-1}$$

is a C^2 positive solution of

$$u_t = \varepsilon^p\,\Delta_p(u^{1/(p-1)})$$

with initial data $f = e^{-\frac{1}{\lambda\varepsilon^{p^*}}\,g}$.

Conclusion: The three following identities are equivalent:

(i) For any $w \in W^{1,p}(\mathbb{R}^n)$ with $\int |w|^p \, dx = 1$,

$$\int |w|^p \log |w| \, dx \leq \frac{n}{p^2} \log \left[\mathcal{L}_p \int |\nabla w|^p \, dx \right]$$

(ii) Let P_t^p be the semigroup associated $u_t = \Delta_p(u^{1/(p-1)})$:

$$\|P_t^p f\|_\beta \leq \|f\|_\alpha \, A(n,p,\alpha,\beta) \, t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}}$$

(iii) Let Q_t^p be the semigroup associated to $v_t + \frac{1}{p} |\nabla v|^p = 0$:

$$\|e^{Q_t^p g}\|_\beta \leq \|e^g\|_\alpha \, B(n,p,\alpha,\beta) \, t^{-\frac{n}{p}\frac{\beta-\alpha}{\alpha\beta}}$$

IV — Inequalities and transport

[Cordero-Erausquin, Gangbo, Houdré, Nazaret, Villani],
[Agueh, Ghoussoub, Kang]

- Sobolev inequality: $\|f\|_{L^{2^*}} \leq S\,\|\nabla f\|_{L^2}$
- (Standard) logarithmic Sobolev inequality
- Logarithmic Sobolev inequality in $W^{1,p}(\mathbb{R}^N)$

Sobolev inequalities

$$\|f\|_{L^{2^*}} \leq S\,\|\nabla f\|_{L^2}$$

$N \geq 3$. Optimal function: $f(x) = (\sigma + |x|^2)^{-(N-2)/2}$.

A proof based on mass transportation:

$$\inf\left\{\frac{1}{2\lambda^2}\int_{\mathbf{R}^N} |\nabla f|^2\,dx \;:\; \int_{\mathbf{R}^N} |f|^{2^*}\,dx = 1\right\}$$

$$= \tfrac{n(n-2)}{2(n-1)}\,\sup\left\{\textstyle\int_{\mathbf{R}^N} |g|^{2^*(1-\frac{1}{n})}dy - \frac{\lambda^2}{2}\int_{\mathbf{R}^N} |y|^2\,|g|^{2^*}dy \;:\; \int_{\mathbf{R}^N} |g|^{2^*}dy = 1\right\}$$

Mass transportation: basic results

μ and ν two Borel probability measures on $\mathbb{R}^N$. $T : \mathbb{R}^N \to \mathbb{R}^N$
$T\#\mu = \nu \iff \nu(A) = \mu(T^{-1}(A))$ for any Borel measurable set A.

Theorem 12 (Brenier, McCann) *$\exists T = \nabla\phi$ such that $T\#\mu = \nu$ and ϕ is convex.*

$\mu = F(x)\,dx$, $\nu = G(x)\,dx$, $\int_{\mathbf{R}^N} F(x)\,dx = \int_{\mathbf{R}^N} G(y)\,dy = 1$

$$\forall\, b \in C(\mathbb{R}^N, \mathbb{R}^+) \quad \int_{\mathbf{R}^N} b(y)G(y)\,dy = \int_{\mathbf{R}^N} b(\nabla\phi(x))F(x)\,dx$$

Under technical assumptions: $\phi \in C^2$, supp(F) or supp(G) is convex... [Caffarelli] ϕ solves the Monge-Ampère equation

$$G(\nabla\phi)\,\det \mathrm{Hess}(\phi) = F$$

A proof of the Sobolev inequality

$$G(\nabla\phi)^{-\frac{1}{n}} = (\det \mathrm{Hess}(\phi))^{\frac{1}{n}} F^{-\frac{1}{n}} \leq \frac{1}{n} \Delta\phi\, F^{-\frac{1}{n}}$$

$$\int G(y)^{1-\frac{1}{n}} dy \leq \frac{1}{n} \int G(\nabla\phi(x))^{1-\frac{1}{n}} (\det \mathrm{Hess}(\phi))^{\frac{1}{n}} \Delta\phi\, dx$$

$$= \frac{1}{n} \int F^{1-\frac{1}{n}} \Delta\phi\, dx = -\frac{1}{n} \int \nabla(F^{1-\frac{1}{n}}) \cdot \nabla\phi\, dx$$

by the arithmetic-geometric inequality. $F = |f|^{2^*}$, $G = |g|^{2^*}$

$$\int |g|^{2^*(1-\frac{1}{n})} dy \leq -\frac{2(n-1)}{n(n-2)} \int (f^{\frac{n}{n-2}}) \nabla f \cdot \nabla\phi\, dx$$

$$\frac{n(n-2)}{2(n-1)} \int |g|^{2^*(1-\frac{1}{n})} dy \leq \frac{2}{\lambda^2} \int |\nabla f|^2 dx + \frac{\lambda^2}{2} \int |f|^{2^*} |\nabla\phi|^2 dx$$

by Young's inequality. Use: $\int F\, |\nabla\phi|^2\, dx = \int G |y|^2 dy$

A proof of the standard logarithmic Sobolev inequality

$G(y) = e^{-|y|^2/2}$, $F(x) = f(x)\, e^{-|x|^2/2}$, $\nabla\phi \# F\, dx = G\, dy$.

$$e^{-|\nabla\phi|^2/2} \det \mathrm{Hess}(\phi) = f(x)\, e^{-|x|^2/2}$$

$\theta(x) = \phi(x) - \frac{1}{2}|x|^2$

$$f(x)\, e^{-|x|^2/2} = \det\left(\mathrm{Id} + \mathrm{Hess}(\theta)\right) e^{-|x+\nabla\theta(x)|^2/2}$$

$$\begin{aligned} \log f - |x|^2/2 &= -|x+\nabla\theta(x)|^2/2 + \log\Big[\det\left(\mathrm{Id} + \mathrm{Hess}(\theta)\right)\Big] \\ &\le -|x+\nabla\theta(x)|^2/2 + \Delta\theta \end{aligned}$$

(use $\log(1+t) \le t$). Let $d\mu(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$.

$$\log f \le -\frac{1}{2}|\nabla\theta|^2 - x\cdot\nabla\theta + \Delta\theta$$

$$\int f \log f\, d\mu \le -\frac{1}{2}\int \left|\sqrt{f}\,\nabla\theta + \frac{\nabla f}{\sqrt{f}}\right|^2 d\mu + \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu \le \frac{1}{2}\int \frac{|\nabla f|^2}{f}\, d\mu$$

LOGARITHMIC SOBOLEV INEQUALITY IN $W^{1,p}(\mathbb{R}^N)$

$G(y) = c_{p,n}\, e^{-\frac{p}{p-1}|y|^{p/(p-1)}} =: f_\infty(y)$, $F(x) = f(x)\, c_{p,n}\, e^{-\frac{p}{p-1}|x|^{p/(p-1)}}$
$\nabla\phi \# F\, dx = G\, dy$, $d\mu(x) = f_\infty^p(x)\, dx$

$$f(x)\, e^{-\frac{p}{p-1}|x|^{p/(p-1)}} = \det\left(\mathrm{Hess}(\phi)\right) e^{-\frac{p}{p-1}|x+\nabla\theta(x)|^{p/(p-1)}}$$

$$f^p(x) = f_\infty^p(\nabla\phi)\det\left(\mathrm{Id} + \mathrm{Hess}(\phi)\right)$$

$$\int f^p \log f^p\, d\mu = \int f^p \log f_\infty^p\, d\mu + \int (\Delta\phi - n) f^p d\mu$$

$$\int \Delta\phi\, f^p d\mu = -p \int f^{p-1}\nabla f \cdot \nabla\phi\, d\mu \le \frac{\lambda^{-q}}{q} \int |f|^p\, |\nabla\phi|^{p/(p-1)} + \frac{\lambda^p}{p} \int |\nabla f|^p d\mu$$

using Young's inequality: $X = f^{p-1}\nabla\phi$, $Y = \nabla f$

$$\int X \cdot Y\, d\mu \le \frac{\lambda^{-q}}{q}\|X\|_q^q + \frac{\lambda^p}{p}\|Y\|_p^p$$