Fast diffusions and generalized entropies

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Abstract

An overview of the connections between functional inequalities, nonlinear diffusions, (transport theory) and generalized entropy functionals

- From functional inequalities to rates in nonlinear diffusions (porous medium equation)

- Functional inequalities and gradient flows

- Large time asymptotics of nonlinear diffusions (fast diffusion equation)
Contents

- $L^q$ Poincaré inequalities and application
  - characterization of some $L^q$ Poincaré inequalities
  - applications to a porous medium equation

- The Bakry-Emery method for generalized Poincaré inequalities
  - a non-local condition (linear case)
  - extension to a porous medium equation

- Remarks on entropies, transport and distances between measures

- The fast diffusion equation
  - intermediate asymptotics and interpolation
  - extensions (finite mass case)
  - the infinite mass regime and Hardy-Poincaré inequalities
Poincaré inequalities for general measures, porous media equation

J.D., Ivan Gentil, Arnaud Guillin and Feng-Yu Wang
Goal

$L^q$-Poincaré inequalities, $q \in (1/2, 1]$

$$\left[\text{Var}_\mu(f^q)\right]^{1/q} := \left[\int f^{2q} d\mu - \left(\int f^q d\mu\right)^2\right]^{1/q} \leq C_P \int |\nabla f|^2 d\nu$$

Application to the weighted porous media equation, $m \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla \psi \cdot \nabla u^m, \quad t \geq 0, \quad x \in \mathbb{R}^d$$

(Ornstein-Uhlenbeck form). With $d\mu = d\nu = d\mu_\psi = e^{-\psi}dx/\int e^{-\psi}dx$

$$\frac{d}{dt}\text{Var}_{\mu_\psi}(u) = -\frac{8}{(m+1)^2} \int |\nabla u^{\frac{m+1}{2}}|^2 d\mu_\psi$$
Outline

Equivalence between the following properties:

- $L^q$-Poincaré inequality
- Capacity-measure criterion
- Weak Poincaré inequality
- BCR (Barthe-Cattiaux-Roberto) criterion

In dimension $d = 1$, there are necessary and sufficient conditions to satisfy the BCR criterion.

Motivation: large time asymptotics in connection with functional inequalities.
**$L^q$-Poincaré inequality**

We shall say that $(\mu, \nu)$ satisfies a $L^q$-Poincaré inequality with constant $C_P$ if for all non-negative functions $f \in C^1(M)$ one has

$$[\text{Var}_\mu(f^q)]^{1/q} \leq C_P \int |\nabla f|^2 \, d\nu$$

$q \in (0, 1]$ (false for $q > 1$ unless $\mu$ is a Dirac measure)

$$\text{Var}_\mu(g^2) = \int g^2 \, d\mu - (\int g \, d\mu)^2 = \mu(g^2) - \mu(g)^2$$

$q \mapsto [\text{Var}_\mu(f^q)]^{1/q}$ increasing wrt $q \in (0, 1]$: $L^q$-Poincaré inequalities form a hierarchy
Capacity-measure criterion

Capacity $\text{Cap}_\nu(A, \Omega)$ of two measurable sets $A$ and $\Omega$ such that $A \subset \Omega \subset M$

$$\text{Cap}_\nu(A, \Omega) := \inf \left\{ \int |\nabla f|^2 \, d\nu : f \in C^1(M), \mathbb{1}_A \leq f \leq \mathbb{1}_\Omega \right\}$$

$$\beta_P := \sup \left\{ \sum_{k \in \mathbb{Z}} \frac{[\mu(\Omega_k)]^{1/(1-q)}}{[\text{Cap}_\nu(\Omega_k, \Omega_{k+1})]^{q/(1-q)}} \right\}^{(1-q)/q}$$

over all $\Omega \subset M$ with $\mu(\Omega) \leq 1/2$ and all sequences $(\Omega_k)_{k \in \mathbb{Z}}$ such that for all $k \in \mathbb{Z}$, $\Omega_k \subset \Omega_{k+1} \subset \Omega$

**Theorem 1**

(i) If $q \in [1/2, 1)$, then $\beta_P \leq 2^{1/q} C_P$

(ii) If $q \in (0, 1)$ and $\beta_P < +\infty$, then $C_P \leq \kappa_P \beta_P$
Weak Poincaré inequalities

Definition 2  [Röckner and Wang] $(\mu, \nu)$ satisfies a weak Poincaré inequality if there exists a non-negative non increasing function $\beta_{WP}(s)$ on $(0, 1/4)$ such that, for any bounded function $f \in C^1(M)$,

$$\forall s > 0, \quad \text{Var}_\mu(f) \leq \beta_{WP}(s) \int |\nabla f|^2 d\nu + s \left[ \text{Osc}_\mu(f) \right]^2$$

$$\text{Var}_\mu(f) \leq \mu((f - a)^2) \quad \forall \ a \in \mathbb{R}$$

For $a = (\text{supess}_\mu f + \text{infess}_\mu f)/2$, $\text{Var}_\mu(f) \leq \left[ \text{Osc}_\mu(f) \right]^2/4$: $s \leq 1/4$.

Proposition 3  Let $q \in [1/2, 1)$. If $(\mu, \nu)$ satisfies the $L^q$-Poincaré inequality, then it also satisfies a weak Poincaré inequality with $\beta_{WP}(s) = (11 + 5\sqrt{5}) \beta_P s^{1-1/q}/2$, $K := (11 + 5\sqrt{5})/2$.

$L^q$-Poincaré $\implies$ BCR criterion $\implies$ weak Poincaré
Theorem 4 [Maz’ja] Let $q \in [1/2, 1)$. For all bounded open set $\Omega \subset M$, if $(\Omega_k)_{k \in \mathbb{Z}}$ is a sequence of open sets such that $\Omega_k \subset \Omega_{k+1} \subset \Omega$, then

$$
\sum_{k \in \mathbb{Z}} \frac{\mu(\Omega_k)^{1/(1-q)}}{[\text{Cap}_\nu(\Omega_k, \Omega_{k+1})]^{q/(1-q)}} \leq \frac{1}{1-q} \int_0^{\mu(\Omega)} \left( \frac{t}{\Phi(t)} \right)^{q/(1-q)} \, dt
$$

where $\Phi(t) := \inf \{ \text{Cap}_\nu(A, \Omega) : A \subset \Omega, \mu(A) \geq t \}$

As a consequence: $\beta_P \leq (1 - q)^{-(1-q)/q} \| t/\Phi(t) \|_{L^{q/(1-q)}(0, \mu(\Omega))}$

Corollary 5 Let $q \in [1/2, 1)$. If $(\mu, \nu)$ satisfies a weak Poincaré inequality with function $\beta_{WP}$, then it satisfies a $L^q$-Poincaré inequality with

$$
\beta_P \leq \frac{11 + 5\sqrt{5}}{2} \left( \frac{4}{1-q} \right)^{1-q} \| \beta_{WP}(\cdot/4) \|_{L^{1-q}(0,1/2)}
$$

$L^q$-Poincaré $\implies$ Weak Poincaré with $\beta_{WP}(s) = C s^{q-1}$ $\implies$ $L^{q'}$-Poincaré $\forall q' \in (0, q)$
BCR criterion (1/2)

A variant of two results of [Barthe, Cattiaux, Roberto, 2005] (no absolute continuity of the measure $\mu$ with respect to the volume measure)

**Theorem 6** [BCR] Let $\mu$ be a probability measure and $\nu$ a positive measure on $M$ such that $(\mu, \nu)$ satisfies a weak Poincaré inequality with function $\beta_{\text{WP}}(s)$. Then for every measurable subsets $A, B$ of $M$ such that $A \subset B$ and $\mu(B) \leq 1/2$,

$$\text{Cap}_\nu(A, B) \geq \frac{\mu(A)}{\gamma(\mu(A))} \quad \text{with} \quad \gamma(s) := 4\beta_{\text{WP}}(s/4)$$

**Proof** □ Take $f$ such that $\mathbb{I}_A \leq f \leq \mathbb{I}_B$: $\text{Osc}_\mu(f) \leq 1$

By Cauchy-Schwarz, $(\int f \, d\mu)^2 \leq \mu(B) \int f^2 \, d\mu \leq \frac{1}{2} \int f^2 \, d\mu$

$$\beta_{\text{WP}}(s) \int |\nabla f|^2 \, d\nu + s \geq \text{Var}_\mu(f) \geq \frac{1}{2} \int f^2 \, d\mu \geq \frac{\mu(A)}{2}$$

$$\frac{a}{\gamma(a)} = \frac{a}{4\beta_{\text{WP}}(a/4)} \leq \sup_{s \in (0, 1/4)} \frac{a/2-s}{\beta_{\text{WP}}(s)} \quad \text{with} \quad a/2 = \mu(A)/2 \leq 1/4 \quad \triangleright$$
**Lemma 7** Take $\mu$ and $\nu$ as before, $\theta \in (0, 1)$, $\gamma$ a positive non increasing function on $(0, \theta)$. If $\forall A, B \subset M$ such that $A \subset B$ are measurable and $\mu(B) \leq \theta$,

$$\text{Cap}_\nu(A, B) \geq \frac{\mu(A)}{\gamma(\mu(A))}$$

then for every function $f \in \mathcal{C}^1(M)$ such that $\mu(\Omega_+) \leq \theta$, $\Omega_+ := \{ f > 0 \}$

$$\int f_+^2 \leq \frac{11 + 5\sqrt{5}}{2} \gamma(s) \int_{\Omega_+} |\nabla f|^2 d\nu + s \left[ \text{supess}_\mu f \right]^2 \quad \forall \ s \in (0, 1)$$

**Theorem 8** Same assumptions, $\theta = 1/2$. Then $\forall f \in \mathcal{C}^1(M)$

$$\text{Var}_\mu(f) \leq \frac{11 + 5\sqrt{5}}{2} \gamma(s) \int |\nabla f|^2 d\nu + s \left[ \text{Osc}_\mu(f) \right] \quad \forall \ s \in (0, 1/4)$$

$\theta = 1/2$: use the median $m_\mu(f)$, $\mu(f \geq m_\mu(f)) \geq 1/2$, $\mu(f \leq m_\mu(f)) \geq 1/2$
Using the BCR criterion: a “Hardy condition”

\[ M = \mathbb{R}, \; d\mu = \rho_\nu \, dx \text{ with median } m_\mu, \; d\nu = \rho_\nu \, dx \]

\[ R(x) := \mu([x, +\infty)) \quad \text{and} \quad L(x) := \mu((-\infty, x]) \]

\[ r(x) := \int_{m_\mu}^{x} \frac{1}{\rho_\nu} \, dx \quad \text{and} \quad \ell(x) := \int_{x}^{m_\mu} \frac{1}{\rho_\nu} \, dx \]

**Proposition 9**  Let \( q \in [1/2, 1] \). \((\mu, \nu)\) satisfies a \( L^q\)-Poincaré inequality if

\[
\int_{m_\mu}^{\infty} |r \, R|^{q/(1-q)} \, d\mu < \infty \quad \text{and} \quad \int_{-\infty}^{m_\mu} |\ell \, L|^{q/(1-q)} \, d\mu < \infty
\]
**Proof**

Method: \( \text{Var}_\mu(f) \leq \mu(|F_-|^2) + \mu(|F_+|^2) \) with \( g = (f - f(m_\mu)) \pm \) and prove that

\[
\mu(|g|^2) \leq \frac{11 + 5\sqrt{5}}{2} \gamma(s) \int |\nabla g|^2 \, d\nu + s \left[ \sup_{\mu} g \right]^2 \quad \forall \ s \in (0, 1/2)
\]

Let \( A \subset B \subset M = (m_\mu, \infty) \) such that \( A \subset B \) and \( \mu(B) \leq 1/2 \)

\[
\text{Cap}_\nu(A, B) \geq \text{Cap}_\nu(A, (m_\mu, \infty)) = \text{Cap}_\nu((a, \infty), (m_\mu, \infty)) = \frac{1}{r(a)}
\]

where \( a = \inf A \). Change variables: \( t = R(a) \) and choose

\[
\gamma(t) := t (r \circ R)^{-1}(t) \text{ for any } t \in (0, 1/2)
\]

\( \triangleright \)
With $\psi \in C^2(\mathbb{R}^d)$, $d\mu_\psi := \frac{e^{-\psi} dx}{Z_\psi}$, define $\mathcal{L}$ on $C^2(\mathbb{R}^d)$ by

$$\forall f \in C^2(\mathbb{R}^d) \quad \mathcal{L}f := \Delta f - \nabla \psi \cdot \nabla f$$

Such a generator $\mathcal{L}$ is symmetric in $L^2_{\mu_\psi}(\mathbb{R}^d)$,

$$\forall f, g \in C^1(\mathbb{R}^d) \quad \int f \mathcal{L}g \, d\mu_\psi = -\int \nabla f \cdot \nabla g \, d\mu_\psi$$

Consider for $m > 1$ the weighted porous media equation

$$\left\{ \begin{array}{ll}
\frac{\partial u}{\partial t} = \mathcal{L} u^m & \text{in } Q \\
u(\cdot, 0) = u_0 & \text{in } \Omega \\
n \cdot \nabla u = 0 & \text{on } \Sigma
\end{array} \right.$$  

$\Omega \subset \mathbb{R}^d$, $Q = \Omega \times [0, +\infty)$, $\Sigma = \partial \Omega \times [0, +\infty)$

$u \in C^2$, $L^1$-contraction, existence and uniqueness
Asymptotic behavior

**Theorem 10** Let $m \geq 1$ and assume that $(\mu_\psi, \mu_\psi)$ satisfies a $L^q$-Poincaré inequality, $q = 2/(m + 1)$

$$\text{Var}_{\mu_\psi}(u(\cdot, t)) \leq \left( \left[ \text{Var}_{\mu_\psi}(u_0) \right]^{-(m-1)/2} + \frac{4m(m - 1)}{(m + 1)^2} C_P \right)^{-2/(m-1)}$$

Reciprocally, if the above inequality is satisfied for any $u_0$, then $(\mu_\psi, \mu_\psi)$ satisfies a $L^q$-Poincaré inequality with constant $C_P$

**Proof**

$$\frac{d}{dt} \text{Var}_{\mu_\psi}(u) = 2 \int u_t u \, d\mu_\psi = 2 \int u \mathcal{L} u^m \, d\mu_\psi = -\frac{8m}{(m + 1)^2} \int |\nabla u^{m+1/2}|^2 \, d\mu_\psi$$

Apply the $L^q$-Poincaré inequality with $u = f^{2/(m+1)}$, $q = 2/(m + 1)$

Reciprocally, a derivation at $t = 0$ gives the $L^q$-Poincaré inequality
A conclusion on $L^q$-Poincaré inequalities

- Observe that we have only algebraic rates

$$
\left( \int f^{2q} \frac{\log f^{2q}}{\int f^{2q} \, d\mu} \, d\mu \right) =: \text{Ent}_\mu (f^{2q})^{1/q} \leq C_{LS} \int |\nabla f|^2 \, d\mu
$$

- Orlicz spaces, duality, connections with mass transport theory
  [Bobkov-Götze, 1999] [Cattiaux-Gentil-Guillin, 2006] [Wang, 2006]
  [Roberto-Zegarlinski, 2003] [Barthe-Cattiaux-Roberto, 2005]
The Bakry-Emery method revisited

J.D., B. Nazaret, G. Savaré
Consider a domain $\Omega \subset \mathbb{R}^d$, $d\gamma = g\, dx$, $g = e^{-F}$ and a generalized Ornstein-Uhlenbeck operator: $\Delta_g v := \Delta v - DF \cdot Dv$

$$\int_\Omega |Dv|^2\, d\gamma = - \int_\Omega v \Delta_g v\, d\gamma \quad \forall v \in H^1_0(\Omega, d\gamma)$$

Let $s := v^{p/2}$ and $\alpha := (2 - p)/p$, $p \in (1, 2]$

$$v_t = \Delta_g v \quad x \in \Omega, \ t \in \mathbb{R}^+$$
$$\nabla v \cdot n = 0 \quad x \in \partial\Omega, \ t \in \mathbb{R}^+$$

$$\mathcal{E}_p(t) := \frac{1}{p-1} \int_\Omega \left[ v^p - 1 - p(v - 1) \right] d\gamma$$

$$\mathcal{I}_p(t) := \frac{4}{p} \int_\Omega |Ds|^2\, d\gamma$$

$$\mathcal{K}_p(t) := \int_\Omega |\Delta_g s|^2\, d\gamma + \alpha \int_\Omega \Delta_g s \frac{|Ds|^2}{s}\, d\gamma$$
Written in terms of $s = v^{p/2}$, the entropy is

$$E_p = \frac{1}{p-1} \int_{\Omega} \left[ s^2 - 1 - p \left( s^{2/p} - 1 \right) \right] d\gamma$$

and the evolution is governed by

$$s_t = \Delta g s + \alpha \frac{|D_s|^2}{s}$$

A simple computation shows that

$$\frac{d}{dt} E_p(t) := -I_p(t)$$
$$\frac{d}{dt} I_p(t) := -\frac{8}{p} K_p(t)$$
Using the commutation relation $[\mathbf{D}, \Delta_g] s = -D^2 F Ds$, we get

$$\int_{\Omega} (\Delta_g s)^2 \, d\gamma = \int_{\Omega} |D^2 s|^2 \, d\gamma + \int_{\Omega} D^2 F Ds \cdot Ds \, d\gamma - \sum_{i,j=1}^{d} \int_{\partial \Omega} \partial_{ij}^2 s \, \partial_i s \, n_j \, \gamma \, d\mathcal{H}^{d-1} \geq 0 \text{ if } \Omega \text{ is convex}$$

Let $z := \sqrt{s}$. Using $2D^2 s \cdot Dz \otimes Dz = D (|Dz|^2) : Dz$ and i.p.p., we get

$$\mathcal{K}_p = \int_{\Omega} |\Delta_g s|^2 \, d\gamma + 4 \alpha \int_{\Omega} \Delta_g s \, |Dz|^2 \, d\gamma$$

$$\geq \int_{\Omega} |D^2 s|^2 \, d\gamma + \int_{\Omega} D^2 F Ds \cdot Ds \, d\gamma$$

$$+ 4^2 \alpha \int_{\Omega} |Dz|^4 \, d\gamma - 2 \cdot 4 \alpha \int_{\Omega} D^2 s : Dz \otimes Dz \, d\gamma$$

$$\geq (1 - \alpha) \int_{\Omega} |D^2 s|^2 \, d\gamma + \int_{\Omega} D^2 F Ds \cdot Ds \, d\gamma$$
An extension of the criterion of Bakry-Emery

Let $V(x) := \inf_{\xi \in S^{d-1}} (D^2 F(x) \xi, \xi)$ and define

$$\lambda_1(p) := \inf_{w \in \mathcal{D}(\Omega) \setminus \{0\}} \frac{\int_{\Omega} \left( 2 \frac{p-1}{p} |Dw|^2 + V |w|^2 \right) d\gamma}{\int_{\Omega} |w|^2 d\gamma}$$

Theorem 1 Let $F \in C^2(\Omega)$, $\gamma = e^{-F} \in L^1(\Omega)$, and $\Omega$ be a convex domain in $\mathbb{R}^d$. If $\lambda_1(p)$ is positive, then

$$\mathcal{I}_p(t) \leq \mathcal{I}_p(0) e^{-2 \lambda_1(p) t}$$
$$\mathcal{E}_p(t) \leq \mathcal{E}_p(0) e^{-2 \lambda_1(p) t}$$
Generalized entropies

Consider the weighted porous media equation

\[ v_t = \Delta_g v^m \]

\( d\gamma \) is a probability measure, \( p \in (1, 2) \)

\[ E_{m,p}(t) := \frac{1}{m + p - 2} \int_{\Omega} \left[ v^{m+p-1} - 1 \right] d\gamma \]

\[ I_{m,p}(t) := c(m, p) \int_{\Omega} |D_s|^2 d\gamma \]

\[ K_{m,p}(t) := \int_{\Omega} s^{\beta(m-1)} |\Delta_g s|^2 d\gamma + \alpha \int_{\Omega} s^{\beta(m-1)} \Delta_g s \frac{|D_s|^2}{s} d\gamma \]

with \( v =: s^\beta \), \( \beta := \frac{1}{p/2 + m - 1} \), \( \alpha := \frac{2-p}{p+2(m-1)} \) and \( c(m, p) = \frac{4 m (m+p-1)}{(2m+p-2)^2} \)
adapting the Bakry-Emery method...

Written in terms of $s = v^{1/\beta}$, the evolution is governed by

$$\frac{1}{m} s_t = s^{\beta(m-1)} \left[ \Delta_g s + \alpha \frac{|Ds|^2}{s} \right]$$

A computation shows that

$$\frac{d}{dt} \mathcal{E}_{m,p}(t) := -\mathcal{I}_{m,p}(t)$$

$$\frac{1}{m} \frac{d}{dt} \mathcal{I}_{m,p}(t) := -2 c(m, p) \mathcal{K}_{m,p}(t)$$

Exactly as in the linear case, define for any $\theta \in (0, 1)$

$$\lambda_1(m, \theta) := \inf_{w \in H^1(\Omega, d\gamma) \setminus \{0\}} \frac{\int_{\Omega} \left( (1 - \theta) |Dw|^2 + V |w|^2 \right) \, d\gamma}{\int_{\Omega} |w|^2 \, d\gamma}$$
The non-local condition

Assume that for some $\theta \in (0, 1)$, $\lambda_1(m, \theta) > 0$. Admissible parameters $m$ and $p$ correspond to $(m, p) \in E_\theta$, $1 < m < p + 1$, where the set $E_\theta$ is defined by the condition: $b^2 - 4a(\theta)c < 0$. 

![Diagram showing the set $E_\theta$ and the line $p = m - 1$.]
Results for the fast diffusion equation

Lemma 1  With the above notations, if \( \Omega \) is convex and \((m, p) \in E_\theta\) are admissible, then

\[
I_{m,p}^{\frac{4}{3}} \leq \frac{1}{3} \left[ 4 c(m, p) \right]^{\frac{4}{3}} K^{\frac{1}{3}} \left[ (m + p - 2) E_{m,p} + 1 \right]^{\frac{4-3q}{3(2-q)}} K_{m,p}
\]

Theorem 2  Under the above conditions there exists a positive constant \( \kappa \) which depends on \( E_{m,p}(0) \) such that any smooth solution \( u \) of the porous media equation satisfies, for any \( t > 0 \),

\[
I_{m,p}(t) \leq \frac{I_{m,p}(0)}{\left[ 1 + \frac{\kappa}{3} \sqrt[3]{I_{m,p}(0)} t \right]^3}
\]

\[
E_{m,p}(t) \leq \frac{3 \left[ I_{m,p}(0) \right]^{\frac{8}{3}}}{2 \kappa \left[ 1 + \frac{\kappa}{3} \sqrt[3]{I_{m,p}(0)} t \right]^2}
\]
Entropies, transport and distances between measures

J.D., B. Nazaret, G. Savaré
Wasserstein distances

$p > 1$, $\mu_0$ and $\mu_1$ probability measures on $\mathbb{R}^d$

- **Transport plans between $\mu_0$ and $\mu_1$:** $\Gamma(\mu_0, \mu_1)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ having $\mu_0$ and $\mu_1$ as marginals.

- **Wasserstein distance between $\mu_0$ and $\mu_1$**

$$W_p^p(\mu_0, \mu_1) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\Sigma(x, y) : \Sigma \in \Gamma(\mu_0, \mu_1) \right\}$$


$$W_p^p(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |v_t|^p \rho_t dx dt : (\rho_t, v_t)_{t \in [0,1]} \text{ admissible} \right\}$$

where admissible paths $(\rho_t, v_t)_{t \in [0,1]}$ are such that

$$\partial_t \rho_t + \nabla \cdot (\rho_t v_t) = 0, \rho_0 = \mu_0, \rho_1 = \mu_1$$
A generalization of the Benamou-Brenier approach

Given a function $h$ on $\mathbb{R}^+$, define the admissible paths by

$$
\begin{align*}
\partial_t \rho_t + \nabla \cdot (h(\rho_t)v_t) &= 0, \\
\rho_0 &= \mu_0, \quad \rho_1 = \mu_1
\end{align*}
$$

and consider the distance

$$
W^p_h(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |v_t|^p h(\rho_t) dx dt : (\rho_t, v_t)_{t \in [0,1]} \text{ admissible} \right\}
$$

$h(\rho) = \rho^\alpha, \ 0 \leq \alpha \leq 1$

- $\alpha = 1$: Wasserstein case
- $\alpha = 0$: homogeneous Sobolev distance on $\dot{W}^{-1,p}$. With $q = \frac{p}{p-1}$

$$
\|\mu_1 - \mu_0\|_{\dot{W}^{-1,p}} = \sup \left\{ \int_{\mathbb{R}^d} \xi d(\mu_1 - \mu_0) : \xi \in C^1_c(\mathbb{R}^d), \int_{\mathbb{R}^d} |\nabla \xi|^q \leq 1 \right\}
$$
**Gradient flows**

- **Jordan-Kinderlehrer-Otto 98**: Formal Riemannian structure on $\mathcal{P}(\mathbb{R}^d)$: the McCann interpolant is a geodesic. For an integral functional such as
  \[
  \mathcal{F}(\rho) = \int_{\mathbb{R}^d} F(\rho(x)) \, dx
  \]
  the gradient flow of $\mathcal{F}$ is
  \[
  \frac{\partial \rho}{\partial t} = \nabla \cdot (\rho \nabla (F'(\rho)))
  \]

- **Ambrosio-Gigli-Savaré 05**: Rigorous framework for JKO’s calculus in the framework of length spaces (based on the optimal transportation)

- **Otto-Westdickenberg 05**: Use the Brenier-Benamou formulation to prove
  \[
  W_2^2(\mu_0^t, \mu_1^t) \leq W_2^2(\mu_0, \mu_1)
  \]
  along the heat flow on a compact Riemannian manifold
The heat equation as gradient flow w.r.t. $\hat{W}_\varphi$

Denote by $S_t$ the semi-group associated to the heat equation. Let $\alpha > 1 - \frac{2}{d}$ and consider the generalized entropy functional

$$\Psi_\alpha(\mu) = \frac{1}{(1 - \alpha)(2 - \alpha)} \int_{\mathbb{R}^d} \rho^{2-\alpha}(x)dx, \text{ if } \mu = \rho \mathcal{L}^d$$

**Theorem 1** If $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\Psi_\alpha(\mu) < +\infty$, then $\Psi_\alpha(S_t\mu) < +\infty$ for all $t > 0$ and

$$\frac{1}{2} \frac{d}{dt} W_\alpha^2(S_t\mu, \sigma) + \Psi_\alpha(S_t\mu) \leq \Psi_\alpha(\sigma)$$

**Corollary 2** $\Psi_\alpha$ is geodesically convex w.r.t. $W_\alpha$
Fast diffusion equations: entropy methods and functional inequalities

\[ u_t = \Delta u^m \quad x \in \mathbb{R}^d, \quad t > 0 \]

- Entropy methods for fast diffusion and porous media equations: intermediate asymptotics
- Entropy methods and functional inequalities
Porous media / fast diffusion equations

Generalized entropies and nonlinear diffusions (EDP, uncomplete):
[Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani],
[Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban],
[Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo,
Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],...
[del Pino, Sáez], [Daskalopulos, Sesum]...

1) [J.D., del Pino] relate entropy and entropy-production by
Gagliardo-Nirenberg inequalities

Various alternative approaches:
2) “entropy – entropy-production method”
3) mass transport techniques
4) hypercontractivity for appropriate semi-groups
Heat equation, porous media & fast diffusion equation

\[ u_t = \Delta u^m \]
\[ x \in \mathbb{R}^d \]

heat equation

fast diffusion equation

global existence in \( L^1 \)

extinction in finite time

exist in finite time

porous media equation

Existence theory, critical values of the parameter \( m \)
Intermediate asymptotics for fast diffusion & porous media

\[
\begin{align*}
  u_t &= \Delta u^m \quad \text{in } \mathbb{R}^d \\
  u|_{t=0} &= u_0 \geq 0 \\
  u_0(1 + |x|^2) &\in L^1, \quad u_0^m \in L^1
\end{align*}
\]

Intermediate asymptotics: \( u_0 \in L^\infty, \int u_0 \, dx = M > 0 \)

Self-similar (Barenblatt) function: \( U(t) = O(t^{-d/(2-d(1-m))}) \)

[Friedmann, Kamin, 1980] As \( t \to +\infty \)

\[
\|u(t, \cdot) - U(t, \cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})
\]

\[\Rightarrow\quad \text{What about } \|u(t, \cdot) - U(t, \cdot)\|_{L^1} ?\]
Time-dependent rescaling

Take $u(t, x) = R^{-d(t)} v(\tau(t), x/R(t))$ where

$$\dot{R} = R^{d(1-m)-1}, \quad R(0) = 1, \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x v), \quad v|_{\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: Entropy or Free energy

$$\Sigma[v] = \int \left( \frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0$$

$$\frac{d}{d\tau} \Sigma[v] = -I[v], \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2 dx$$
Entropy and entropy production

Stationary solution: choose $C$ such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left( C + \frac{1-m}{2m} |x|^2 \right)^{-1/(1-m)} +$$

Fix $\Sigma_0$ so that $\Sigma[v_\infty] = 0$. The entropy can be put in an $m$-homogeneous form

$$\Sigma[v] = \int \psi \left( \frac{v}{v_\infty} \right) v_\infty^m \, dx \quad \text{with} \quad \psi(t) = \frac{t^m - 1 - m(t-1)}{m-1}$$

**Theorem 1** $d \geq 3$, $m \in \left[ \frac{d-1}{d}, +\infty \right)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2 \Sigma[v]$$
An equivalent formulation

$$\Sigma[v] = \int \left( \frac{v^m}{m-1} + \frac{1}{2} |x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int v |\nabla v^m| + x^2 dx = \frac{1}{2} I[v]$$

$$p = \frac{1}{2m-1}, v = w^{2p}, v^m = w^{p+1}$$

$$\frac{1}{2} \left( \frac{2m}{2m-1} \right)^2 \int |\nabla w|^2 dx + \left( \frac{1}{1-m} - d \right) \int |w|^{1+p} dx + K \geq 0$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$ and, for some $\gamma$, $K$ can be written as

$$K = K_0 \left( \int v dx = \int w^{2p} dx \right)^{\gamma}$$

$w = w_\infty = v_\infty^{1/2p}$ is optimal

$m = \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev
Gagliardo-Nirenberg inequalities

**Theorem 2** [Del Pino, J.D.]  
Assume that $1 < p \leq \frac{d}{d-2}$ and $d \geq 3$

\[ \| w \|_{2p} \leq A \| \nabla w \|^{\theta} \| w \|^{1-\theta}_{p+1} \]

\[ A = \left( \frac{y(p-1)^2}{2\pi d} \right)^{\frac{\theta}{2}} \left( \frac{2y-d}{2y} \right)^{\frac{1}{2p}} \left( \frac{\Gamma(y)}{\Gamma(y - \frac{d}{2})} \right)^{\frac{\theta}{d}} \]

\[ \theta = \frac{d(p-1)}{p(d+2-(d-2)p)}, \quad y = \frac{p+1}{p-1} \]

Similar results for $0 < p < 1$

Uses [Serrin-Pucci], [Serrin-Tang]

$1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \iff$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$

$0 < p < 1 \iff$ Porous medium case: $m > 1$
Intermediate asymptotics

$$\Sigma[v] \leq \Sigma[u_0] e^{-2\tau} + \text{Csiszár-Kullback inequalities}$$

**Theorem 3** [Del Pino, J.D.]

(i) \( \frac{d-1}{d} < m < 1 \) if \( d \geq 3 \)

\[
\limsup_{t \to +\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \| u^m - u_\infty^m \|_{L^1} < +\infty
\]

(ii) \( 1 < m < 2 \)

\[
\limsup_{t \to +\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \| [u - u_\infty] u_\infty^{m-1} \|_{L^1} < +\infty
\]

\[u_\infty(t, x) = R^{-d}(t) v_\infty \left( x/R(t) \right)\]
Fast diffusion equations: the finite mass regime

If $m \geq 1$: porous medium regime or $m_1 := \frac{d-1}{d} \leq m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality.

If $m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in $L^1$ and the Barenblatt self-similar solution has finite mass.
The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance.

Displacement convexity holds in the same range of exponents, $m \in ((d-1)/d, 1)$, as for the Gagliardo-Nirenberg inequalities.

$\Rightarrow$ How to extend to $m_c < m < m_1$ what has been done for $m \geq m_1$?
Fast diffusion: finite mass regime

Inequalities...

Sobolev

logarithmic Sobolev

Gagliardo-Nirenberg

\[ \frac{d-2}{d}, \quad \frac{d}{d+2}, \quad \frac{d-1}{d}, \quad 1 \]

\[ v^m \in L^1, \quad |x|^2 v \in L^1 \]

Bakry-Emery method (relative entropy)

global existence in \( L^1 \)

... existence of solutions of \( u_t = \Delta u^m \)
Extensions and related results

- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg] [J.D.-Nazaret-Savaré], etc
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev]
If $m > m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in $L^1$ and the Barenblatt self-similar solution has finite mass.

For $m \leq m_c$, the Barenblatt self-similar solution has infinite mass.

⇒ How to extend to $m \leq m_c$ what has been done for $m > m_c$? Work in relative variables!
Fast diffusion: infinite mass regime

\[ \Sigma[V_{D_1}|V_{D_0}] = \infty \]
\[ v_0 - V_{D*} \in L^1 \]
\[ V_{D_1} - V_{D_0} \not\in L^1 \]

\[ \Sigma[V_{D_1}|V_{D_0}] < \infty \]
\[ v_0, V_{D} \in L^1 \]

\[ m \]

\[ \frac{d-4}{d-2} \]
\[ \frac{d-2}{d} \]
\[ \frac{d}{d+2} \]
\[ \frac{d-1}{d} \]
\[ \frac{1}{Gagliardo-Nirenberg} \]
\[ v^m \in L^1, |x|^2 v \in L^1 \]

\[ m_* \]
\[ m_c \]
\[ m_1 \]

Bakry-Emery method (relative entropy)

Global existence in \( L^1 \)

Fast diffusion equations: Entropy methods and linearization, intermediate asymptotics, vanishing – p.2/19
A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez

- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

⇒ Extend the domain of validity of the method to the price of a restriction of the set of admissible solutions
Setting of the problem

We consider the solutions $u(\tau, y)$ of

$$
\begin{cases}
\partial_\tau u = \Delta u^m \\
u(0, \cdot) = u_0
\end{cases}
$$

where $m \in (0, 1)$ (fast diffusion) and $(\tau, y) \in Q_T = (0, T) \times \mathbb{R}^d$

Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where

$$m_c := \frac{d - 2}{d}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time

$$\lim_{\tau \nearrow T} u(\tau, y) = 0$$
Barenblatt solutions

\[ U_{D,T}(\tau, y) := \frac{1}{R(\tau)^d} \left( D + \frac{1-m}{2m} \left| \frac{y}{R(\tau)} \right|^2 \right)^{-\frac{1}{1-m}} \]

with

\[ R(\tau) := \left[ d (m - m_c) (\tau + T) \right]^{\frac{1}{d(m-m_c)}} \text{ if } m_c < m < 1 \]

(vanishing in finite time) if \( 0 < m < m_c \)

\[ R(\tau) := \left[ d (m_c - m) (T - \tau) \right]^{-\frac{1}{d(m_c-m)}} \]

Time-dependent rescaling: \( t := \log \left( \frac{R(\tau)}{R(0)} \right) \) and \( x := \frac{y}{R(\tau)} \). The function \( v(t, x) := R(\tau)^d u(\tau, y) \) solves a nonlinear Fokker-Planck type equation

\[
\begin{cases}
\partial_t v(t, x) = \Delta v^m(t, x) + \nabla \cdot (x v(t, x)) & (t, x) \in (0, +\infty) \times \mathbb{R}^d \\
v(0, x) = v_0(x) = R(0)^d u_0(R(0) x) & x \in \mathbb{R}^d
\end{cases}
\]
Assumptions

(H1) $u_0$ is a non-negative function in $L^1_{\text{loc}}(\mathbb{R}^d)$ and there exist positive constants $T$ and $D_0 > D_1$ such that

$$U_{D_0,T}(0,y) \leq u_0(y) \leq U_{D_1,T}(0,y) \quad \forall \ y \in \mathbb{R}^d$$

(H2) If $m \in (0, m_*)$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$u_0(y) = U_{D_*,T}(0,y) + f(y) \quad \forall \ y \in \mathbb{R}^d$$

(H1') $v_0$ is a non-negative function in $L^1_{\text{loc}}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(x) \leq v_0(x) \leq V_{D_1}(x) \quad \forall \ x \in \mathbb{R}^d$$

(H2') If $m \in (0, m_*)$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x) \quad \forall \ x \in \mathbb{R}^d$$
Convergence to the asymptotic profile (without rate)

\[ m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{2}, \quad p(m) := \frac{d(1-m)}{2(2-m)} \]

**Theorem 1** Let \( d \geq 3 \), \( m \in (0, 1) \). Consider a solution \( v \) with initial data satisfying (H1’)-(H2’)

(i) For any \( m > m_* \), there exists a unique \( D_* \) such that
\[ \int_{\mathbb{R}^d} (v(t) - V_{D_*}) \, dx = 0 \text{ for any } t > 0. \]
Moreover, for any \( p \in (p(m), \infty] \),
\[ \lim_{t \to \infty} \int_{\mathbb{R}^d} |v(t) - V_{D_*}|^p \, dx = 0 \]

(ii) For \( m \leq m_* \), \( v(t) - V_{D_*} \) is integrable,
\[ \int_{\mathbb{R}^d} (v(t) - V_{D_*}) \, dx = \int_{\mathbb{R}^d} f \, dx \]
and \( v(t) \) converges to \( V_{D_*} \) in \( L^p(\mathbb{R}^d) \) as \( t \to \infty \), for any \( p \in (1, \infty] \)

(iii) (Convergence in Relative Error) For any \( p \in (d/2, \infty] \),
\[ \lim_{t \to \infty} \| v(t)/V_{D_*} - 1 \|_p = 0 \]

[Daskalopoulos-Sesum, 06], [Blanchet-Bonforte-J.D.-Grillo-Vázquez, 06]
Convergence with rate

\[ q_\ast := \frac{2 \, d \,(1 - m)}{2 \,(2 - m) + d \,(1 - m)} \]

**Theorem 2**  If \( m \neq m_\ast \), there exist \( t_0 \geq 0 \) and \( \lambda_{m,d} > 0 \) such that

(i) For any \( q \in (q_\ast, \infty] \), there exists a positive constant \( C_q \) such that

\[
\|v(t) - V_{D_\ast}\|_q \leq C_q \, e^{-\lambda_{m,d} \, t} \quad \forall \, t \geq t_0
\]

(ii) For any \( \vartheta \in [0, (2 - m)/(1 - m)) \), there exists a positive constant \( C_{\vartheta} \) such that

\[
\| |x|^{\vartheta} (v(t) - V_{D_\ast}) \|_2 \leq C_{\vartheta} \, e^{-\lambda_{m,d} \, t} \quad \forall \, t \geq t_0
\]

(iii) For any \( j \in \mathbb{N} \), there exists a positive constant \( H_j \) such that

\[
\|v(t) - V_{D_\ast}\|_{C_j(\mathbb{R}^d)} \leq H_j \, e^{-\frac{\lambda_{m,d}}{d+2(j+1)} \, t} \quad \forall \, t \geq t_0
\]
Corollary 3  Let $d \geq 3$, $m \in (0, 1)$, $m \neq m_\ast$. Consider a solution $u$ with initial data satisfying (H1)-(H2). For $\tau$ large enough, for any $q \in (q_\ast, \infty]$, there exists a positive constant $C$ such that

$$
\|u(\tau) - U_{D_\ast}(\tau)\|_q \leq C R(\tau)^{-\alpha}
$$

where $\alpha = \lambda_{m_d} + d(q - 1)/q$ and large means $T - \tau > 0$, small, if $m < m_c$, and $\tau \to \infty$ if $m \geq m_c$

For any $p \in (d/2, \infty]$, there exists a positive constant $C$ and $\gamma > 0$ such that

$$
\left\| v(t)/V_{D_\ast} - 1 \right\|_{L^p(\mathbb{R}^d)} \leq C e^{-\gamma t} \quad \forall \ t \geq 0
$$
Rewriting the equation in relative variables

$L^1$-contraction, Maximum Principle, conservation of relative mass...

Passing to the quotient: the function $w(t, x) := \frac{v(t, x)}{V_{D_*}(x)}$ solves

$$\begin{cases} 
    w_t = \frac{1}{V_{D_*}} \nabla \cdot \left[ w V_{D_*} \nabla \left( \frac{m}{m - 1} \left( w^{m-1} - 1 \right) V_{D_*}^{m-1} \right) \right] & \text{in } (0, +\infty) \times \mathbb{R}^d \\
    w(0, \cdot) = w_0 := \frac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d 
\end{cases}$$

with

$$0 < \inf_{x \in \mathbb{R}^d} \frac{V_{D_0}}{V_{D_*}} \leq w(t, x) \leq \sup_{x \in \mathbb{R}^d} \frac{V_{D_1}}{V_{D_*}} < \infty$$

... Harnack Principle

$$\|w(t)\|_{C^k(\mathbb{R}^d)} \leq H_k < +\infty \quad \forall \ t \geq t_0$$

$\exists \ t_0 \geq 0 \ \text{s.t. (H1) holds if} \ \exists \ R > 0, \sup_{|y| > R} u_0(y) |y|^{\frac{2}{1-m}} < \infty, \text{ and } m > m_c$
Relative entropy

Relative entropy

\[ F[w] := \frac{1}{1 - m} \int_{\mathbb{R}^d} \left[ (w - 1) - \frac{1}{m} (w^m - 1) \right] V_D^m \, dx \]

Relative Fisher information

\[ J[w] := \frac{m}{(m - 1)^2} \int_{\mathbb{R}^d} | \nabla \left[ (w^{m-1} - 1) V_{D^*}^{m-1} \right] |^2 w V_{D^*} \, dx \]

Proposition 1  Under assumptions (H1)-(H2),

\[ \frac{d}{dt} F[w(t)] = - J[w(t)] \]

Proposition 2  Under assumptions (H1)-(H2), there exists a constant \( \lambda > 0 \) such that

\[ F[w(t)] \leq \lambda^{-1} J[w(t)] \]
Heuristics: linearization

Take \( w(t, x) = 1 + \varepsilon \frac{g(t, x)}{V_{D^*}^{m-1}(x)} \) and formally consider the limit \( \varepsilon \to 0 \) in

\[
\begin{align*}
wt &= \frac{1}{V_{D^*}} \nabla \cdot \left[ w V_{D^*} \nabla \left( \frac{m}{m-1} (w^{m-1} - 1) V_{D^*}^{m-1} \right) \right] \quad \text{in } (0, +\infty) \times \mathbb{R}^d \\
w(0, \cdot) &= w_0 := \frac{v_0}{V_{D^*}} \quad \text{in } \mathbb{R}^d
\end{align*}
\]

Then \( g \) solves

\[
g_t = m V_{D^*}^{m-2}(x) \nabla \cdot [V_{D^*}(x) \nabla g(t, x)]
\]

and the entropy and Fisher information functionals

\[
F[g] := \frac{1}{2} \int_{\mathbb{R}^d} |g|^2 V_{D^*}^{2-m} \, dx \quad \text{and} \quad I[g] := m \int_{\mathbb{R}^d} |\nabla g|^2 V_{D^*} \, dx
\]

consistently verify \( \frac{d}{dt} F[g(t)] = -I[g(t)] \)
Comparison of the functionals

**Lemma 3** Let $m \in (0, 1)$ and assume that $u_0$ satisfies (H1)-(H2)

[Relative entropy]

\[
C_1 \int_{\mathbb{R}^d} |w - 1|^2 V_{D^*}^m \, dx \leq \mathcal{F}[w] \leq C_2 \int_{\mathbb{R}^d} |w - 1|^2 V_{D^*}^m \, dx
\]

[Fisher information]

\[
l[g] \leq \beta_1 \mathcal{J}[w] + \beta_2 \mathcal{F}[g] \quad \text{with} \quad g := (w - 1) V_{D^*}^{m-1}
\]

**Theorem 4 (Hardy-Poincaré)** There exists a positive constant $\lambda_{m,d}$ such that for any $m \neq m_* = (d - 4)/(d - 2)$, $m \in (0, 1)$, for any $g \in \mathcal{D}(\mathbb{R}^d)$,

\[
\int_{\mathbb{R}^d} |g - \bar{g}|^2 V_{D^*}^{2-m} \, dx \leq C_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 V_{D^*} \, dx
\]

with $\bar{g} = \int_{\mathbb{R}^d} g V_{D^*}^{2-m} \, dx$ if $m > m_*$, $\bar{g} = 0$ otherwise
Hardy-Poincaré inequalities

With $\alpha = \frac{1}{m-1}$, $\alpha_* = \frac{1}{m_*-1} = 1 - \frac{d}{2}$

**Theorem 5** Assume that $d \geq 3$, $\alpha \in \mathbb{R} \setminus \{\alpha^*\}$, $d\mu_\alpha(x) := h_\alpha(x) \, dx$, $h_\alpha(x) := (1 + |x|^2)^\alpha$. Then

$$\int_{\mathbb{R}^d} \frac{|v|^2}{1 + |x|^2} \, d\mu_\alpha \leq C_{\alpha,d} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\mu_\alpha$$

holds for some positive constant $C_{\alpha,d}$, for any $v \in \mathcal{D}(\mathbb{R}^d)$, under the additional condition $\int_{\mathbb{R}^d} v \, d\mu_{\alpha-1} = 0$ if $\alpha \in (-\infty, \alpha^*)$

... Hardy-Poincaré inequalities $\equiv$ weighted Poincaré inequalities corresponding to generalized Cauchy distributions (fat tails)... [Bobkov, Ledoux] [Cattiaux, Gozlan, Guillin, Roberto]
Limit cases

Poincaré inequality: take $\alpha = -1/\varepsilon^2$ to $v_\varepsilon(x) := \varepsilon^{-d/2} v(x/\varepsilon)$ and let $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} |v|^2 \, dv_\infty \leq \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v|^2 \, dv_\infty \quad \text{with} \quad dv_\infty(x) := e^{-|x|^2} \, dx$$

... under the additional condition $\int_{\mathbb{R}^d} v \, e^{-|x|^2} \, dx = 0$

Hardy’s inequality: take $v_{1/\varepsilon}(x) := \varepsilon^{d/2} v(\varepsilon \, x)$ and let $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, d\nu_{0,\alpha} \leq \frac{1}{(\alpha - \alpha_*)^2} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\nu_{0,\alpha} \quad \text{with} \quad d\nu_{0,\alpha}(x) := |x|^{2\alpha} \, dx$$

... under the additional condition $\bar{v}_\alpha := \int_{\mathbb{R}^d} v \, d\nu_{0,\alpha} = 0$ if $\alpha < \alpha^*$
Some estimates of $C_{\alpha,d}$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$-\infty &lt; \alpha \leq -d$</th>
<th>$-d &lt; \alpha &lt; \alpha^*$</th>
<th>$\alpha^* &lt; \alpha \leq 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\alpha,d}$</td>
<td>$\frac{1}{2</td>
<td>\alpha</td>
<td>}$</td>
</tr>
<tr>
<td>Optimality</td>
<td>-</td>
<td>-</td>
<td>yes</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$1 \leq \alpha \leq \bar{\alpha}(d)$</th>
<th>$\bar{\alpha}(d) \leq \alpha \leq d$</th>
<th>$d$</th>
<th>$\alpha &gt; d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_{\alpha,d}$</td>
<td>$\frac{4}{d(d+2\alpha-2)}$</td>
<td>$\frac{1}{\alpha(d+\alpha-2)}$</td>
<td>$\frac{1}{2d(d-1)}$</td>
<td>$\frac{1}{d(d+\alpha-2)}$</td>
</tr>
<tr>
<td>Optimality</td>
<td>-</td>
<td>-</td>
<td>yes</td>
<td>-</td>
</tr>
</tbody>
</table>

$\alpha^* = -\frac{d-2}{2}$, $\bar{\alpha}(d) \in (1, d)$
\[-(1 + |x|^2)^{1-\alpha} \nabla \cdot ((1 + |x|^2)^\alpha \nabla) \text{ in } L^2((1 + |x|^2)^{\alpha-1} \, dx)\]

Taken from [J. Denzler & R. J. McCann, PNAS 100 (2003)], $p = \frac{2}{2-m} - d$
Hardy’s inequality: the “completing the square method”

Let $v \in \mathcal{D}(\mathbb{R}^d)$ with $\text{supp}(v) \subset \mathbb{R}^d \setminus \{0\}$ if $\alpha < \alpha^*$

$$0 \leq \int_{\mathbb{R}^d} \left| \nabla v + \lambda \frac{x}{|x|^2} v \right|^2 |x|^{2\alpha} \, dx$$

$$= \int_{\mathbb{R}^d} |\nabla v|^2 |x|^{2\alpha} \, dx + \left[ \lambda^2 - \lambda (d + 2\alpha - 2) \right] \int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, |x|^{2\alpha} \, dx$$

An optimization of the right hand side with respect to $\lambda$ gives $\lambda = \alpha - \alpha^*$, that is $(d + 2\alpha - 2)^2 / 4 = \lambda^2$. Such an inequality is optimal, with optimal constant $\lambda^2$, as follows by considering the test functions:

1) if $\alpha > \alpha^*$: $v_\varepsilon(x) = \min\{\varepsilon^{-\lambda}, (|x|^{-\lambda} - \varepsilon^\lambda)_+\}$

2) if $\alpha < \alpha^*$: $v_\varepsilon(x) = |x|^{1-\alpha-d/2+\varepsilon}$ for $|x| < 1$
   
   $v_\varepsilon(x) = (2 - |x|)_+$ for $|x| \geq 1$

and letting $\varepsilon \to 0$ in both cases
An optimality case

**Proposition 4** Let $d \geq 3$, $\alpha \in (\alpha^*, \infty)$. Then the Hardy-Poincaré inequality holds for any $v \in \mathcal{D}(\mathbb{R}^d)$ with $C_{\alpha,d} := 4/(d - 2 + 2 \alpha)^2$ if $\alpha \in (\alpha^*, 1]$ and $C_{\alpha,d} := 4/[d(d - 2 + 2 \alpha)]$ if $\alpha \geq 1$. The constant $C_{\alpha,d}$ is optimal for any $\alpha \in (\alpha^*, 1]$.

**Proof [Davies]:**

$h_\alpha = (1 + |x|^2)^\alpha$, $\nabla h_\alpha = 2\alpha x h_{\alpha-1}$,

$\Delta h_\alpha = 2\alpha h_{\alpha-2} [d + 2(\alpha - \alpha^*) |x|^2] > 0$

By Cauchy-Schwarz

$$\left| \int_{\mathbb{R}^d} |v|^2 \Delta h_\alpha \, dx \right|^2 \leq 4 \left( \int_{\mathbb{R}^d} |v| |\nabla v| |\nabla h_\alpha| \, dx \right)^2$$

$$\leq 4 \int_{\mathbb{R}^d} |v|^2 |\Delta h_\alpha| \, dx \int_{\mathbb{R}^d} |\nabla v|^2 |\nabla h_\alpha|^2 |\Delta h_\alpha|^{-1} \, dx$$

$$|\Delta h_\alpha| \geq 2 |\alpha| \min\{d, (d - 2 + 2 \alpha)\} \frac{h_\alpha(x)}{1+|x|^2}$$

$$\frac{|\nabla h_\alpha|^2}{|\Delta h_\alpha|} \leq \frac{2 |\alpha|}{d-2+2 \alpha} h_\alpha(x)$$