Fast diffusions and generalized entropies

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PARIS-LONDON ANALYSIS SEMINAR

October 3, 2008

Abstract

An overview of the connections between functional inequalities, nonlinear diffusions, (transport theory) and generalized entropy functionals

- From functional inequalities to rates in nonlinear diffusions (porous medium equation)
- Functional inequalities and gradient flows
- Large time asymptotics of nonlinear diffusions (fast diffusion equation)

Contents

- L^q Poincaré inequalities and application
  - characterization of some L^q Poincaré inequalities
  - applications to a porous medium equation
- The Bakry-Emery method for generalized Poincaré inequalities
  - a non-local condition (linear case)
  - extension to a porous medium equation
- Remarks on entropies, transport and distances between measures
- The fast diffusion equation
  - intermediate asymptotics and interpolation
  - extensions (finite mass case)
  - the infinite mass regime and Hardy-Poincaré inequalities

L^q Poincaré inequalities for general measures, porous media equation

J.D., Ivan Gentil, Arnaud Guillin and Feng-Yu Wang

Goal

L^q-Poincaré inequalities, $q \in (1/2, 1]$

$$\left[\mathbf{Var}_\mu(f^q)\,\right]^{1/q} := \left[\int f^{2q}\,d\mu - \left(\int f^q\,d\mu\right)^2\right]^{1/q} \leqslant \mathrm{C_P}\int |\nabla f|^2\,d\nu$$

Application to the weighted porous media equation, $m \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla\psi\cdot\nabla u^m\,, \quad t \geqslant 0\,, \quad x \in \mathbb{R}^d$$

(Ornstein-Uhlenbeck form). With $d\mu = d\nu = d\mu_\psi = e^{-\psi}\,dx/\int e^{-\psi}\,dx$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = -\frac{8}{(m+1)^2}\int |\,\nabla u^{\frac{m+1}{2}}\,|^2\,d\mu_\psi$$

Outline

Equivalence between the following properties:

- L^q-Poincaré inequality
- Capacity-measure criterion
- Weak Poincaré inequality
- BCR (Barthe-Cattiaux-Roberto) criterion

In dimension $d = 1$, there are necessary and sufficient conditions to satisfy the BCR criterion

Motivation: large time asymptotics in connection with functional inequalities

L^q-Poincaré inequality

We shall say that (μ, ν) satisfies a L^q-Poincaré inequality with constant $\mathrm{C_P}$ if for all non-negative functions $f \in \mathcal{C}^1(M)$ one has

$$\left[\mathbf{Var}_\mu(f^q)\right]^{1/q} \leqslant \mathrm{C_P} \int |\nabla f|^2 \, d\nu$$

$q \in (0,1]$ (false for $q > 1$ unless μ is a Dirac measure)

$\mathbf{Var}_\mu(g^2) = \int g^2 \, d\mu - \left(\int g \, d\mu\right)^2 = \mu(g^2) - \mu(g)^2$

$q \mapsto \left[\mathbf{Var}_\mu(f^q)\right]^{1/q}$ increasing wrt $q \in (0,1]$: L^q-Poincaré inequalities form a hierarchy

Capacity-measure criterion

Capacity $\mathrm{Cap}_\nu(A,\Omega)$ of two measurable sets A and Ω such that $A \subset \Omega \subset M$

$$\mathrm{Cap}_\nu(A,\Omega) := \inf \left\{ \int |\nabla f|^2 \, d\nu \ : \ f \in \mathcal{C}^1(M)\,,\ \mathbb{I}_A \leqslant f \leqslant \mathbb{I}_\Omega \right\}$$

$$\beta_{\mathrm{P}} := \sup \left\{ \sum_{k\in\mathbb{Z}} \frac{\left[\mu(\Omega_k)\right]^{1/(1-q)}}{\left[\mathrm{Cap}_\nu(\Omega_k, \Omega_{k+1})\right]^{q/(1-q)}} \right\}^{(1-q)/q}$$

over all $\Omega \subset M$ with $\mu(\Omega) \leq 1/2$ and all sequences $(\Omega_k)_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$, $\Omega_k \subset \Omega_{k+1} \subset \Omega$

Theorem 1 (i) *If* $q \in [1/2, 1)$, *then* $\beta_{\mathrm{P}} \leqslant 2^{1/q}\, \mathrm{C}_{\mathrm{P}}$

(ii) *If* $q \in (0,1)$ *and* $\beta_{\mathrm{P}} < +\infty$, *then* $\mathrm{C}_{\mathrm{P}} \leqslant \kappa_{\mathrm{P}}\, \beta_{\mathrm{P}}$

Weak Poincaré inequalities

Definition 2 [Röckner and Wang] (μ, ν) *satisfies a weak Poincaré inequality if there exists a non-negative non increasing function* $\beta_{\mathrm{WP}}(s)$ *on* $(0, 1/4)$ *such that, for any bounded function* $f \in \mathcal{C}^1(M)$,

$$\forall\, s > 0\,, \quad \mathbf{Var}_\mu(f) \leqslant \beta_{\mathrm{WP}}(s) \int |\nabla f|^2\, d\nu + s\left[\mathbf{Osc}_\mu(f)\right]^2$$

$\mathbf{Var}_\mu(f) \leqslant \mu((f-a)^2)\ \forall\, a \in \mathbb{R}$

For $a = (\mathrm{supess}_\mu f + \mathrm{infess}_\mu f)/2$, $\mathbf{Var}_\mu(f) \leqslant \left[\mathbf{Osc}_\mu(f)\right]^2/4$: $s \leqslant 1/4$.

Proposition 3 *Let* $q \in [1/2, 1)$. *If* (μ, ν) *satisfies the* L^q*-Poincaré inequality, then it also satisfies a weak Poincaré inequality with* $\beta_{\mathrm{WP}}(s) = (11 + 5\sqrt{5})\, \beta_{\mathrm{P}}\, s^{1-1/q}/2$, $K := (11 + 5\sqrt{5})/2$.

L^q-Poincaré $\Longrightarrow$ BCR criterion $\Longrightarrow$ weak Poincaré

Theorem 4 [Maz'ja] *Let $q \in [1/2, 1)$. For all bounded open set $\Omega \subset M$, if $(\Omega_k)_{k\in\mathbb{Z}}$ is a sequence of open sets such that $\Omega_k \subset \Omega_{k+1} \subset \Omega$, then*

$$\sum_{k\in\mathbb{Z}} \frac{\mu(\Omega_k)^{1/(1-q)}}{[\mathrm{Cap}_\nu(\Omega_k, \Omega_{k+1})]^{q/(1-q)}} \leqslant \frac{1}{1-q} \int_0^{\mu(\Omega)} \left(\frac{t}{\Phi(t)}\right)^{q/(1-q)} dt$$

where $\Phi(t) := \inf \{\mathrm{Cap}_\nu(A, \Omega) \,:\, A \subset \Omega\,,\ \mu(A) \geqslant t\}$

As a consequence: $\beta_{\mathrm{P}} \leqslant (1-q)^{-(1-q)/q} \, \|t/\Phi(t)\|_{L^{q/(1-q)}(0,\mu(\Omega))}$

Corollary 5 *Let $q \in [1/2, 1)$. If (μ, ν) satisfies a weak Poincaré inequality with function β_{WP}, then it satisfies a L^q-Poincaré inequality with*

$$\beta_{\mathrm{P}} \leqslant \frac{11 + 5\sqrt{5}}{2} \left(\frac{4}{1-q}\right)^{\frac{1-q}{q}} \|\beta_{\mathrm{WP}}(\cdot/4)\|_{L^{\frac{q}{1-q}}(0,1/2)}$$

L^q-Poincaré	$\Longrightarrow$	Weak Poincaré with $\beta_{\mathrm{WP}}(s) = C\, s^{\frac{q-1}{q}}$	$\Longrightarrow$	$L^{q'}$-Poincaré $\forall\, q' \in (0, q)$

BCR criterion (1/2)

A variant of two results of [Barthe, Cattiaux, Roberto, 2005] (no absolute continuity of the measure μ with respect to the volume measure)

Theorem 6 [BCR] *Let μ be a probability measure and ν a positive measure on M such that (μ, ν) satisfies a weak Poincaré inequality with function $\beta_{\mathrm{WP}}(s)$. Then for every measurable subsets A, B of M such that $A \subset B$ and $\mu(B) \leqslant 1/2$,*

$$\mathrm{Cap}_\nu(A, B) \geq \frac{\mu(A)}{\gamma(\mu(A))} \quad \textit{with} \quad \gamma(s) := 4\,\beta_{\mathrm{WP}}(s/4)$$

Proof $\triangleleft$ Take f such that $\mathbb{I}_A \leqslant f \leqslant \mathbb{I}_B$: $\mathbf{Osc}_\mu(f) \leqslant 1$

By Cauchy-Schwarz, $\left(\int f\,d\mu\right)^2 \leqslant \mu(B) \int f^2\,d\mu \leqslant \frac{1}{2} \int f^2\,d\mu$

$$\beta_{\mathrm{WP}}(s) \int |\nabla f|^2\,d\nu + s \geqslant \mathbf{Var}_\mu(f) \geq \frac{1}{2} \int f^2\,d\mu \geq \frac{\mu(A)}{2}$$

$\frac{a}{\gamma(a)} = \frac{a}{4\,\beta_{\mathrm{WP}}(a/4)} \leqslant \sup_{s\in(0,1/4)} \frac{a/2-s}{\beta_{\mathrm{WP}}(s)}$ with $a/2 = \mu(A)/2 \leqslant 1/4$ $\triangleright$

BCR criterion (2/2)

Lemma 7 *Take μ and ν as before, $\theta \in (0,1)$, γ a positive non increasing function on $(0,\theta)$. If $\forall\ A, B \subset M$ such that $A \subset B$ are measurable and $\mu(B) \leqslant \theta$,*

$$\mathrm{Cap}_\nu(A,B) \geq \frac{\mu(A)}{\gamma(\mu(A))}$$

then for every function $f \in \mathcal{C}^1(M)$ such that $\mu(\Omega_+) \leqslant \theta$, $\Omega_+ := \{f > 0\}$

$$\int f_+^2 \leq \frac{11+5\sqrt{5}}{2}\gamma(s)\int_{\Omega_+} |\nabla f|^2\, d\nu + s\left[\text{supess}_\mu\, f\right]^2 \quad \forall\ s \in (0,1)$$

Theorem 8 *Same assumptions, $\theta = 1/2$. Then $\forall f \in \mathcal{C}^1(M)$*

$$\mathbf{Var}_\mu(f) \leq \frac{11+5\sqrt{5}}{2}\gamma(s)\int |\nabla f|^2\, d\nu + s\left[\mathbf{Osc}_\mu(f)\right] \quad \forall\ s \in (0,1/4)$$

$\theta = 1/2$: use the median $m_\mu(f)$, $\mu(f \geqslant m_\mu(f)) \geqslant 1/2$, $\mu(f \leqslant m_\mu(f)) \geqslant 1/2$

Using the BCR criterion: a "Hardy condition"

[Muckenhoupt, 1972] [Bobkov-Götze, 1999] [Barthe-Roberto, 2003] [Barthe-Cattiaux-Roberto, 2005]

$M = \mathbb{R}$, $d\mu = \rho_\nu \, dx$ with median m_μ, $d\nu = \rho_\nu \, dx$

$$R(x) := \mu([x, +\infty)) \ , \quad L(x) := \mu((-\infty, x])$$

$$r(x) := \int_{m_\mu}^{x} \frac{1}{\rho_\nu} \, dx \quad \text{and} \quad \ell(x) := \int_{x}^{m_\mu} \frac{1}{\rho_\nu} \, dx$$

Proposition 9 *Let $q \in [1/2, 1]$. (μ, ν) satisfies a L^q-Poincaré inequality if*

$$\int_{m_\mu}^{\infty} |\, r\, R\,|^{q/(1-q)} \, d\mu < \infty \quad \textit{and} \quad \int_{-\infty}^{m_\mu} |\, \ell\, L\,|^{q/(1-q)} \, d\mu < \infty$$

Proof

Proof $\lhd$ Method: $\mathbf{Var}_\mu(f) \leqslant \mu(|F_-|^2) + \mu(|F_+|^2))$ with $g = (f - f(m_\mu))_\pm$ and prove that

$$\mu(|g|^2) \leqslant \frac{11 + 5\sqrt{5}}{2}\,\gamma(s)\int |\nabla g|^2\,d\nu + s\left[\text{supess}_\mu g\right]^2 \quad \forall\, s \in (0, 1/2)$$

Let $A \subset B \subset M = (m_\mu, \infty)$ such that $A \subset B$ and $\mu(B) \leqslant 1/2$

$$\mathrm{Cap}_\nu(A, B) \geqslant \mathrm{Cap}_\nu\big(A, (m_\mu, \infty)\big) = \mathrm{Cap}_\nu\big((a, \infty), (m_\mu, \infty)\big) = \frac{1}{r(a)}$$

where $a = \inf A$. Change variables: $t = R(a)$ and choose $\gamma(t) := t\,(r \circ R)^{-1}(t)$ for any $t \in (0, 1/2)$ $\rhd$

Porous media equation

With $\psi \in \mathcal{C}^2(\mathbb{R}^d)$, $d\mu_\psi := \frac{e^{-\psi}\,dx}{Z_\psi}$, define $\mathcal{L}$ on $\mathcal{C}^2(\mathbb{R}^d)$ by

$$\forall\ f \in \mathcal{C}^2(\mathbb{R}^d) \quad \mathcal{L}f := \Delta f - \nabla\psi \cdot \nabla f$$

Such a generator $\mathcal{L}$ is symmetric in $L^2_{\mu_\psi}(\mathbb{R}^d)$,

$$\forall\ f,\ g \in \mathcal{C}^1(\mathbb{R}^d) \quad \int f\,\mathcal{L}g\,d\mu_\psi = -\int \nabla f \cdot \nabla g\,d\mu_\psi$$

Consider for $m > 1$ the weighted porous media equation

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}\,u^m & \text{in} \quad \mathrm{Q} \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega \\ n \cdot \nabla u = 0 & \text{on} \quad \Sigma \end{cases}$$

$\Omega \subset \mathbb{R}^d$, $Q = \Omega \times [0, +\infty)$, $\Sigma = \partial\Omega \times [0, +\infty)$

$u \in \mathcal{C}^2$, L^1-contraction, existence and uniqueness

Asymptotic behavior

Theorem 10 *Let $m \geqslant 1$ and assume that (μ_ψ, μ_ψ) satisfies a L^q-Poincaré inequality, $q = 2/(m+1)$*

$$\mathbf{Var}_{\mu_\psi}(u(\cdot,t)) \leqslant \left(\left[\mathbf{Var}_{\mu_\psi}(u_0) \right]^{-(m-1)/2} + \frac{4\,m\,(m-1)}{(m+1)^2}\,\mathrm{C_P}\,t \right)^{-2/(m-1)}$$

Reciprocally, if the above inequality is satisfied for any u_0, then (μ_ψ, μ_ψ) satisfies a L^q-Poincaré inequality with constant $\mathrm{C_P}$

Proof $\triangleleft$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = 2\int u_t\,u\,d\mu_\psi = 2\int u\,\mathcal{L}u^m\,d\mu_\psi = -\frac{8m}{(m+1)^2}\int |\nabla u^{\frac{m+1}{2}}|^2\,d\mu_\psi$$

Apply the L^q-Poincaré inequality with $u = f^{2/(m+1)}$, $q = 2/(m+1)$

Reciprocally, a derivation at $t = 0$ gives the L^q-Poincaré inequality $\triangleright$

A conclusion on L^q-Poincaré inequalities

- Observe that we have only algebraic rates
- Weak logarithmic Sobolev inequalities [Cattiaux-Gentil-Guillin, 2006], L^q-logarithmic Sobolev inequalities [D.-Gentil-Guillin-Wang, 2006]

$$\left(\int f^{2q} \frac{\log f^{2q}}{\int f^{2q}\,d\mu}\,d\mu\right) =: \mathbf{Ent}_\mu\left(f^{2q}\right)^{1/q} \leq \mathrm{C_{LS}} \int |\nabla f|^2\,d\mu$$

- Orlicz spaces, duality, connections with mass transport theory [Bobkov-Götze, 1999] [Cattiaux-Gentil-Guillin, 2006] [Wang, 2006] [Roberto-Zegarlinski, 2003] [Barthe-Cattiaux-Roberto, 2005]

The Bakry-Emery method revisited

J.D., B. Nazaret, G. Savaré

Consider a domain $\Omega \subset \mathbb{R}^d$, $d\gamma = g\,dx$, $g = e^{-F}$ and a generalized Ornstein-Uhlenbeck operator: $\Delta_g v := \Delta v - \mathrm{D}F \cdot \mathrm{D}v$

$$\int_\Omega |\mathrm{D}v|^2\,d\gamma = -\int_\Omega v\,\Delta_g v\,d\gamma \quad \forall\, v \in H^1_0(\Omega, d\gamma)$$

Let $s := v^{p/2}$ and $\alpha := (2-p)/p$, $p \in (1,2]$

$$v_t = \Delta_g v \quad x \in \Omega\,,\ t \in \mathbb{R}^+$$

$$\nabla v \cdot n = 0 \quad x \in \partial\Omega\,,\ t \in \mathbb{R}^+$$

$$\mathcal{E}_p(t) \quad := \quad \frac{1}{p-1}\int_\Omega \Big[v^p - 1 - p\,(v-1)\Big]\,d\gamma$$

$$\mathcal{I}_p(t) \quad := \quad \frac{4}{p}\int_\Omega |\mathrm{D}s|^2\,d\gamma$$

$$\mathcal{K}_p(t) \quad := \quad \int_\Omega |\Delta_g s|^2\,d\gamma + \alpha\int_\Omega \Delta_g s\,\frac{|\mathrm{D}s|^2}{s}\,d\gamma$$

Written in terms of $s = v^{p/2}$, the entropy is

$$\mathcal{E}_p = \frac{1}{p-1} \int_\Omega \left[s^2 - 1 - p \left(s^{2/p} - 1 \right) \right] d\gamma$$

and the evolution is governed by

$$s_t = \Delta_g s + \alpha \, \frac{|\mathrm{D} s|^2}{s}$$

A simple computation shows that

$$\begin{aligned} \frac{d}{dt} \mathcal{E}_p(t) &:= -\mathcal{I}_p(t) \\ \frac{d}{dt} \mathcal{I}_p(t) &:= -\frac{8}{p} \, \mathcal{K}_p(t) \end{aligned}$$

Using the commutation relation $[\mathrm{D}, \Delta_g]\, s = -\mathrm{D}^2 F\, \mathrm{D} s$, we get

$$\int_\Omega (\Delta_g s)^2 \, d\gamma = \int_\Omega |\mathrm{D}^2 s|^2 \, d\gamma + \int_\Omega \mathrm{D}^2 F\, \mathrm{D}s \cdot \mathrm{D}s \, d\gamma \underbrace{- \sum_{i,j=1}^{d} \int_{\partial\Omega} \partial^2_{ij} s\, \partial_i s\, n_j \; g\, d\mathcal{H}^{d-1}}_{\geq 0 \text{ if } \Omega \text{ is convex}}$$

Let $z := \sqrt{s}$. Using : $2\,\mathrm{D}^2 s \cdot \mathrm{D}z \otimes \mathrm{D}z = \mathrm{D}\left(|\mathrm{D}z|^2\right) : \mathrm{D}z$ and i.p.p., we get

$$\begin{aligned}
\mathcal{K}_p &= \int_\Omega |\Delta_g s|^2 \, d\gamma + 4\,\alpha \int_\Omega \Delta_g s\, |\mathrm{D}z|^2 \, d\gamma \\
&\geq \int_\Omega |\mathrm{D}^2 s|^2 \, d\gamma + \int_\Omega \mathrm{D}^2 F\, \mathrm{D}s \cdot \mathrm{D}s \, d\gamma \\
&\qquad + 4^2\,\alpha \int_\Omega |Dz|^4 \, d\gamma - 2 \cdot 4\,\alpha \int_\Omega \mathrm{D}^2 s \;:\; \mathrm{D}z \otimes \mathrm{D}z \, d\gamma \\
&\geq (1-\alpha) \int_\Omega |\mathrm{D}^2 s|^2 \, d\gamma + \int_\Omega \mathrm{D}^2 F\, \mathrm{D}s \cdot \mathrm{D}s \, d\gamma
\end{aligned}$$

An extension of the criterion of Bakry-Emery

Let $V(x) := \inf_{\xi \in S^{d-1}} \left(\mathrm{D}^2 F(x)\,\xi,\xi\right)$ and define

$$\lambda_1(p) := \inf_{w \in \mathcal{D}(\Omega)\setminus\{0\}} \frac{\int_\Omega \left(2\,\frac{p-1}{p}\,|\mathrm{D}w|^2 + V\,|w|^2\right)\,d\gamma}{\int_\Omega |w|^2\,d\gamma}$$

Theorem 1 *Let $F \in C^2(\Omega)$, $\gamma = e^{-F} \in L^1(\Omega)$, and Ω be a convex domain in $\mathbb{R}^d$. If $\lambda_1(p)$ is positive, then*

$$\mathcal{I}_p(t) \leq \mathcal{I}_p(0)\,e^{-2\,\lambda_1(p)\,t}$$

$$\mathcal{E}_p(t) \leq \mathcal{E}_p(0)\,e^{-2\,\lambda_1(p)\,t}$$

Generalized entropies

Consider the weighted porous media equation

$$v_t = \Delta_g v^m$$

$d\gamma$ is a probability measure, $p \in (1,2)$

$$\mathcal{E}_{m,p}(t) := \frac{1}{m+p-2} \int_\Omega \left[v^{m+p-1} - 1 \right] d\gamma$$

$$\mathcal{I}_{m,p}(t) := c(m,p) \int_\Omega |\mathrm{D}s|^2 \, d\gamma$$

$$\mathcal{K}_{m,p}(t) := \int_\Omega s^{\beta(m-1)} \, |\Delta_g s|^2 \, d\gamma + \alpha \int_\Omega s^{\beta(m-1)} \, \Delta_g s \, \frac{|\mathrm{D}s|^2}{s} \, d\gamma$$

with $v =: s^\beta$, $\beta := \frac{1}{p/2+m-1}$, $\alpha := \frac{2-p}{p+2(m-1)}$ and $c(m,p) = \frac{4\,m\,(m+p-1)}{(2m+p-2)^2}$

adapting the Bakry-Emery method...

Written in terms of $s = v^{1/\beta}$, the evolution is governed by

$$\frac{1}{m}\, s_t = s^{\beta(m-1)} \left[\Delta_g s + \alpha\, \frac{|\mathrm{D} s|^2}{s} \right]$$

A computation shows that

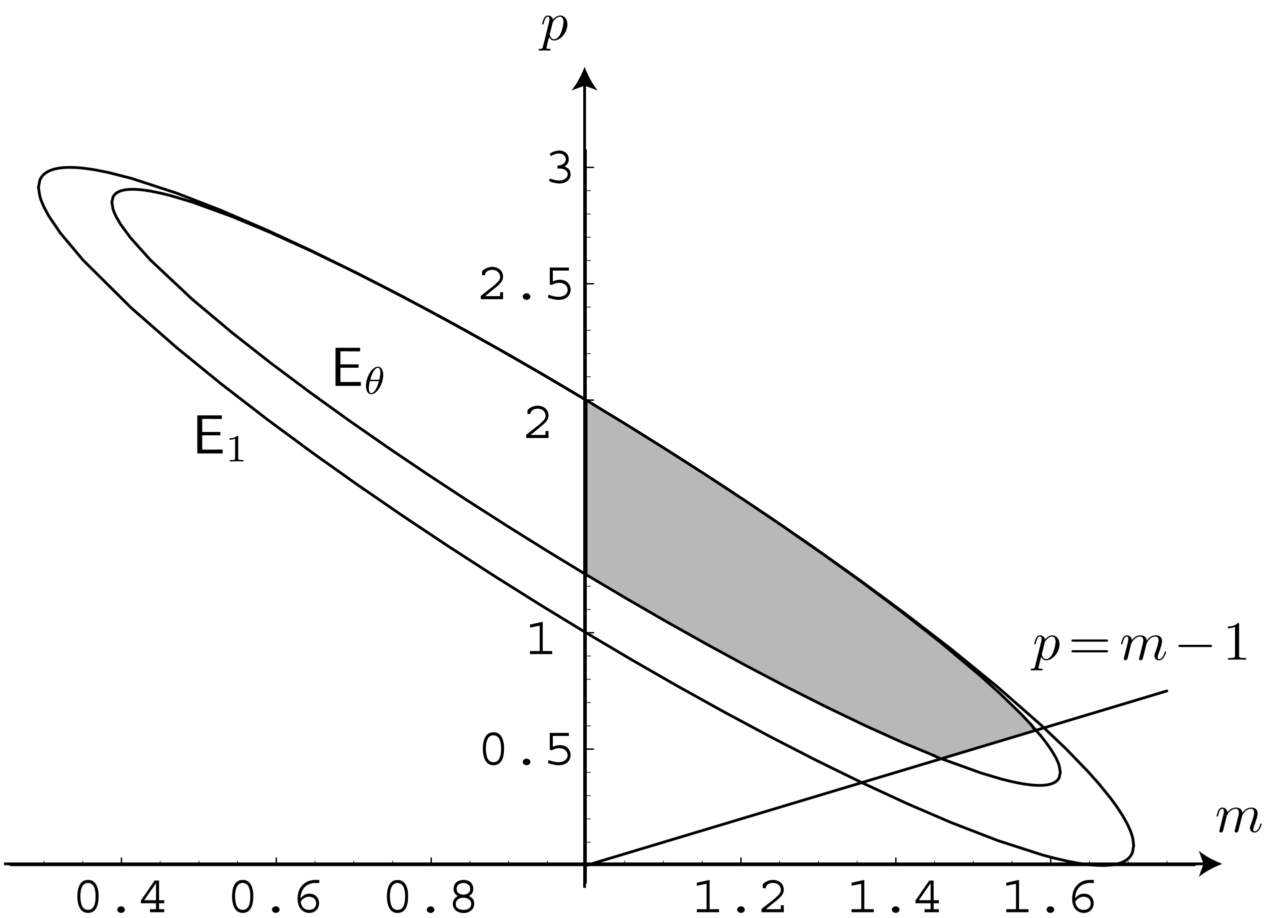
$$\begin{aligned} \frac{d}{dt}\mathcal{E}_{m,p}(t) &:= -\mathcal{I}_{m,p}(t) \\ \frac{1}{m}\,\frac{d}{dt}\mathcal{I}_{m,p}(t) &:= -2\, c(m,p)\, \mathcal{K}_{m,p}(t) \end{aligned}$$

Exactly as in the linear case, define for any $\theta \in (0,1)$

$$\lambda_1(m,\theta) := \inf_{w \in H^1(\Omega, d\gamma) \setminus \{0\}} \frac{\int_\Omega \Big((1-\theta)\, |\mathrm{D} w|^2 + V\, |w|^2 \Big)\, d\gamma}{\int_\Omega |w|^2\, d\gamma}$$

The non-local condition

Assume that for some $\theta \in (0,1)$, $\lambda_1(m,\theta) > 0$. Admissible parameters m and p correspond to $(m,p) \in \mathsf{E}_\theta$, $1 < m < p+1$, where the set E_θ is defined by the condition: $\mathsf{b}^2 - 4\,\mathsf{a}(\theta)\,\mathsf{c} < 0$.



Results for the fast diffusion equation

Lemma 1 *With the above notations, if Ω is convex and $(m,p) \in \mathsf{E}_\theta$ are admissible, then*

$$\mathcal{I}_{m,p}^{\frac{4}{3}} \leq \frac{1}{3}\left[4\,c(m,p)\right]^{\frac{4}{3}} \mathsf{K}^{\frac{1}{3}} \Big[(m+p-2)\,\mathcal{E}_{m,p}+1\Big]^{\frac{4-3q}{3(2-q)}} \mathcal{K}_{m,p}$$

Theorem 2 *Under the above conditions there exists a positive constant κ which depends on $\mathcal{E}_{m,p}(0)$ such that any smooth solution u of the porous media equation satisfies, for any $t>0$,*

$$\mathcal{I}_{m,p}(t) \leq \frac{\mathcal{I}_{m,p}(0)}{\left[1+\frac{\kappa}{3}\,\sqrt[3]{\mathcal{I}_{m,p}(0)}\,t\right]^3}$$

$$\mathcal{E}_{m,p}(t) \leq \frac{3\left[\mathcal{I}_{m,p}(0)\right]^{\frac{8}{3}}}{2\,\kappa\left[1+\frac{\kappa}{3}\,\sqrt[3]{\mathcal{I}_{m,p}(0)}\,t\right]^2}$$

Entropies, transport and distances between measures

J.D., B. Nazaret, G. Savaré

Wasserstein distances

$p > 1$, μ_0 and μ_1 probability measures on $\mathbb{R}^d$

- Transport plans between μ_0 and μ_1 : $\Gamma(\mu_0, \mu_1)$ is the set of probability measures on $\mathbb{R}^d \times \mathbb{R}^d$ having μ_0 and μ_1 as marginals.
- Wasserstein distance between μ_0 ans μ_1

$$W_p^p(\mu_0, \mu_1) = \inf \left\{ \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\Sigma(x, y) \ : \ \Sigma \in \Gamma(\mu_0, \mu_1) \right\}$$

- The Benamou-Brenier characterization (2000)

$$W_p^p(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |\mathbf{v}_t|^p \rho_t dx dt \ : \ (\rho_t, \mathbf{v}_t)_{t \in [0,1]} \text{ admissible} \right\}$$

where admissible paths $(\rho_t, \mathbf{v}_t)_{t\in[0,1]}$ are such that
$\partial_t \rho_t + \nabla \cdot (\rho_t \mathbf{v}_t) = 0$, $\rho_0 = \mu_0$, $\rho_1 = \mu_1$

A generalization of the Benamou-Brenier approach

Given a function h on $\mathbb{R}^+$, define the admissible paths by

$$\begin{cases} \partial_t \rho_t + \nabla \cdot (h(\rho_t)\mathbf{v}_t) = 0, \\ \rho_0 = \mu_0, \ \rho_1 = \mu_1 \end{cases}$$

and consider the distance

$$W_h^p(\mu_0, \mu_1) = \inf \left\{ \int_0^1 \int_{\mathbb{R}^d} |\mathbf{v}_t|^p h(\rho_t) dx dt \ : \ (\rho_t, \mathbf{v}_t)_{t \in [0,1]} \text{ admissible} \right\}$$

$$h(\rho) = \rho^\alpha, \ 0 \leq \alpha \leq 1$$

- $\alpha = 1$: Wasserstein case
- $\alpha = 0$: homogeneous Sobolev distance on $\dot{W}^{-1,p}$. With $q = \frac{p}{p-1}$

$$\|\mu_1 - \mu_0\|_{\dot{W}^{-1,p}} = \sup \left\{ \int_{\mathbb{R}^d} \xi d(\mu_1 - \mu_0) \ : \ \xi \in \mathcal{C}_c^1(\mathbb{R}^d), \ \int_{\mathbb{R}^d} |\nabla \xi|^q \leq 1 \right\}$$

Gradient flows

- Jordan-Kinderlehrer-Otto 98 : Formal Riemannian structure on $\mathcal{P}(\mathbb{R}^d)$: the McCann interpolant is a geodesic. For an integral functional such as

$$\mathcal{F}(\rho) = \int_{\mathbb{R}^d} F(\rho(x))dx$$

the gradient flow of $\mathcal{F}$ is

$$\frac{\partial \rho}{\partial t} = \nabla \cdot \left(\rho \nabla \left(F'(\rho)\right)\right)$$

- Ambrosio-Gigli-Savaré 05 : Rigorous framework for JKO's calculus in the framework of length spaces (based on the optimal transportation)
- Otto-Westdickenberg 05 : Use the Brenier-Benamou formulation to prove

$$W_2^2(\mu_0^t, \mu_1^t) \leq W_2^2(\mu_0, \mu_1)$$

along the heat flow on a compact Riemannian manifold

The heat equation as gradient flow w.r.t. W_φ

Denote by S_t the semi-group associated to the heat equation. Let $\alpha > 1 - \frac{2}{d}$ and consider the generalized entropy functional

$$\Psi_\alpha(\mu) = \frac{1}{(1-\alpha)(2-\alpha)} \int_{\mathbb{R}^d} \rho^{2-\alpha}(x)dx, \text{ if } \mu = \rho\mathcal{L}^d$$

Theorem 1 *If $\mu \in \mathcal{P}(\mathbb{R}^d)$, $\Psi_\alpha(\mu) < +\infty$, then $\Psi_\alpha(S_t\mu) < +\infty$ for all $t > 0$ and*

$$\frac{1}{2}\frac{d}{dt}W_\alpha^2(S_t\mu, \sigma) + \Psi_\alpha(S_t\mu) \leq \Psi_\alpha(\sigma)$$

Corollary 2 *Ψ_α is geodesically convex w.r.t. W_α*

Fast diffusion equations: entropy methods and functional inequalities

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\; t > 0$$

- Entropy methods for fast diffusion and porous media equations: intermediate asymptotics
- Entropy methods and functional inequalities

Porous media / fast diffusion equations

Generalized entropies and nonlinear diffusions (EDP, uncomplete):
[Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],...
[del Pino, Sáez], [Daskalopulos, Sesum]...

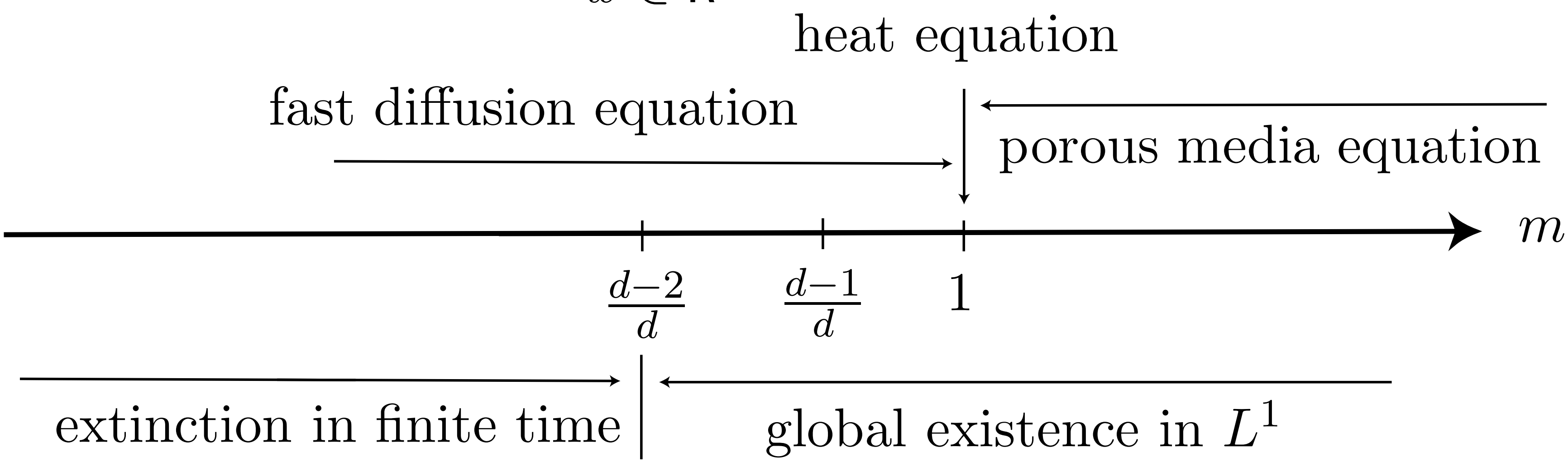
1) [J.D., del Pino] relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Various alternative approaches:
2) “entropy – entropy-production method”
3) mass transport techniques
4) hypercontractivity for appropriate semi-groups

Heat equation, porous media & fast diffusion equation

$$u_t = \Delta u^m$$

$$x \in \mathbb{R}^d$$



Existence theory, critical values of the parameter m

Intermediate asymptotics for fast diffusion & porous media

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^d$$

$$u_{|t=0} = u_0 \geq 0$$

$$u_0(1+|x|^2) \in L^1 \ , \quad u_0^m \in L^1$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0 \, dx = M > 0$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$
[Friedmann, Kamin, 1980] As $t \to +\infty$

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$$

$\Longrightarrow$ What about $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^1}$?

Time-dependent rescaling

Take $u(t,x) = R^{-d}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{d(1-m)-1}\ , \quad R(0) = 1\ , \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x\, v)\ , \quad v_{|\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: Entropy or Free energy

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right)\, dx - \Sigma_0$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\ , \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Entropy and entropy production

Stationary solution: choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m} \, |x|^2 \right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form

$$\Sigma[v] = \int \psi\left(\tfrac{v}{v_\infty}\right) \, v_\infty^m \, dx \quad with \ \psi(t) = \tfrac{t^m - 1 - m\,(t-1)}{m-1}$$

Theorem 1 $d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int |w|^{1+p} dx + K \geq 0$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$ and, for some γ, K can be written as

$$K = K_0 \left(\int v\, dx = \int w^{2p}\, dx \right)^\gamma$$

$w = w_\infty = v_\infty^{1/2p}$ is optimal

$m = \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

Gagliardo-Nirenberg inequalities

Theorem 2 [Del Pino, J.D.] *Assume that* $1 < p \leq \frac{d}{d-2}$ *and* $d \geq 3$

$$\|w\|_{2p} \leq A \, \|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}$$

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)} \,, \quad y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$

Uses [Serrin-Pucci], [Serrin-Tang]

$1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \Longleftrightarrow$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$

$0 < p < 1 \Longleftrightarrow$ Porous medium case: $m > 1$

Intermediate asymptotics

$\Sigma[v] \le \Sigma[u_0]\, e^{-2\tau}$+ Csiszár-Kullback inequalities

Theorem 3 [Del Pino, J.D.]

(i) $\frac{d-1}{d} < m < 1$ *if* $d \ge 3$

$$\limsup_{t\to+\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t\to+\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \|\, [u - u_\infty]\, u_\infty^{m-1}\, \|_{L^1} < +\infty$$

$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\left(x/R(t)\right)$$

Fast diffusion equations: the finite mass regime

- If $m \geq 1$: porous medium regime or $m_1 := \frac{d-1}{d} \leq m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality
- If $m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass

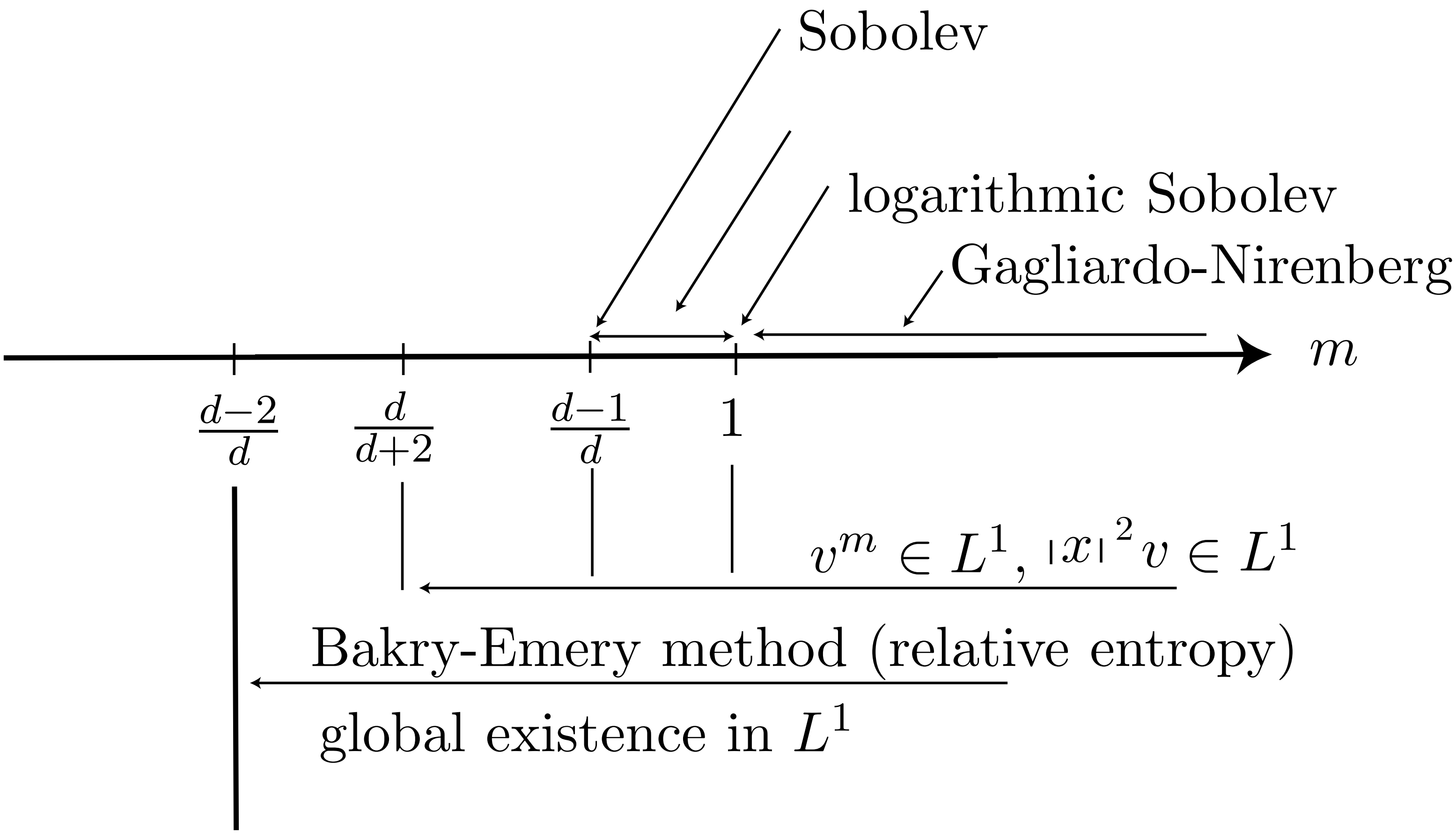
A remark on the mass transport approach

- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance
- Displacement convexity holds in the same range of exponents, $m \in ((d-1)/d, 1)$, as for the Gagliardo-Nirenberg inequalities

$\Rightarrow$ How to extend to $m_c < m < m_1$ what has been done for $m \geq m_1$?

Fast diffusion: finite mass regime

Inequalities...



... existence of solutions of $u_t = \Delta u^m$

Extensions and related results

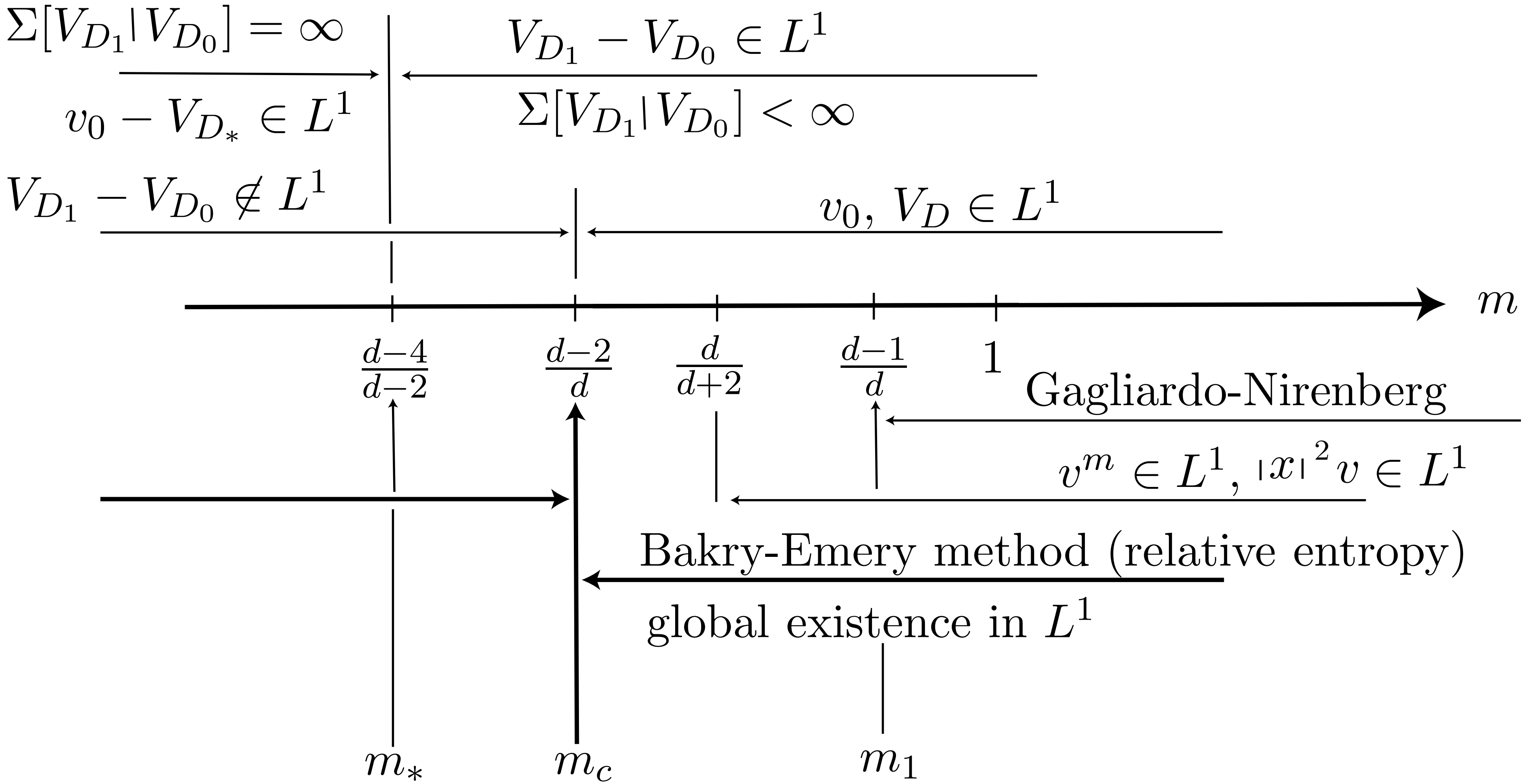
- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg] [J.D.-Nazaret-Savaré], etc
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev]

Fast diffusion equations: the infinite mass regime

- If $m > m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass.
- For $m \leq m_c$, the Barenblatt self-similar solution has infinite mass

$\Rightarrow$ How to extend to $m \leq m_c$ what has been done for $m > m_c$? Work in relative variables !

Fast diffusion: infinite mass regime



Entropy methods and linearization...

... intermediate asymptotics, vanishing

A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez

- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

$\Longrightarrow$ Extend the domain of validity of the method to the price of a restriction

of the set of admissible solutions

Setting of the problem

We consider the solutions $u(\tau, y)$ of

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0,\cdot) = u_0 \end{cases}$$

where $m \in (0,1)$ (fast diffusion) and $(\tau, y) \in Q_T = (0,T) \times \mathbb{R}^d$
Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where

$$m_c := \frac{d-2}{d}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time

$$\lim_{\tau \nearrow T} u(\tau, y) = 0$$

Barenblatt solutions

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}$$

with

- $R(\tau) := \left[d\,(m-m_c)\,(\tau+T)\right]^{\frac{1}{d\,(m-m_c)}}$ if $m_c < m < 1$
- (vanishing in finite time) if $0 < m < m_c$

$$R(\tau) := \left[d\,(m_c-m)\,(T-\tau)\right]^{-\frac{1}{d\,(m_c-m)}}$$

Time-dependent rescaling: $t := \log\left(\frac{R(\tau)}{R(0)}\right)$ and $x := \frac{y}{R(\tau)}$. The function $v(t,x) := R(\tau)^d\,u(\tau,y)$ solves a nonlinear *Fokker-Planck type equation*

$$\begin{cases} \partial_t v(t,x) = \Delta v^m(t,x) + \nabla\cdot(x\,v(t,x)) & (t,x)\in(0,+\infty)\times\mathbb{R}^d \\ v(0,x) = v_0(x) = R(0)^d\,u_0(R(0)\,x) & x\in\mathbb{R}^d \end{cases}$$

Assumptions

(H1) u_0 is a non-negative function in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist positive constants T and $D_0 > D_1$ such that

$$U_{D_0,T}(0,y) \le u_0(y) \le U_{D_1,T}(0,y) \quad \forall\, y \in \mathbb{R}^d$$

(H2) If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$u_0(y) = U_{D_*,T}(0,y) + f(y) \quad \forall\, y \in \mathbb{R}^d$$

(H1') v_0 is a non-negative function in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(x) \le v_0(x) \le V_{D_1}(x) \quad \forall\, x \in \mathbb{R}^d$$

(H2') If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x) \quad \forall\, x \in \mathbb{R}^d$$

Convergence to the asymptotic profile (without rate)

$$m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{2}\,, \quad p(m) := \frac{d\,(1-m)}{2\,(2-m)}$$

Theorem 1 *Let $d \geq 3$, $m \in (0,1)$. Consider a solution v with initial data satisfying (H1')-(H2')*

(i) For any $m > m_*$, there exists a unique D_* such that $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,dx = 0$ for any $t > 0$. Moreover, for any $p \in (p(m), \infty]$, $\lim_{t\to\infty} \int_{\mathbb{R}^d} |v(t) - V_{D_*}|^p\,dx = 0$

(ii) For $m \leq m_*$, $v(t) - V_{D_*}$ is integrable, $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,dx = \int_{\mathbb{R}^d} f\,dx$ and $v(t)$ converges to V_{D_*} in $\mathrm{L}^p(\mathbb{R}^d)$ as $t \to \infty$, for any $p \in (1, \infty]$

(iii) (Convergence in Relative Error) For any $p \in (d/2, \infty]$,

$$\lim_{t\to\infty} \|\, v(t)/\,V_{D_*} - 1\,\|_p = \ 0$$

[Daskalopoulos-Sesum, 06], [Blanchet-Bonforte-J.D.-Grillo-Vázquez, 06]

Convergence with rate

$$q_* := \frac{2\,d\,(1-m)}{2\,(2-m)+d\,(1-m)}$$

Theorem 2 *If $m \neq m_*$, there exist $t_0 \geq 0$ and $\lambda_{m,d} > 0$ such that*

(i) *For any $q \in (q_*, \infty]$, there exists a positive constant C_q such that*

$$\|v(t) - V_{D_*}\|_q \leq C_q\, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0$$

(ii) *For any $\vartheta \in [0, (2-m)/(1-m))$, there exists a positive constant C_ϑ such that*

$$\big\|\, |x|^\vartheta (v(t) - V_{D_*}) \big\|_2 \leq C_\vartheta\, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0$$

(iii) *For any $j \in \mathbb{N}$, there exists a positive constant H_j such that*

$$\|v(t) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq H_j\, \mathrm{e}^{-\frac{\lambda_{m,d}}{d+2(j+1)}\, t} \quad \forall\, t \geq t_0$$

Intermediate asymptotics

Corollary 3 *Let* $d \geq 3$, $m \in (0,1)$, $m \neq m_*$. *Consider a solution* u *with initial data satisfying* (H1)-(H2). *For* τ *large enough, for any* $q \in (q_*, \infty]$, *there exists a positive constant* C *such that*

$$\|u(\tau) - U_{D_*}(\tau)\|_q \leq C\, R(\tau)^{-\alpha}$$

where $\alpha = \lambda_{m,d} + d(q-1)/q$ *and large means* $T - \tau > 0$, *small, if* $m < m_c$, *and* $\tau \to \infty$ *if* $m \geq m_c$

For any $p \in (d/2, \infty]$, *there exists a positive constant* $\mathcal{C}$ *and* $\gamma > 0$ *such that*

$$\left\| \, v(t)/\, V_{D_*} - 1 \, \right\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \mathcal{C}\, \mathrm{e}^{-\gamma\, t} \quad \forall\, t \geq 0$$

Rewriting the equation in relative variables

L^1-contraction, Maximum Principle, conservation of relative mass...

Passing to the quotient: the function $w(t,x) := \frac{v(t,x)}{V_{D_*}(x)}$ solves

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}} \nabla \cdot \left[w\, V_{D_*} \nabla \left(\dfrac{m}{m-1} (w^{m-1} - 1)\, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

with

$$0 < \inf_{x \in \mathbb{R}^d} \frac{V_{D_0}}{V_{D_*}} \le w(t,x) \le \sup_{x \in \mathbb{R}^d} \frac{V_{D_1}}{V_{D_*}} < \infty$$

... Harnack Principle

$$\|w(t)\|_{C^k(\mathbb{R}^d)} \le \overline{H}_k < +\infty \quad \forall\, t \ge t_0$$

$\exists\, t_0 \ge 0$ s.t. (H1) holds if $\exists\, R > 0$, $\sup_{|y|>R} u_0(y)\, |y|^{\frac{2}{1-m}} < \infty$, and $m > m_c$

Relative entropy

Relative entropy

$$\mathcal{F}[w] := \frac{1}{1-m} \int_{\mathbb{R}^d} \left[(w-1) - \frac{1}{m}(w^m - 1) \right] V_{D_*}^m \, dx$$

Relative Fisher information

$$\mathcal{J}[w] := \frac{m}{(m-1)^2} \int_{\mathbb{R}^d} \left| \nabla \left[\left(w^{m-1} - 1 \right) V_{D_*}^{m-1} \right] \right|^2 w \, V_{D_*} \, dx$$

Proposition 1 *Under assumptions (H1)-(H2),*

$$\frac{d}{dt} \mathcal{F}[w(t)] = - \mathcal{J}[w(t)]$$

Proposition 2 *Under assumptions (H1)-(H2), there exists a constant* $\lambda > 0$ *such that*

$$\mathcal{F}[w(t)] \leq \lambda^{-1} \, \mathcal{J}[w(t)]$$

Heuristics: linearization

Take $w(t,x) = 1 + \varepsilon\,\frac{g(t,x)}{V_{D_*}^{m-1}(x)}$ and formally consider the limit $\varepsilon \to 0$ in

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}}\,\nabla\cdot\left[w\,V_{D_*}\,\nabla\left(\dfrac{m}{m-1}\,(w^{m-1}-1)\,V_{D_*}^{m-1}\right)\right] & \text{in } (0,+\infty)\times\mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

Then g solves

$$g_t = m\,V_{D_*}^{m-2}(x)\,\nabla\cdot\left[V_{D_*}(x)\,\nabla g(t,x)\right]$$

and the entropy and Fisher information functionals

$$\mathsf{F}[g] := \frac{1}{2}\int_{\mathbb{R}^d} |g|^2\,V_{D_*}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\int_{\mathbb{R}^d} |\nabla g|^2\,V_{D_*}\,dx$$

consistently verify $\dfrac{d}{dt}\,\mathsf{F}[g(t)] = -\,\mathsf{I}[g(t)]$

Comparison of the functionals

Lemma 3 *Let $m \in (0,1)$ and assume that u_0 satisfies* (H1)-(H2)
[Relative entropy]

$$C_1 \int_{\mathbb{R}^d} |w-1|^2\, V_{D_*}^m\, dx \le \mathcal{F}[w] \le C_2 \int_{\mathbb{R}^d} |w-1|^2\, V_{D_*}^m\, dx$$

[Fisher information]

$$I[g] \le \beta_1\, \mathcal{J}[w] + \beta_2\;\; F[g] \quad \textit{with} \quad g := (w-1)\, V_{D_*}^{m-1}$$

Theorem 4 (Hardy-Poincaré) *There exists a positive constant $\lambda_{m,d}$ such that for any $m \neq m_* = (d-4)/(d-2)$, $m \in (0,1)$, for any $g \in \mathcal{D}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |g-\overline{g}|^2\; V_{D_*}^{2-m}\, dx \le \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2\, V_{D_*}\, dx$$

with $\overline{g} = \int_{\mathbb{R}^d} g\, V_{D_*}^{2-m}\, dx$ if $m > m_*$, $\overline{g} = 0$ otherwise

Hardy-Poincaré inequalities

With $\alpha = \frac{1}{m-1}$, $\alpha_* = \frac{1}{m_*-1} = 1 - \frac{d}{2}$

Theorem 5 *Assume that* $d \geq 3$, $\alpha \in \mathbb{R} \setminus \{\alpha^*\}$, $d\mu_\alpha(x) := h_\alpha(x)\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$. *Then*

$$\int_{\mathbb{R}^d} \frac{|v|^2}{1+|x|^2}\,d\mu_\alpha \leq \mathcal{C}_{\alpha,d} \int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha$$

holds for some positive constant $\mathcal{C}_{\alpha,d}$*, for any* $v \in \mathcal{D}(\mathbb{R}^d)$*, under the additional condition* $\int_{\mathbb{R}^d} v\,d\mu_{\alpha-1} = 0$ *if* $\alpha \in (-\infty, \alpha^*)$

. . . Hardy-Poincaré inequalities $\equiv$ weighted Poincaré inequalities corresponding to generalized Cauchy distributions (fat tails)... [Bobkov, Ledoux] [Cattiaux, Gozlan, Guillin, Roberto]

Limit cases

Poincaré inequality: take $\alpha = -1/\varepsilon^2$ to $v_\varepsilon(x) := \varepsilon^{-d/2}\, v(x/\varepsilon)$ and let $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} |v|^2\, d\nu_\infty \le \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v|^2\, d\nu_\infty \quad \text{with} \quad d\nu_\infty(x) := e^{-|x|^2} dx$$

... under the additional condition $\int_{\mathbb{R}^d} v\; e^{-|x|^2} dx = 0$

Hardy's inequality: take $v_{1/\varepsilon}(x) := \varepsilon^{d/2}\, v(\varepsilon\, x)$ and let $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2}\, d\nu_{0,\alpha} \le \frac{1}{(\alpha - \alpha_*)^2} \int_{\mathbb{R}^d} |\nabla v|^2\, d\nu_{0,\alpha} \quad \text{with} \quad d\nu_{0,\alpha}(x) := |x|^{2\alpha}\, dx$$

... under the additional condition $\bar{v}_\alpha := \int_{\mathbb{R}^d} v\, d\nu_{0,\alpha} = 0$ if $\alpha < \alpha^*$

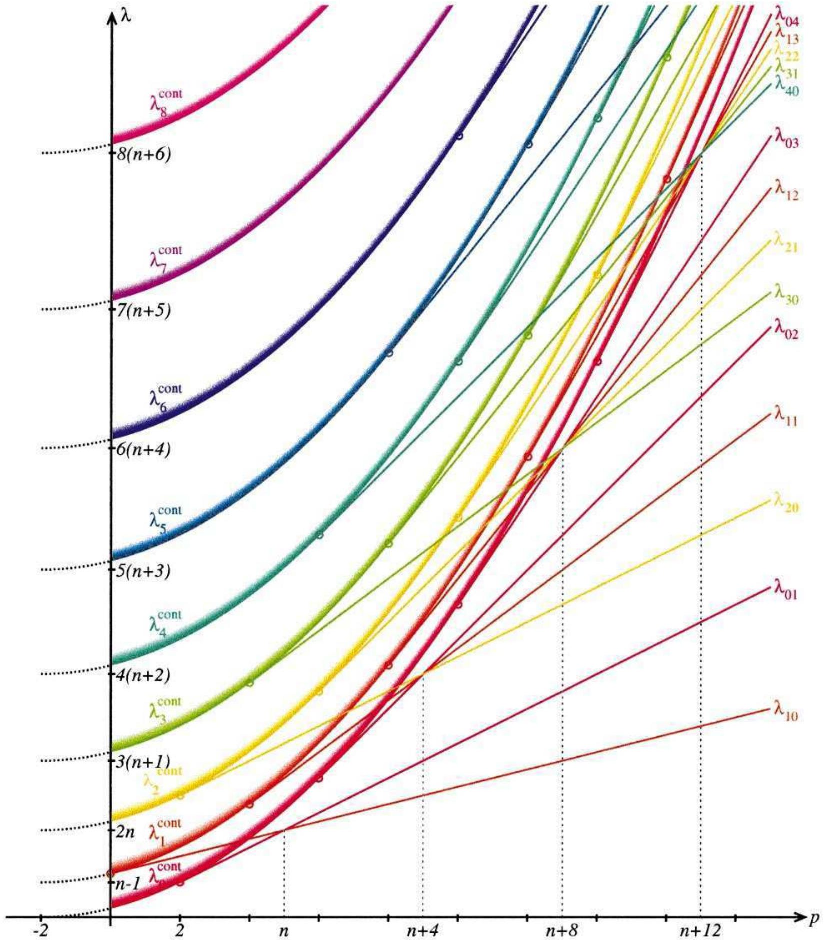
Some estimates of $\mathcal{C}_{\alpha,d}$

α	$-\infty < \alpha \le -d$	$-d < \alpha < \alpha^*$	$\alpha^* < \alpha \le 1$
$\mathcal{C}_{\alpha,d}$	$\frac{1}{2\,\lvert\alpha\rvert}$	$\mathcal{C}_{\alpha,d} \ge \frac{4}{(d+2\alpha-2)^2}$	$\frac{4}{(d+2\alpha-2)^2}$
Optimality	-	-	yes

α	$1 \le \alpha \le \bar{\alpha}(d)$	$\bar{\alpha}(d) \le \alpha \le d$	d	$\alpha > d$
$\mathcal{C}_{\alpha,d}$	$\frac{4}{d(d+2\alpha-2)}$	$\frac{1}{\alpha(d+\alpha-2)}$	$\frac{1}{2d(d-1)}$	$\frac{1}{d(d+\alpha-2)}$
Optimality	-	-	yes	-

$\alpha_* = -\frac{d-2}{2}$, $\bar{\alpha}(d) \in (1,d)$

$-(1+|x|^2)^{1-\alpha}\nabla\cdot((1+|x|^2)^\alpha\nabla)$ in $L^2((1+|x|^2)^{\alpha-1}dx)$



Taken from [J. Denzler & R. J. McCann, PNAS 100 (2003)], $p = \frac{2}{2-m} - d$

Hardy's inequality: the "completing the square method"

Let $v \in \mathcal{D}(\mathbb{R}^d)$ with $\mathrm{supp}(v) \subset \mathbb{R}^d \setminus \{0\}$ if $\alpha < \alpha^*$

$$\begin{aligned} 0 &\leq \int_{\mathbb{R}^d} \left| \nabla v + \lambda \, \frac{x}{|x|^2} \, v \right|^2 |x|^{2\alpha} \, dx \\ &= \int_{\mathbb{R}^d} |\nabla v|^2 \, |x|^{2\alpha} \, dx + \left[\lambda^2 - \lambda \, (d + 2\alpha - 2) \right] \int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, |x|^{2\alpha} \, dx \end{aligned}$$

An optimization of the right hand side with respect to λ gives $\lambda = \alpha - \alpha^*$, that is $(d + 2\alpha - 2)^2/4 = \lambda^2$. Such an inequality is optimal, with optimal constant λ^2, as follows by considering the test functions:

1) if $\alpha > \alpha^*$: $v_\varepsilon(x) = \min\{\varepsilon^{-\lambda}, (|x|^{-\lambda} - \varepsilon^{\lambda})_+\}$

2) if $\alpha < \alpha^*$: $v_\varepsilon(x) = |x|^{1-\alpha-d/2+\varepsilon}$ for $|x| < 1$
$v_\varepsilon(x) = (2 - |x|)_+$ for $|x| \geq 1$

and letting $\varepsilon \to 0$ in both cases

An optimality case

Proposition 4 *Let $d \geq 3$, $\alpha \in (\alpha^*, \infty)$. Then the Hardy-Poincaré inequality holds for any $v \in \mathcal{D}(\mathbb{R}^d)$ with $\mathcal{C}_{\alpha,d} := 4/(d-2+2\,\alpha)^2$ if $\alpha \in (\alpha^*, 1]$ and $\mathcal{C}_{\alpha,d} := 4/[d\,(d-2+2\,\alpha)]$ if $\alpha \geq 1$. The constant $\mathcal{C}_{\alpha,d}$ is optimal for any $\alpha \in (\alpha^*, 1]$.*

Proof [Davies]: $h_\alpha = (1+|x|^2)^\alpha$, $\nabla h_\alpha = 2\alpha\, x\, h_{\alpha-1}$,
$\Delta h_\alpha = 2\alpha\, h_{\alpha-2}[d + 2(\alpha-\alpha^*)\,|x|^2] > 0$
By Cauchy-Schwarz

$$\begin{aligned}\left|\int_{\mathbb{R}^d} |v|^2\, \Delta h_\alpha\; dx\right|^2 &\leq 4\left(\int_{\mathbb{R}^d} |v|\,|\nabla v|\,|\nabla h_\alpha|\; dx\right)^2\\ &\leq 4\int_{\mathbb{R}^d} |v|^2\,|\Delta h_\alpha|\; dx \int_{\mathbb{R}^d} |\nabla v|^2\,|\nabla h_\alpha|^2\,|\Delta h_\alpha|^{-1}\; dx\end{aligned}$$

$|\Delta h_\alpha| \geq 2\,|\alpha|\;\min\{d, (d-2+2\,\alpha)\}\,\frac{h_\alpha(x)}{1+|x|^2}$
$\frac{|\nabla h_\alpha|^2}{|\Delta h_\alpha|} \leq \frac{2\,|\alpha|}{d-2+2\,\alpha}\,h_\alpha(x)$