Entropy methods and applications to diffusions

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Outline

1. A brief introduction to entropy methods
   - Entropies for the Fokker-Planck equation on a domain in $\mathbb{R}^d$
   - A proof of the interpolation inequalities on $\mathbb{S}^d$ by the carré du champ method
2. Stability, fast diffusion equation and entropy methods
   - Entropy methods and fast diffusion in $\mathbb{R}^d$
   - The threshold time and the improved entropy – entropy production inequality (subcritical case)
   - Stability results (subcritical and critical case)
3. Symmetry and symmetry breaking
   - Caffarelli-Kohn-Nirenberg inequalities
   - Sharp symmetry versus symmetry breaking results
   - Scheme of the proof

A brief introduction to entropy methods

▷ The Bakry-Emery method: Fokker-Planck equation on $\mathbb{R}^d$

▷ Entropies and flows on the sphere $\mathbb{S}^d$: bifurcations, rigidity, inequalities

Three points of view

- decay rates in diffusion equations
- entropy - entropy production inequalities and functional inequalities
- rigidity problems in elliptic equations, bifurcation problems

The Fokker-Planck equation (domain in $\mathbb{R}^d$)

The linear Fokker-Planck (FP) equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \phi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \phi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \phi \cdot \nabla v =: \mathscr{L} v$$

[Bakry, Emery, 1985], [Arnold, Markowich, Toscani, Unterreiter, 2001]
With mass normalized to 1, the unique stationary solution of (FP) is

$$u_s = \frac{e^{-\phi}}{\int_\Omega e^{-\phi}\,dx} \quad \Longleftrightarrow \quad v_s = 1$$

The Bakry-Emery method (domain in $\mathbb{R}^d$)

With $d\gamma = u_s\,dx$ and v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$, the *q-entropy* is defined by

$$\mathcal{E}_q[v] := \frac{1}{q-1}\int_\Omega \big(v^q - 1 - q\,(v-1)\big)\,d\gamma$$

Under the action of (OU), with $w = v^{q/2}$, $\mathcal{I}_q[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$,

$$\frac{d}{dt}\mathcal{E}_q[v(t,\cdot)] = -\,\mathcal{I}_q[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\Big(\mathcal{I}_q[v] - 2\,\lambda\,\mathcal{E}_q[v]\Big) \le 0$$

$$\text{with} \quad \lambda := \inf_{w\in H^1(\Omega,d\gamma)\setminus\{0\}} \frac{\int_\Omega \big(2\,\frac{q-1}{q}\,\|\mathrm{Hess}\,w\|^2 + \mathrm{Hess}\,\phi : \nabla w\otimes\nabla w\big)\,d\gamma}{\int_\Omega |\nabla w|^2\,d\gamma}$$

Proposition

[Bakry, Emery, 1984] [JD, Nazaret, Savaré, 2008] *Let Ω be convex. If $\lambda > 0$ and v is a solution of (OU), then $\mathcal{I}_q[v(t,\cdot)] \le \mathcal{I}_q[v(0,\cdot)]\,e^{-2\lambda t}$ and $\mathcal{E}_q[v(t,\cdot)] \le \mathcal{E}_q[v(0,\cdot)]\,e^{-2\lambda t}$ for any $t \ge 0$ and, as a consequence,*

$$\mathcal{I}_q[v] \ge 2\,\lambda\,\mathcal{E}_q[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma) \quad \text{(Entropy-entropy production ineq.)}$$

A bifurcation problem on the sphere $\mathbb{S}^d$

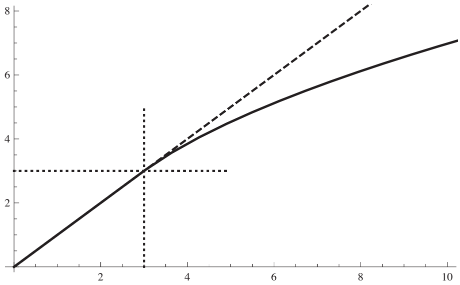


Figure: $(p-2)\,\lambda \mapsto (p-2)\,\mu(\lambda)$ *with* $d=3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p-2}$$

▷ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry, Emery, 1985] [Beckner, 1993], [Bidaut-Véron, Véron, 1991, Corollary 6.1]

Interpolation inequalities and a rigidity result on $\mathbb{S}^d$

$p \in [1,2) \cup (2,2^*]$ if $d \ge 3$, $2^* = \frac{2d}{d-2}$; $p \in [1,2) \cup (2,+\infty)$ if $d = 1, 2$

▷ *Optimal Gagliardo-Nirenberg-Sobolev interpolation inequalities*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

▷ *A result of uniqueness on a classical example*

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

Theorem

If $\lambda \le d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas, Spruck, 1981], [Bidaut-Véron, Véron, 1991]

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

[Bakry, Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$,

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^{\#} := \frac{2\,d^2+1}{(d-1)^2}$

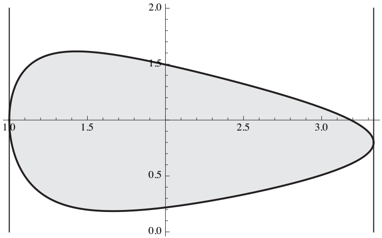
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\frac{d}{dt}\Big(\mathscr{I}_p[\rho] - d\,\mathscr{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region,* $d = 5$

Computation of the admissible region

With $\rho = |u|^{\beta p}$ and $m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$, $\kappa = \beta\,(p-2) + 1$, with the *trace free Hessian*

$$\mathrm{L}u := \mathrm{H}u - \frac{1}{d-1}\,(\Delta u)\,g_d$$

and the trace free tensor

$$\mathrm{M}u := \frac{\nabla u \otimes \nabla u}{u} - \frac{1}{d-1}\,\frac{|\nabla u|^2}{u}\,g_d$$

we have

$$\frac{d}{dt}\Big(\mathscr{I}_p[\rho] - d\,\mathscr{E}_p[\rho]\Big) = -\frac{d}{d-1}\left(\mathsf{a}\,\|\mathrm{L}u\|^2 - 2\,\mathsf{b}\,\mathrm{L}u : \mathrm{M}u + \mathsf{c}\,\|\mathrm{M}u\|^2\right)$$

$$\mathsf{a} = 1\,,\quad \mathsf{b} = (\kappa + \beta - 1)\,\frac{d-1}{d+2}\,,\quad \mathsf{c} = (\kappa + \beta - 1)\,\frac{d}{d+2} + \kappa\,(\beta - 1)$$

so that the *admissible region* is defined by $\mathsf{b}^2 - \mathsf{a}\,\mathsf{c} < 0$

Improved inequalities

▷ the monotonicity result

$$\frac{d}{dt}\Big(\mathscr{I}_p[\rho]-d\,\mathscr{E}_p[\rho]\Big)=-\frac{d}{d-1}\,\mathsf{a}\,\Big\|\mathrm{L}u-\frac{\mathsf{b}}{\mathsf{a}}\,\mathrm{M}\Big\|^2-\frac{d}{d-1}\left(\mathsf{c}-\frac{\mathsf{b}^2}{\mathsf{a}}\right)\|\mathrm{M}u\|^2$$

▷ improved inequalities [Arnold, JD, 2005], [JD, Nazaret, Savaré, 2008], [JD, Toscani, 2013], [JD, Esteban, Kowalczyk, Loss, 2014], [JD, Esteban, 2020]

$$\mathscr{I}_p[\rho]\ge d\,\Phi\Big(\mathscr{E}_p[\rho]\Big)$$

for some convex Φ with $\Phi(0)=0$ and $\Phi'(0)=1$

Application: with $d\ge2$, $2-p\neq\gamma:=\left(\frac{d-1}{d+2}\right)^2(p-1)\,(2^\#-p)>0$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge\frac{d}{2-p-\gamma}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\|u\|^{2-\frac{2\gamma}{2-p}}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{\frac{2\gamma}{2-p}}_{\mathrm{L}^2(\mathbb{S}^d)}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

Stability, a joint project with M. Bonforte, B. Nazaret and N. Simonov

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

The fast diffusion equation in original variables

Consider the *fast diffusion* equation in $\mathbb{R}^d$, $d \ge 1$, $m \in (0,1)$

$$\frac{\partial u}{\partial t} = \Delta u^m$$

with initial datum $u(t=0,x) = u_0(x) \ge 0$ such that

$$\int_{\mathbb{R}^d} u_0 \, dx = \mathscr{M} > 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2 \, u_0 \, dx < +\infty$$

The large time behavior is governed by the self-similar Barenblatt solutions

$$B(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathscr{B}\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where $\mu := 2 + d\,(m-1)$ and $\mathscr{B}$ is the Barenblatt profile with $\int_{\mathbb{R}^d} \mathscr{B} \, dx = \mathscr{M}$

$$\mathscr{B}(x) := \left(1 + |x|^2\right)^{-\frac{1}{1-m}}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The variation of the Fisher information

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $1 - \frac{1}{d} \le m < 1$, then

$$\mathrm{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v\,|\nabla \mathrm{p}|^2\,dx = -2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathrm{p}\|^2 + (m-1)\,(\Delta \mathrm{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathrm{p}\|^2 - \frac{1}{d}\,(\Delta \mathrm{p})^2 = \left\| \mathrm{D}^2 \mathrm{p} - \frac{1}{d}\,\Delta \mathrm{p}\,\mathrm{Id} \right\|^2$$

.... there are no boundary terms in the integrations by parts ?

The concavity property

Theorem

[Toscani, Savaré] *Assume that* $m \geq 1 - \frac{1}{d}$ *if* $d > 1$ *and* $m > 0$ *if* $d = 1$. *Then* $F(t)$ *is increasing,* $(1-m)\,\mathsf{F}''(t) \leq 0$ *and*

$$\lim_{t \to +\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t \to +\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[JD, Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_2^\theta\,\|w\|_{q+1}^{1-\theta} \geq \mathsf{C}_{\rm GN}\,\|w\|_{2q}$$

if $1 - \frac{1}{d} \leq m < 1$. Correspondance: $v^{m-1/2} = \frac{w}{\|w\|_{2q}}$, $q = \frac{1}{2\,m-1}$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x) = \frac{1}{\kappa^d R^d}\, v\left(\tau, \frac{x}{\kappa R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}, \quad \tau(t) := \tfrac{1}{2}\log R(t)$$

$\frac{\partial u}{\partial t} = \Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left[v \left(\nabla v^{m-1} - 2\,x \right) \right] = 0 \qquad (r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathscr{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathscr{B}^m - m\,\mathscr{B}^{m-1}\,(v - \mathscr{B}) \right) dx$$

$$\mathscr{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 dx$$

are such that $\mathscr{I}[v] \geq 4\,\mathscr{F}[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathscr{F}[v(t,\cdot)] \leq \mathscr{F}[v_0]\, e^{-4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,,\quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

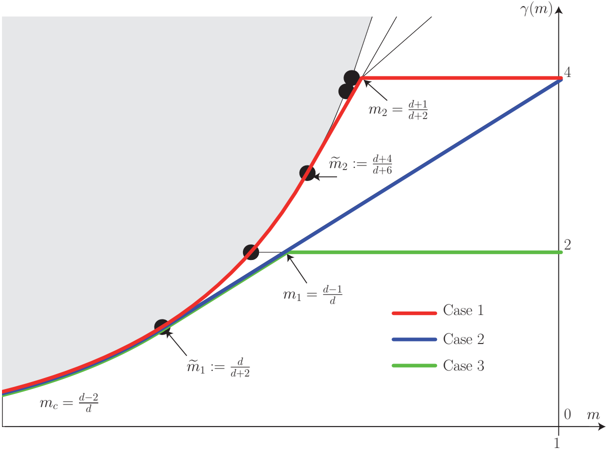
Lemma

Under assumption (H),

$$\mathscr{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t\ge 0\,,\quad \gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m):=2$ *if* $1-1/d\le m<1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,dx$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ *if* $d \ge 2$, $m \in (1/3, 1)$ *if* $d = 1$, $\eta = 2\,(d\,m - d + 1)$ *and* $\chi = m/(266 + 56\,m)$. *If* $\int_{\mathbb{R}^d} v\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ *and*

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, *then*

$$\mathscr{I}[v] \ge (4+\eta)\,\mathscr{F}[v]$$

The initial time layer improvement: backward estimate

Hint: for some strictly convex function ψ with $\psi(0) = \psi'(0) = 0$, we have

$$\mathcal{I} - 4\mathcal{F} \geq 4\left(\psi(\mathcal{F}) - \mathcal{F}\right) \geq 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

Rephrasing the *carré du champ* method, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}\left(\mathcal{Q} - 4\right)$$

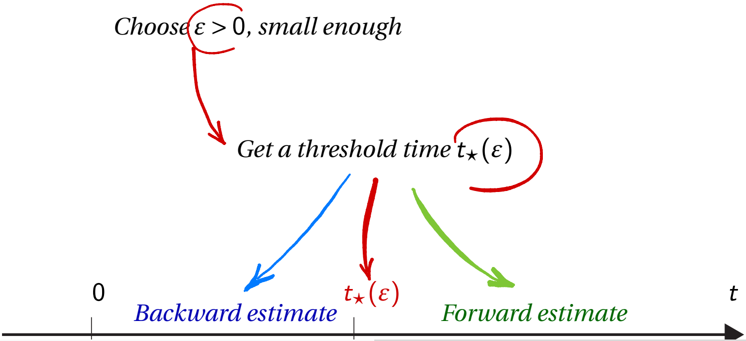
Lemma

Assume that $m > m_1$ and v is a solution to (r FDE) with nonnegative initial datum v_0. If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star, \cdot)] \geq 4 + \eta$, then

$$\mathcal{Q}[v(t,\cdot)] \geq 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0, t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (r FDE)

(and in particular for the initial datum)

If v is a solves (r FDE) for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} v_0\,dx \le A < \infty \qquad (\mathrm{H}_A)$$

then

$$(1-\varepsilon)\mathscr{B} \le v(t,\cdot) \le (1+\varepsilon)\mathscr{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and A

Global Harnack Principle

The *Global Harnack Principle* holds if for some $t>0$ large enough

$$\mathscr{B}_{M_1}(t-\tau_1,x) \le u(t,x) \le \mathscr{B}_{M_2}(t+\tau_2,x) \qquad \text{(GHP)}$$

[Vázquez, 2003], [Bonforte, Vázquez, 2006]: (GHP) holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$
[Vázquez, 2003], [Bonforte, Simonov, 2020]: (GHP) holds if

$$\mathsf{A}[u_0] := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d \setminus B_R(0)} |u_0|\,dx < \infty$$

Theorem

[Bonforte, Simonov, 2020] If $M + \mathsf{A}[u_0] < \infty$, then

$$\lim_{t\to\infty} \left\| \frac{u(t)-B(t)}{B(t)} \right\|_\infty = 0$$

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $T \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d(m-m_c)}{(1-m)}} \int_{|x|>r} u_0 \, dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0 \, dx = \int_{\mathbb{R}^d} B \, dx = \mathcal{M}$ *and* $\mathcal{F}[u_0] \leq G$, *then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall \, t \geq T$$

The threshold time

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ and $G > 0$

$$T = \mathsf{c}_\star \, \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a} = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$, $\alpha = d\,(m - m_c)$ *and* $\vartheta = \nu/(d+\nu)$

$$\mathsf{c}_\star = \mathsf{c}_\star(m,d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max\left\{\varepsilon\,\kappa_1(\varepsilon,m), \varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m), \varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1}, \frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon,m) := \frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathcal{I}[v] \geq (4+\zeta)\,\mathcal{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v] = G$, $\int_{\mathbb{R}^d} v\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0, 2\varepsilon_\star)\,, \quad \varepsilon_\star := \frac{1}{2}\min\left\{\varepsilon_{m,d}, \chi\,\eta\right\} \quad \text{with} \quad t_\star = t_\star(\varepsilon) = \frac{1}{2}\log R(T)$$

$$(1-\varepsilon)\,\mathcal{B} \leq v(t,\cdot) \leq (1+\varepsilon)\,\mathcal{B} \quad \forall\, t \geq t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathcal{I}[v(t,.)] \geq (4+\zeta)\,\mathcal{F}[v(t,.)] \quad \forall\, t \in [0, t_\star] \quad \text{where} \quad \zeta = \frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}$$

Two consequences

$$\zeta = \mathsf{Z}\big(A, \mathscr{F}[u_0]\big)\,, \quad \mathsf{Z}(A,G) := \frac{\zeta_\star}{1 + A^{(1-m)\frac{2}{\alpha}} + G}\,, \quad \zeta_\star := \frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^{a}}{2\,\alpha\,c_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (r FDE) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0\,dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\,v_0\,dx = 0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)] \le \mathscr{F}[v_0]\,e^{-(4+\zeta)\,t} \quad \forall\, t \ge 0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v] - 4\,\mathscr{F}[v] \ge \frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \ge 3$$

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}\,,\quad \kappa[f] := \frac{\mathcal{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathsf{A}[f] := \frac{\mathcal{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,f^{p+1}-\mathsf{g}^{p+1}-\frac{1+p}{2p}\,\mathsf{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,f^{2p}-\mathsf{g}^{2p}\right)\right)dx$$

$$\mathfrak{S}[f] := \frac{\mathcal{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}(\mathsf{A}[f],\mathsf{E}[f])$$

Theorem

Let $d\ge1$, $p\in(1,p^*)$

If $f\in\mathcal{W}_p(\mathbb{R}^d) := \left\{f\in\mathrm{L}^{2p}(\mathbb{R}^d)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d),\ |x|\,f^p\in\mathrm{L}^2(\mathbb{R}^d)\right\}$,

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma}-\left(\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\right)^{2p\gamma}\ge\mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

With $\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Theorem

Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathscr{C} = \mathscr{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}(\mathbb{R}^d,(1+|x|^2)\,dx)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{A}[f^{2p}] < \infty$,

$$\delta[f] \ge \mathscr{C}[f] \inf_{\varphi \in \mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\nabla f + f^p \nabla \varphi^{1-p}\right|^2 dx$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathrm{A}[f^{2p}]$ and $\mathscr{F}[f^{2p}]$ is explicit and does not degenerate if $f \in \mathfrak{M}$

▷ Can we remove the condition $\mathrm{A}[f^{2p}] < \infty$?

Stability in Sobolev's inequality (critical case)

- ▷ A constructive stability result
- ▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathscr{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\,f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Theorem

Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathscr{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \le A$$

we have

$$\delta[f] := \|\nabla f\|_2^2 - \mathsf{S}_d^2\, \|f\|_{2^*}^2 \ge \frac{\mathscr{C}_\star(A)}{4 + \mathscr{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathscr{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2d)}\right)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on d*

Peculiarities of the critical case

▷ We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\right)$$

the *Bianchi-Egnell type result*

$$\delta[f] \geq \frac{\mathfrak{C}_\star\,Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathscr{J}[f|g]$$

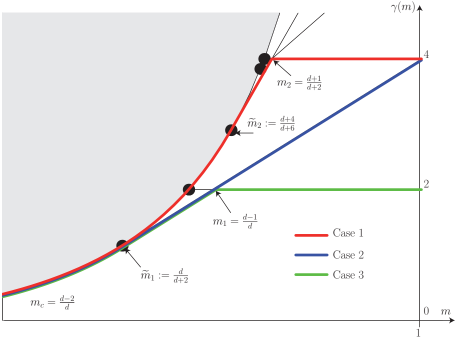
with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

▷ Notion of time delay [JD, Toscani, 2014 & 2015]

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x) = \lambda^{-d/2}\,\mathscr{B}\big(x/\sqrt{\lambda}\big)$ in order to match $\int_{\mathbb{R}^d} |x|^2\,v\,dx$ where the function v solves (r FDE) or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathscr{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0)=0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

Symmetry and symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

Joint work with M.J. Esteban and M. Loss

Caffarelli-Kohn-Nirenberg inequalities

Let $\mathscr{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) : |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathscr{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

▷ *An optimal function among radial functions:*

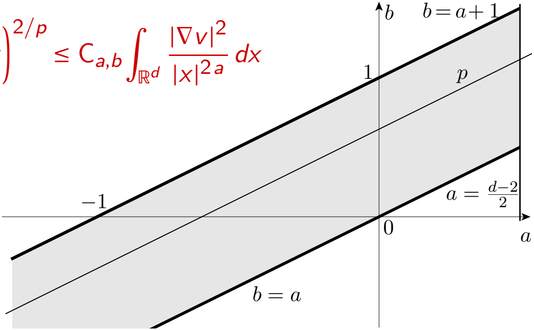
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b}\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$
$a < b \le a+1$ if $d=2$, $a+1/2 < b \le a+1$ if $d=1$
and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

[Glaser, Martin, Grosse, Thirring (1976)]
[F. Catrina, Z.-Q. Wang (2001)]

▷ Proving symmetry breaking
[F. Catrina, Z.-Q. Wang], [V. Felli, M. Schneider (2003)]
[J.D., Esteban, Loss, Tarantello, 2009] There is a curve...

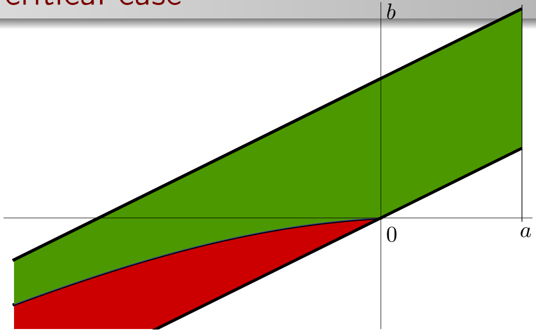
▷ Moving planes and symmetrization techniques
[Chou, Chu], [Horiuchi]
[Betta, Brock, Mercaldo, Posteraro]
+ Perturbation results: [CS Lin, ZQ Wang], [Smets, Willem], [JD, Esteban, Tarantello 2007], [J.D., Esteban, Loss, Tarantello, 2009]

▷ Linear instability of radial minimizers: the Felli-Schneider curve
[Catrina, Wang], [Felli, Schneider]

▷ Direct spectral estimates
[J.D., Esteban, Loss, 2011]: sharp interpolation on the sphere and a Keller-Lieb-Thirring spectral estimate on the line

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss, 2016]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The symmetry proof in one slide

- A change of variables: $v(|x|^{\alpha-1}x) = w(x)$, $\mathrm{D}_\alpha v = \left(\alpha \frac{\partial v}{\partial s}, \frac{1}{s} \nabla_\omega v\right)$

$$\|v\|_{2p,d-n} \le \mathsf{K}_{\alpha,n,p} \, \|\mathrm{D}_\alpha v\|_{2,d-n}^{\vartheta} \, \|v\|_{p+1,d-n}^{1-\vartheta} \quad \forall\, v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

The Felli & Schneider condition becomes $\alpha > \alpha_{\mathrm{FS}} := \sqrt{\frac{d-1}{n-1}}$ and $p = \frac{2n}{n-2}$

- Concavity of the Rényi entropy power: with $\mathscr{L}_\alpha = -\mathrm{D}_\alpha^* \, \mathrm{D}_\alpha = \alpha^2 \left(u'' + \frac{n-1}{s} \, u'\right) + \frac{1}{s^2} \, \Delta_\omega \, u$ and $\frac{\partial u}{\partial t} = \mathscr{L}_\alpha u^m$

$$-\frac{d}{dt} \mathscr{G}[u(t,\cdot)] \left(\int_{\mathbb{R}^d} u^m \, |x|^{n-d} \, dx\right)^{1-\sigma}$$
$$\ge +2 \int_{\mathbb{R}^d} \left(\alpha^4 \left(1 - \frac{1}{n}\right) \left|\mathrm{P}'' - \frac{\mathrm{P}'}{s} - \frac{\Delta_\omega \, \mathrm{P}}{\alpha^2 \, (n-1) \, s^2}\right|^2 + \frac{2\,\alpha^2}{s^2} \left|\nabla_\omega \mathrm{P}' - \frac{\nabla_\omega \mathrm{P}}{s}\right|^2\right) u^m \, |x|^{n-d} \, dx$$
$$+2 \int_{\mathbb{R}^d} \left((n-2) \left(\alpha_{\mathrm{FS}}^2 - \alpha^2\right) |\nabla_\omega \mathrm{P}|^2 + c(n,m,d) \, \frac{|\nabla_\omega \mathrm{P}|^4}{\mathrm{P}^2}\right) u^m \, |x|^{n-d} \, dx$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

The variational problem on the cylinder

▷ *With the Emden-Fowler transformation*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad\omega=\frac{x}{r}$$

the variational problem becomes

$$\Lambda\mapsto\mu(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathscr{C})}\frac{\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathscr{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathscr{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathscr{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathscr{C})}}$$

is a concave increasing function

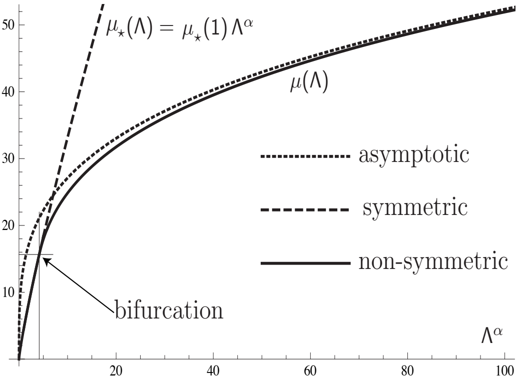
Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathbb{R})}\frac{\|\partial_s\varphi\|^2_2+\Lambda\,\|\varphi\|^2_2}{\|\varphi\|^2_p}=\mu_\star(1)\,\Lambda^\alpha$$

Symmetry means $\mu(\Lambda)=\mu_\star(\Lambda)$

Symmetry breaking means $\mu(\Lambda)<\mu_\star(\Lambda)$

Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as shown by F. Catrina and Z.-Q. Wang*

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Thank you for your attention !