Symmetry breaking in Caffarelli-Kohn-Nirenberg inequalities

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IN COLLABORATION WITH

M. DEL PINO, M. ESTEBAN, S. FILIPPAS, M. LOSS, G. TARANTELLO, A. TERTIKAS

13/2:2012

Pavia

- Some slides related to this talk:

  http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/

- A review of known results:
  Jean Dolbeault and Maria J. Esteban
  About existence, symmetry and symmetry breaking for extremal functions of some interpolation functional inequalities

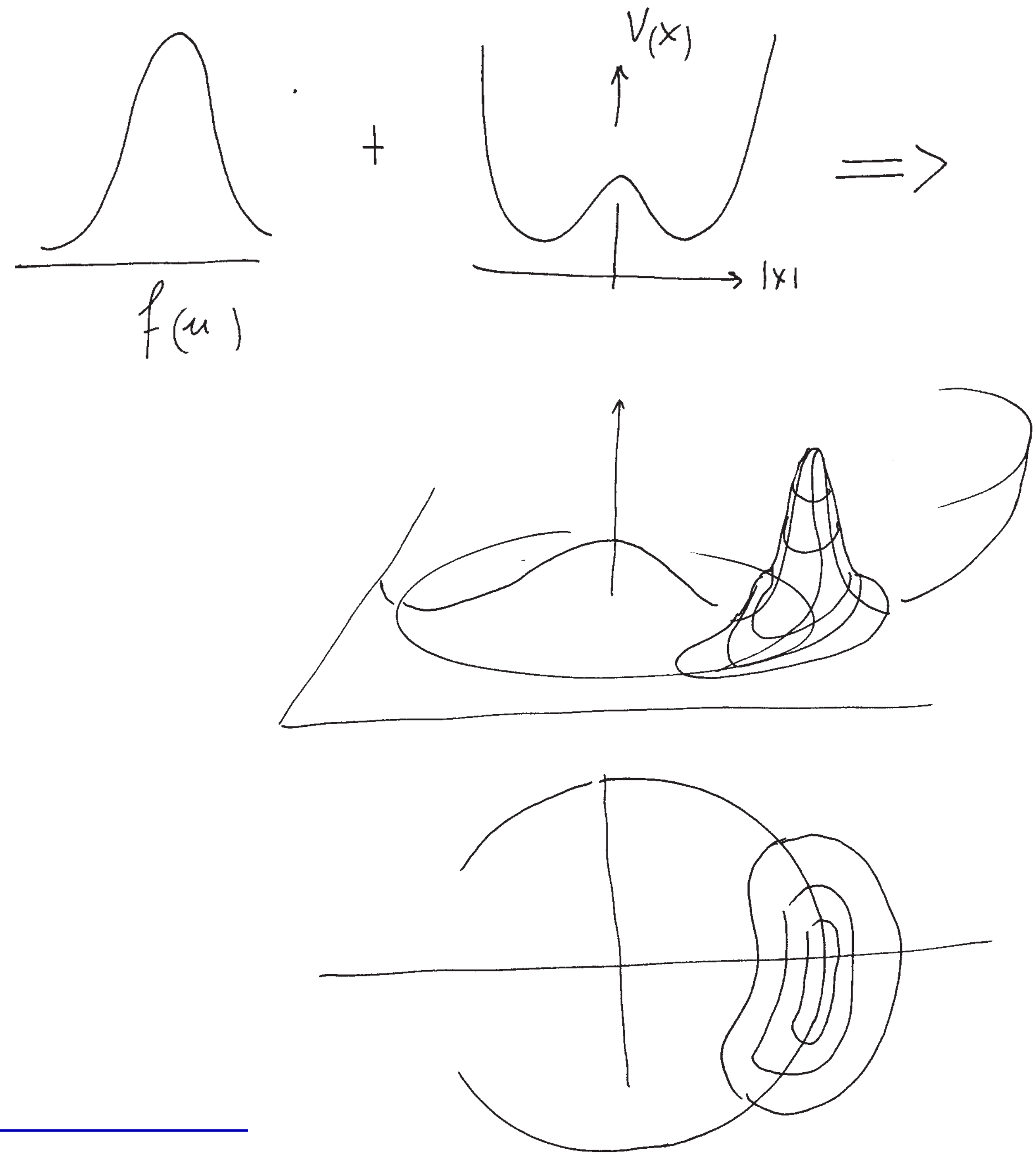
  http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/

- A preprint:
  Jean Dolbeault, Maria J. Esteban and Michael Loss
  Symmetry of extremals of functional inequalities via spectral estimates for linear operators

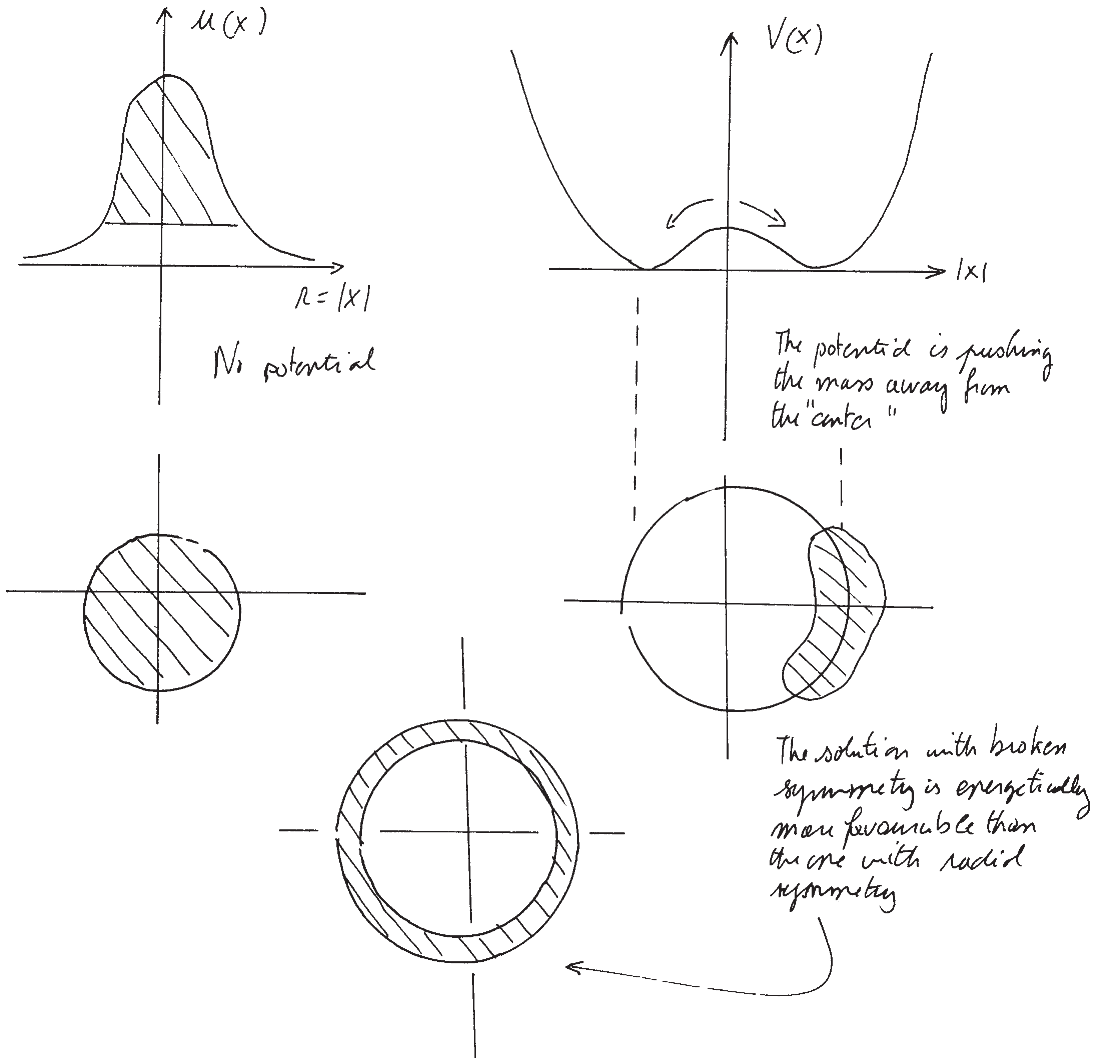
  http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/

Introduction

A symmetry breaking mechanism



The energy point of view (ground state)



Caffarelli-Kohn-Nirenberg inequalities (Part I)

Joint work(s) with M. Esteban, M. Loss and G. Tarantello

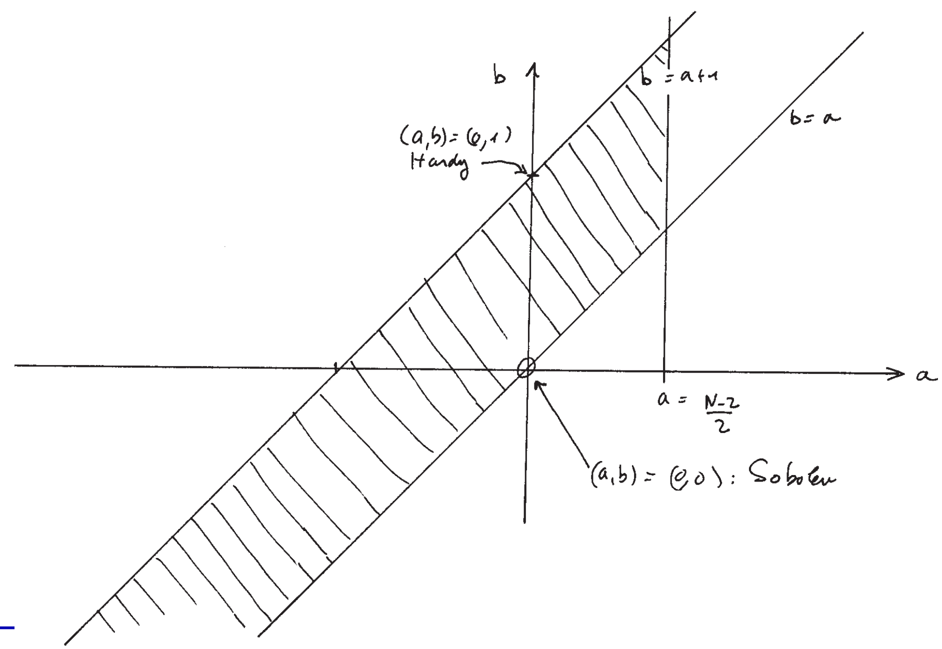
Caffarelli-Kohn-Nirenberg (CKN) inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\, u \in \mathcal{D}_{a,b}$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, and $a \ne \frac{d-2}{2} =: a_c$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$\mathcal{D}_{a,b} := \left\{\, |x|^{-b}\,u \in L^p(\mathbb{R}^d, dx) \;:\; |x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^d, dx) \right\}$$



The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\; u \in \mathcal{D}_{a,b}$$

$\mathsf{C}_{a,b}$ = best constant for general functions u
$\mathsf{C}^*_{a,b}$ = best constant for radially symmetric functions u

$$\mathsf{C}^*_{a,b} \le C_{a,b}$$

Up to scalar multiplication and dilation, the optimal radial function is

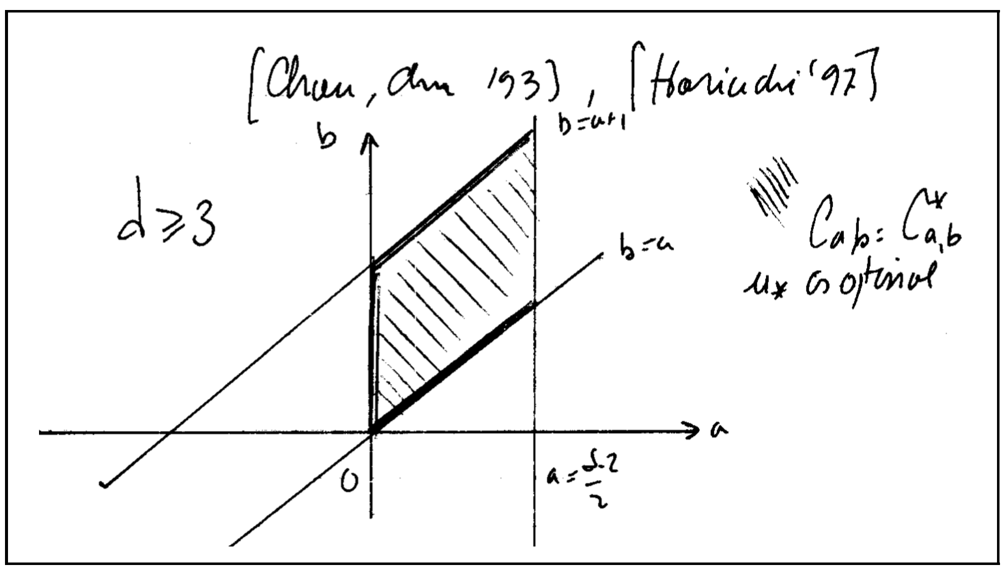
$$u^*_{a,b}(x) = |x|^{a+\frac{d}{2}\,\frac{b-a}{b-a+1}} \left(1+|x|^2\right)^{-\frac{d-2+2(b-a)}{2(1+a-b)}}$$

Questions: is optimality (equality) achieved ? do we have $u_{a,b} = u^*_{a,b}$?

Known results

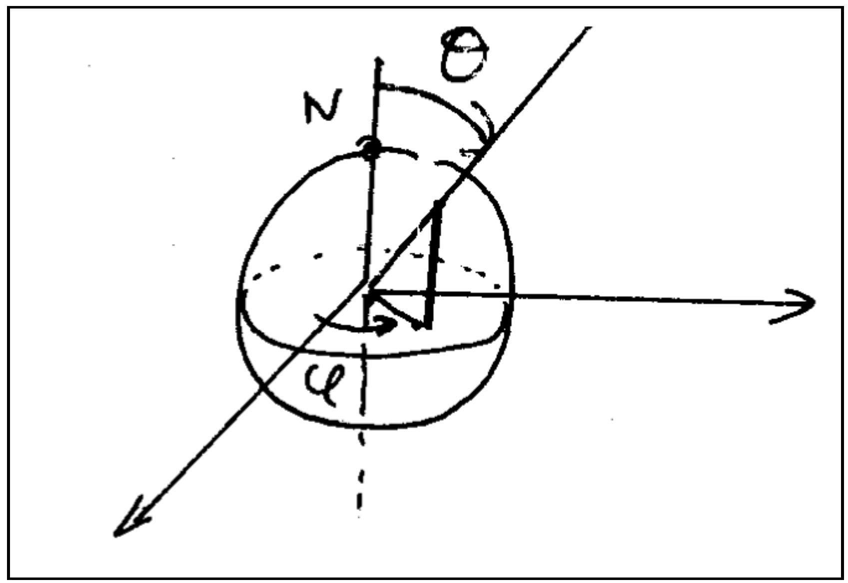
[Aubin, Talenti, Lieb, Chou-Chu, Lions, Catrina-Wang, ...]

- Extremals exist for $a < b < a+1$ and $0 \le a \le \frac{d-2}{2}$, for $a \le b < a+1$ and $a < 0$ if $d \ge 2$
- Optimal constants are never achieved in the following cases
  - “critical / Sobolev” case: for $b = a < 0$, $d \ge 3$
  - “Hardy” case: $b = a+1$, $d \ge 2$
- If $d \ge 3$, $0 \le a < \frac{d-2}{2}$ and $a \le b < a+1$, the extremal functions are radially symmetric ... $u(x) = |x|^a\, v(x)$ + Schwarz symmetrization



More results on symmetry

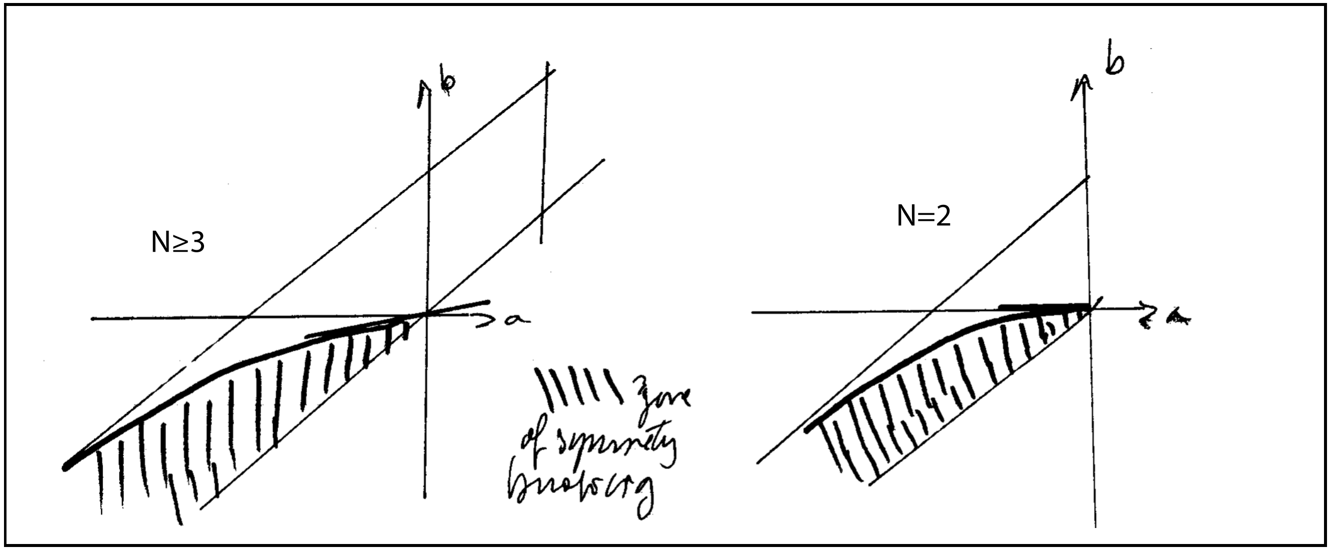
- Radial symmetry has also been established for $d \geq 3$, $a < 0$, $|a|$ small and $0 < b < a + 1$: [Lin-Wang, Smets-Willem]
- Schwarz foliated symmetry [Smets-Willem]



$d = 3$: optimality is achieved among solutions which depend only on the “latitude" θ and on r. Similar results hold in higher dimensions

Symmetry breaking

- [Catrina-Wang, Felli-Schneider] if $a < 0$, $a \le b < b^{FS}(a)$, the extremal functions ARE NOT radially symmetric !



$$b^{FS}(a) = \frac{d\,(d-2-2a)}{2\,\sqrt{(d-2-2a)^2+4(d-1)}} - \frac{1}{2}\,(d-2-2a)$$

- [Catrina-Wang] As $a \to -\infty$, optimal functions look like some decentered optimal functions for some Gagliardo-Nirenberg interpolation inequalities (after some appropriate transformation)

Approaching Onofri's inequality ($d = 2$)

- [J.D., M. Esteban, G. Tarantello] A generalized Onofri inequality

On $\mathbb{R}^2$, consider $d\mu_\alpha = \frac{\alpha+1}{\pi}\,\frac{|x|^{2\alpha}\,dx}{(1+|x|^{2\,(\alpha+1)})^2}$ with $\alpha > -1$

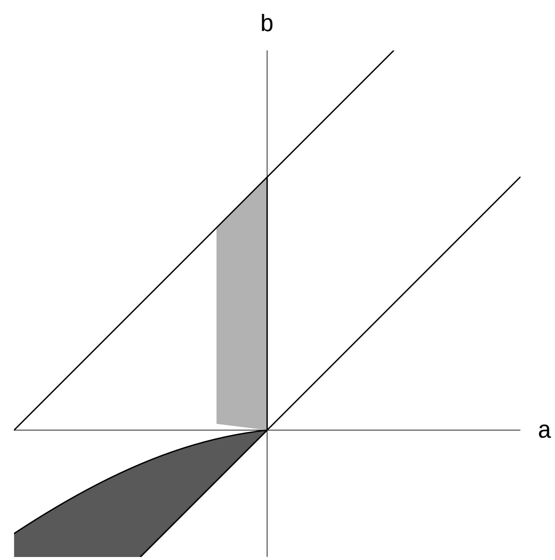
$$\log\left(\int_{\mathbb{R}^2} e^v\,d\mu_\alpha\right) - \int_{\mathbb{R}^2} v\,d\mu_\alpha \le \frac{1}{16\,\pi\,(\alpha+1)}\,\|\nabla v\|^2_{L^2(\mathbb{R}^2,\,dx)}$$

- For $d = 2$, radial symmetry holds if $-\eta < a < 0$ and $-\varepsilon(\eta)\,a \le b < a+1$

Theorem 1. [J.D.-Esteban-Tarantello] *For all* $\varepsilon > 0\,\exists\,\eta > 0$ *s.t. for* $a < 0$, $|a| < \eta$

(i) *if* $|a| > \frac{2}{p-\varepsilon}\,(1+|a|^2)$, *then*
$\mathsf{C}_{a,b} > \mathsf{C}^*_{a,b}$ *(symmetry breaking)*

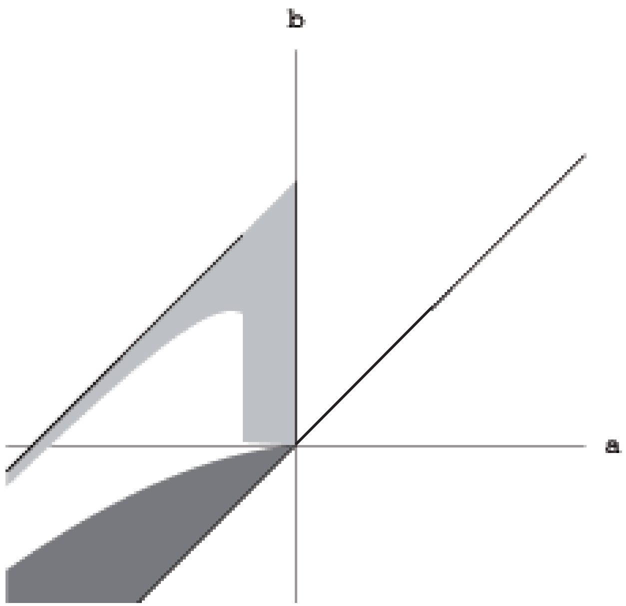
(ii) *if* $|a| < \frac{2}{p+\varepsilon}\,(1+|a|^2)$, *then*
s $\mathsf{C}_{a,b} = \mathsf{C}^*_{a,b}$ *and* $u_{a,b} = u^*_{a,b}$



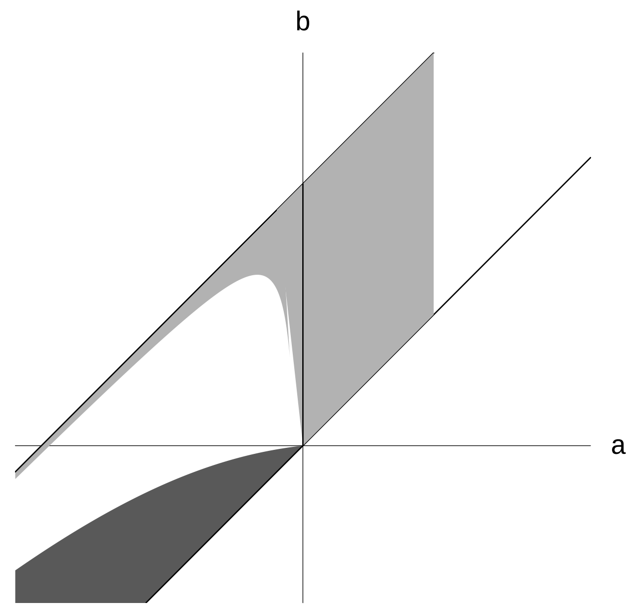
A larger symetry region

- For $d \geq 2$, radial symmetry can be proved when b is close to $a+1$

Theorem 2. [J.D.-Esteban-Loss-Tarantello] *Let $d \geq 2$. For every $A < 0$, there exists $\varepsilon > 0$ such that the extremals are radially symmetric if $a+1-\varepsilon < b < a+1$ and $a \in (A, 0)$. So they are given by $u^*_{a,b}$, up to a scalar multiplication and a dilation*



$d = 2$



$d \geq 3$

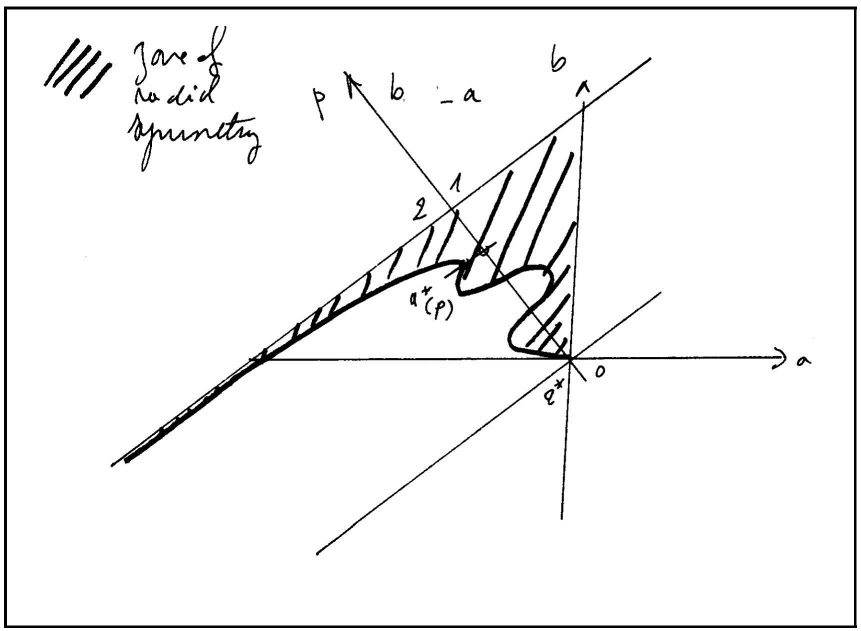
Two regions and a curve

- The symmetry and the symmetry breaking zones are simply connected and separated by a continuous curve

Theorem 3. [J.D.-Esteban-Loss-Tarantello] *For all $d \geq 2$, there exists a continuous function $a^*\colon (2, 2^*) \longrightarrow (-\infty, 0)$ such that $\lim_{p \to 2^*_-} a^*(p) = 0$, $\lim_{p \to 2_+} a^*(p) = -\infty$ and*

(i) If $(a, p) \in \left(a^*(p), \frac{d-2}{2}\right) \times (2, 2^*)$, all extremals radially symmetric

(ii) If $(a, p) \in (-\infty, a^*(p)) \times (2, 2^*)$, none of the extremals is radially symmetric



Open question. Do the curves obtained by Felli-Schneider and ours coincide ?

Emden-Fowler transformation and the cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{d-1}$

$$t = \log |x| \,, \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1}, \quad w(t,\omega) = |x|^{-a}\, v(x) \,, \quad \Lambda = \frac{1}{4}\,(d-2-2a)^2$$

- Caffarelli-Kohn-Nirenberg inequalities rewritten on the cylinder become standard interpolation inequalities of Gagliardo-Nirenberg type

$$\|w\|^2_{L^p(\mathcal{C})} \le C_{\Lambda,p} \left[\|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\, \|w\|^2_{L^2(\mathcal{C})} \right]$$

$$\mathcal{E}_\Lambda[w] := \|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\, \|w\|^2_{L^2(\mathcal{C})}$$

$$C^{-1}_{\Lambda,p} := \mathsf{C}^{-1}_{a,b} = \inf \left\{ \mathcal{E}_\Lambda(w) \;:\; \|w\|^2_{L^p(\mathcal{C})} = 1 \right\}$$

$$a < 0 \implies \Lambda > a_c^2 = \tfrac{1}{4}\,(d-2)^2$$

"critical / Sobolev" case: $b - a \to 0 \iff p \to \dfrac{2d}{d-2}$

"Hardy" case: $b - (a+1) \to 0 \iff p \to 2_+$

Perturbative methods for proving symmetry

- Euler-Lagrange equations
- A priori estimates (use radial extremals)
- Spectral analysis (gap away from the FS region of symmetry breaking)
- Elliptic regularity
- Argue by contradiction

Scaling and consequences

- A scaling property along the axis of the cylinder ($d \geq 2$)

let $w_\sigma(t,\omega) := w(\sigma\, t,\omega)$ for any $\sigma > 0$

$$\mathcal{F}_{\sigma^2 \Lambda,p}(w_\sigma) = \sigma^{1+2/p}\, \mathcal{F}_{\Lambda,p}(w) - \sigma^{-1+2/p}\,(\sigma^2 - 1)\, \frac{\int_{\mathcal{C}} |\nabla_\omega w|^2\, dy}{\left(\int_{\mathcal{C}} |w|^p\, dy\right)^{2/p}}$$

Lemma 4. [JD, Esteban, Loss, Tarantello] *If* $d \geq 2$, $\Lambda > 0$ *and* $p \in (2, 2^*)$

(i) *If* $\mathsf{C}^d_{\Lambda,p} = \mathsf{C}^{d,*}_{\Lambda,p}$, *then* $C^d_{\lambda,p} = \mathsf{C}^{d,*}_{\lambda,p}$ *and* $w_{\lambda,p} = w^*_{\lambda,p}$, *for any* $\lambda \in (0,\Lambda)$

(ii) *If there is a non radially symmetric extremal* $w_{\Lambda,p}$, *then* $\mathsf{C}^d_{\lambda,p} > \mathsf{C}^{d,*}_{\lambda,p}$ *for all* $\lambda > \Lambda$

A curve separates symmetry and symmetry breaking regions

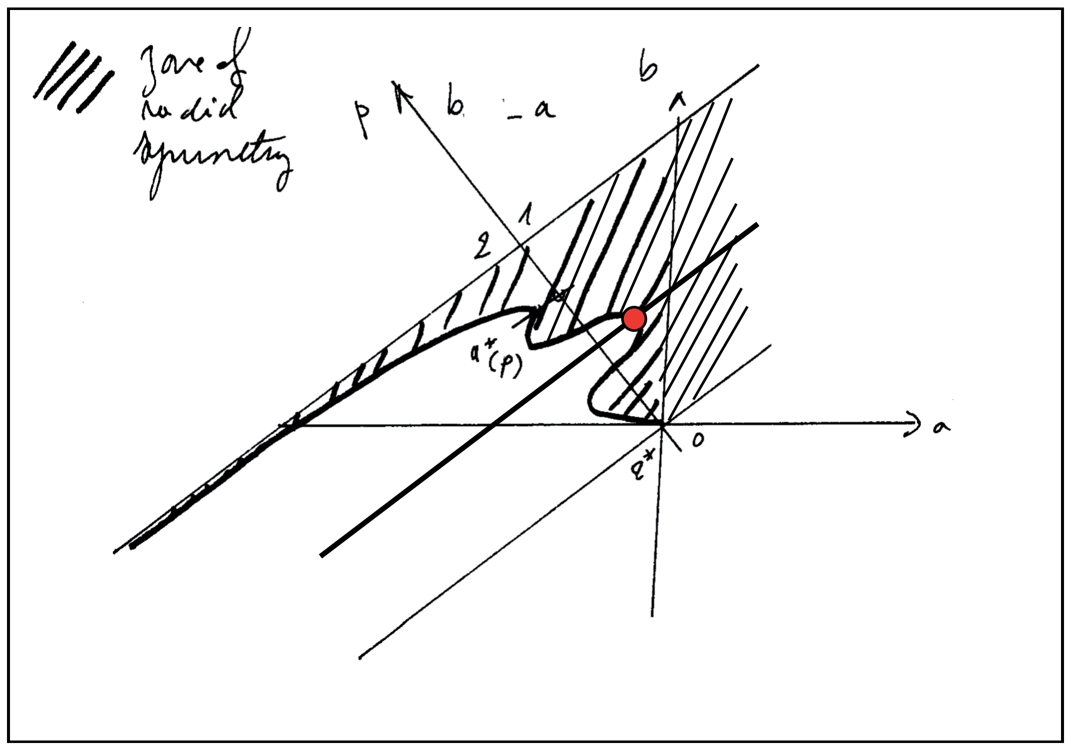
Corollary 5. [JD, Esteban, Loss, Tarantello] *Let* $d \geq 2$. *For all* $p \in (2, 2^*)$, $\Lambda^*(p) \in (0, \Lambda^{\mathrm{FS}}(p)]$ *and*

(i) *If* $\lambda \in (0, \Lambda^*(p))$, *then* $w_{\lambda,p} = w^*_{\lambda,p}$ *and clearly,* $C^d_{\lambda,p} = C^{d,*}_{\lambda,p}$

(ii) *If* $\lambda = \Lambda^*(p)$, *then* $C^d_{\lambda,p} = C^{d,*}_{\lambda,p}$

(iii) *If* $\lambda > \Lambda^*(p)$, *then* $C^d_{\lambda,p} > C^{d,*}_{\lambda,p}$

Upper semicontinuity
is easy to prove
For continuity,
a delicate spectral
analysis is needed



Caffarelli-Kohn-Nirenberg inequalities (Part II) and Logarithmic Hardy inequalities

Joint work with M. del Pino, S. Filippas and A. Tertikas

Generalized Caffarelli-Kohn-Nirenberg inequalities (CKN)

Let $2^* = \infty$ if $d = 1$ or $d = 2$, $2^* = 2d/(d-2)$ if $d \geq 3$ and define

$$\vartheta(p,d) := \frac{d\,(p-2)}{2\,p}$$

Theorem 6. [Caffarelli-Kohn-Nirenberg-84] *Let $d \geq 1$. For any $\theta \in [\vartheta(p,d), 1]$, with $p = \frac{2\,d}{d-2+2\,(b-a)}$, there exists a positive constant $\mathsf{C}_{\mathrm{CKN}}(\theta,p,a)$ such that*

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}_{\mathrm{CKN}}(\theta,p,a) \left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a - a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a)$ and

$$\mathsf{C}_{\mathrm{CKN}}(\theta,p,a) \geq \mathsf{C}^*_{\mathrm{CKN}}(\theta,p,a) = \mathsf{C}^*_{\mathrm{CKN}}(\theta,p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\mathrm{CKN}}(\theta,p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}} \left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}} \left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta} \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]$$

Weighted logarithmic Hardy inequalities (WLH)

- A “logarithmic Hardy inequality”

Theorem 7. [del Pino, J.D. Filippas, Tertikas] *Let* $d \geq 3$*. There exists a constant* $\mathsf{C}_{\rm LH} \in (0, \mathsf{S}]$ *such that, for all* $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,dx = 1$*, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \log\left(|x|^{d-2}|u|^2\right) dx \leq \frac{d}{2} \log\left[\mathsf{C}_{\rm LH} \int_{\mathbb{R}^d} |\nabla u|^2\,dx\right]$$

- A “weighted logarithmic Hardy inequality” (WLH)

Theorem 8. [del Pino, J.D. Filippas, Tertikas] *Let* $d \geq 1$*. Suppose that* $a < (d-2)/2$*,* $\gamma \geq d/4$ *and* $\gamma > 1/2$ *if* $d = 2$*. Then there exists a positive constant* $\mathsf{C}_{\rm WLH}$ *such that, for any* $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ *normalized by* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx = 1$*, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}} \log\left(|x|^{d-2-2\,a}\,|u|^2\right) dx \leq 2\,\gamma \log\left[\mathsf{C}_{\rm WLH} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

Weighted logarithmic Hardy inequalities: radial case

Theorem 9. [del Pino, J.D. Filippas, Tertikas] *Let* $d \geq 1$, $a < (d-2)/2$ *and* $\gamma \geq 1/4$. *If* $u = u(|x|) \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ *is radially symmetric, and* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx = 1$, *then*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}} \log\left(|x|^{d-2-2\,a}\,|u|^2\right)\,dx \leq 2\,\gamma\,\log\left[\mathsf{C}^*_{\rm WLH}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

$$\mathsf{C}^*_{\rm WLH} = \frac{1}{\gamma}\,\frac{\left[\Gamma\left(\frac{d}{2}\right)\right]^{\frac{1}{2\,\gamma}}}{\left(8\,\pi^{d+1}\,e\right)^{\frac{1}{4\,\gamma}}}\left(\frac{4\,\gamma-1}{(d-2-2\,a)^2}\right)^{\frac{4\,\gamma-1}{4\,\gamma}} \quad \text{if} \quad \gamma > \frac{1}{4}$$

$$\mathsf{C}^*_{\rm WLH} = 4\,\frac{\left[\Gamma\left(\frac{d}{2}\right)\right]^2}{8\,\pi^{d+1}\,e} \quad \text{if} \quad \gamma = \frac{1}{4}$$

If $\gamma > \frac{1}{4}$, *equality is achieved by the function*

$$u = \frac{\tilde{u}}{\int_{\mathbb{R}^d}\frac{|\tilde{u}|^2}{|x|^2}\,dx} \quad \text{where} \quad \tilde{u}(x) = |x|^{-\frac{d-2-2\,a}{2}}\exp\left(-\frac{(d-2-2\,a)^2}{4\,(4\,\gamma-1)}\left[\,\log|x|\,\right]^2\right)$$

Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities

Joint work with Maria J. Esteban

First existence result: the sub-critical case

Theorem 10. [J.D. Esteban] *Let $d \geq 2$ and assume that $a \in (-\infty, a_c)$*

(i) *For any $p \in (2, 2^*)$ and any $\theta \in (\vartheta(p,d), 1)$, the Caffarelli-Kohn-Nirenberg inequality (CKN)*

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$

Critical case: there exists a continuous function $a^* : (2, 2^*) \to (-\infty, a_c)$ such that the inequality also admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ if $\theta = \vartheta(p,d)$ and $a \in (a^*(p), a_c)$

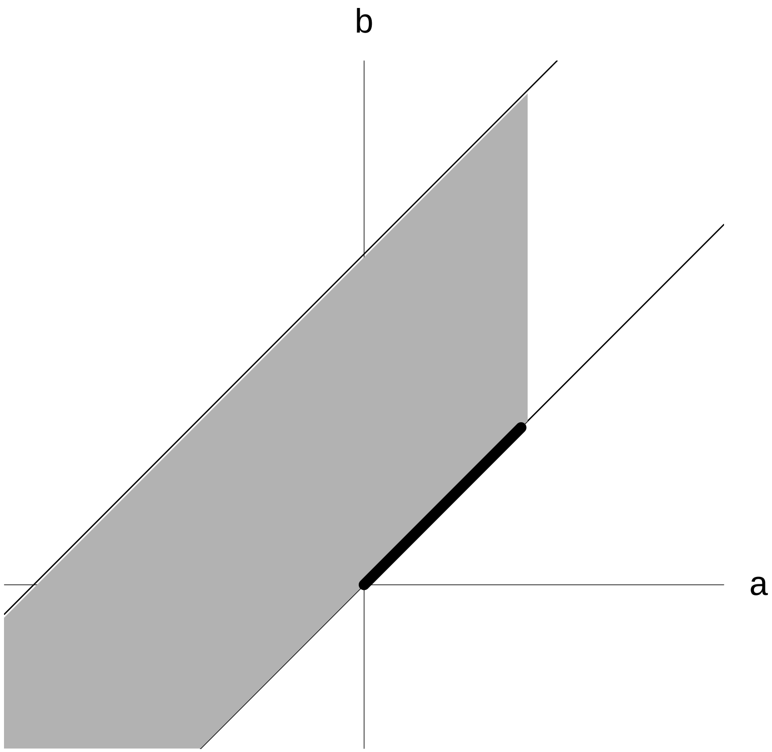
(ii) *For any $\gamma > d/4$, the weighted logarithmic Hardy inequality (WLH)*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,\log\left(|x|^{d-2-2\,a}\,|u|^2\right)\,dx \leq 2\,\gamma\,\log\left[\mathsf{C}_{\mathrm{WLH}}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

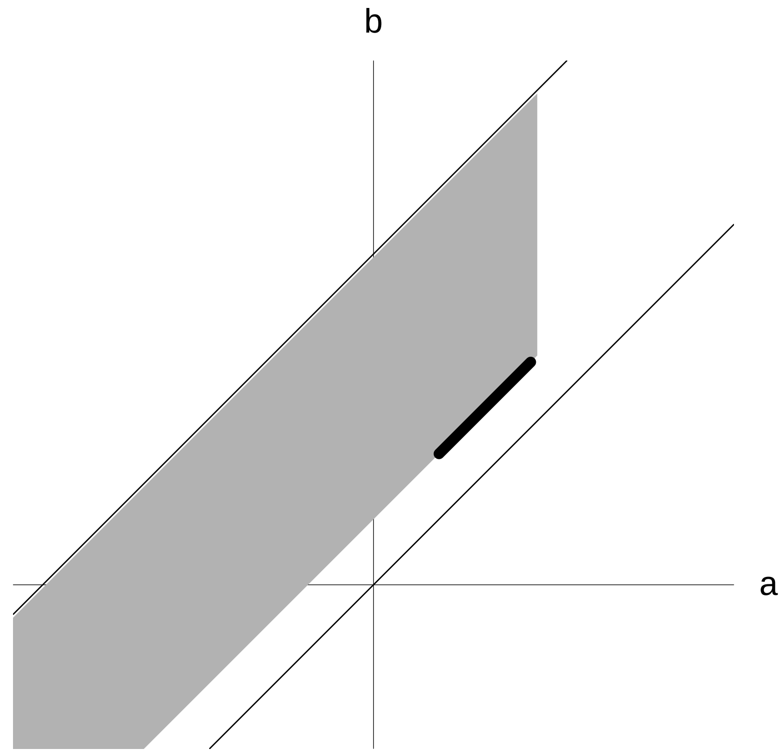
admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$

Critical case: idem if $\gamma = d/4$, $d \geq 3$ and $a \in (a^\star, a_c)$ for some $a^\star \in (-\infty, a_c)$

Existence for CKN



$d = 3$, $\theta = 1$



$d = 3$, $\theta = 0.8$

Second existence result: the critical case

Let

$$a_\star := a_c - \sqrt{(d-1)\, e\, (2^{d+1}\,\pi)^{-1/(d-1)}\, \Gamma(d/2)^{2/(d-1)}}$$

Theorem 11 (Critical cases)**.** [J.D. Esteban]

(i) *if* $\theta = \vartheta(p,d)$ *and* $\mathsf{C}_{\rm GN}(p) < \mathsf{C}_{\rm CKN}(\theta,p,a)$*, then (CKN) admits an extremal function in* $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$*,*

(ii) *if* $\gamma = d/4$*,* $d \ge 3$*, and* $\mathsf{C}_{\rm LS} < \mathsf{C}_{\rm WLH}(\gamma,a)$*, then (WLH) admits an extremal function in* $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$

If $a \in (a_\star, a_c)$ *then*

$$\mathsf{C}_{\rm LS} < \mathsf{C}_{\rm WLH}(d/4,a)$$

Radial symmetry and symmetry breaking

Joint work with
M. del Pino, S. Filippas and A. Tertikas (symmetry breaking)
Maria J. Esteban, Gabriella Tarantello and Achilles Tertikas

Implementing the method of Catrina-Wang / Felli-Schneider

Among functions $w \in H^1(\mathcal{C})$ which depend only on s, the minimum of

$$\mathcal{J}[w] := \int_{\mathcal{C}} \left(|\nabla w|^2 + \tfrac{1}{4}\,(d-2-2\,a)^2\,|w|^2\right)\,dy - [\mathsf{C}^*(\theta,p,a)]^{-\frac{1}{\theta}}\;\frac{\left(\int_{\mathcal{C}} |w|^p\;dy\right)^{\frac{2}{p\,\theta}}}{\left(\int_{\mathcal{C}} |w|^2\;dy\right)^{\frac{1-\theta}{\theta}}}$$

is achieved by $\overline{w}(y) := \left[\cosh(\lambda\,s)\right]^{-\frac{2}{p-2}}$, $y=(s,\omega)\in\mathbb{R}\times\mathbb{S}^{d-1}=\mathcal{C}$ with $\lambda := \frac{1}{4}\,(d-2-2\,a)\,(p-2)\,\sqrt{\frac{p+2}{2\,p\,\theta-(p-2)}}$ as a solution of

$$\lambda^2\,(p-2)^2\,w'' - 4\,w + 2\,p\,|w|^{p-2}\,w = 0$$

Spectrum of $\mathcal{L} := -\Delta + \kappa\,\overline{w}^{p-2} + \mu$ is given for $\sqrt{1+4\,\kappa/\lambda^2} \geq 2\,j+1$ by

$$\lambda_{i,j} = \mu + i\,(d+i-2) - \frac{\lambda^2}{4}\left(\sqrt{1+\frac{4\,\kappa}{\lambda^2}} - (1+2\,j)\right)^2 \quad \forall\, i\,,\ j\in\mathbb{N}$$

- The eigenspace of $\mathcal{L}$ corresponding to $\lambda_{0,0}$ is generated by $\overline{w}$
- The eigenfunction $\phi_{(1,0)}$ associated to $\lambda_{1,0}$ is not radially symmetric and such that $\int_{\mathcal{C}} \overline{w}\,\phi_{(1,0)}\;dy = 0$ and $\int_{\mathcal{C}} \overline{w}^{p-1}\,\phi_{(1,0)}\;dy = 0$
- If $\lambda_{1,0} < 0$, *optimal functions for (CKN) cannot be radially symmetric* and

$$\mathsf{C}(\theta,p,a) > \mathsf{C}^*(\theta,p,a)$$

Schwarz' symmetrization

With $u(x) = |x|^a \, v(x)$, (CKN) is then equivalent to

$$\| |x|^{a-b} \, v \|^2_{\mathrm{L}^p(\mathbb{R}^N)} \le \mathsf{C}_{\mathrm{CKN}}(\theta, p, \Lambda) \, (\mathcal{A} - \lambda \, \mathcal{B})^\theta \; \mathcal{B}^{1-\theta}$$

with $\mathcal{A} := \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^N)}$, $\mathcal{B} := \| |x|^{-1} \, v \|^2_{\mathrm{L}^2(\mathbb{R}^N)}$ and $\lambda := a \, (2 \, a_c - a)$. We observe that the function $B \mapsto h(\mathcal{B}) := (\mathcal{A} - \lambda \, \mathcal{B})^\theta \; \mathcal{B}^{1-\theta}$ satisfies

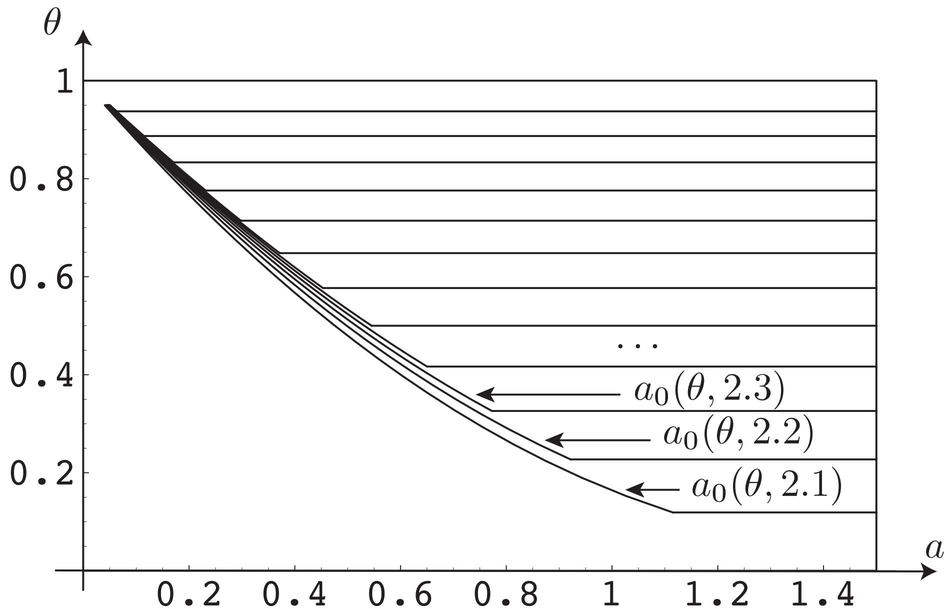
$$\frac{h'(\mathcal{B})}{h(\mathcal{B})} = \frac{1-\theta}{\mathcal{B}} - \frac{\lambda \, \theta}{\mathcal{A} - \lambda \, \mathcal{B}}$$

By Hardy's inequality ($d \ge 3$), we know that

$$\mathcal{A} - \lambda \, \mathcal{B} \ge \inf_{a>0} \big(\mathcal{A} - a \, (2 \, a_c - a) \, \mathcal{B} \big) = \mathcal{A} - a_c^2 \, \mathcal{B} > 0$$

and so $h'(\mathcal{B}) \le 0$ if $(1-\theta) \, \mathcal{A} < \lambda \, \mathcal{B} \Longleftrightarrow \mathcal{A}/\mathcal{B} < \lambda/(1-\theta)$
By interpolation $\mathcal{A}/\mathcal{B}$ is small if $a_c - a > 0$ is small enough, for $\theta > \vartheta(p,d)$ and $d \ge 3$

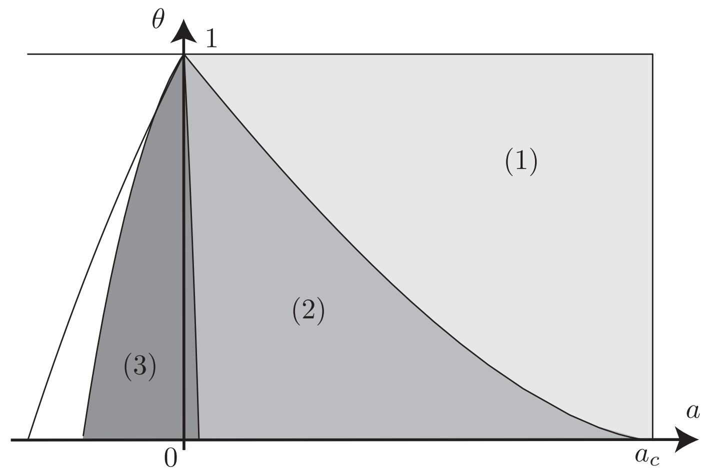
Regions in which Schwarz' symmetrization holds



- Here $d = 5$, $a_c = 1.5$ and $p = 2.1$, 2.2, $\ldots 3.2$
- Symmetry holds if $a \in [a_0(\theta, p), a_c)$, $\theta \in (\vartheta(p, d), 1)$
- Horizontal segments correspond to $\theta = \vartheta(p, d)$
- Hardy's inequality: the above symmetry region is contained in $\theta > (1 - \frac{a}{a_c})^2$

Alternatively, we could prove the symmetry by the moving planes method *in the same region*

Summary (1/2): Existence for (CKN)



The zones in which existence is known are:

- (1) extremals are achieved among radial functions, by the Schwarz symmetrization method
- (1)+(2) this follows from the explicit *a priori* estimates; $\Lambda_1 = (a_c - a_1)^2$
- (1)+(2)+(3) this follows by comparison of the optimal constant for (CKN) with the optimal constant in the corresponding Gagliardo-Nirenberg-Sobolev inequality

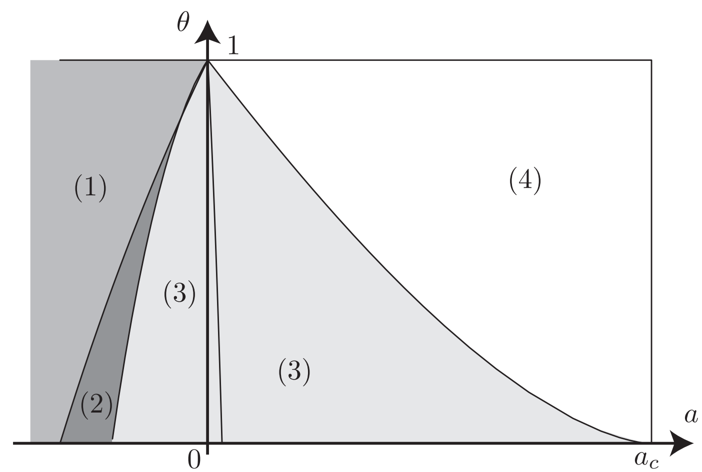
Summary (2/2): Symmetry and symmetry breaking for (CKN)

The zone of symmetry breaking contains:

(1) by linearization around radial extremals

(1)+(2) by comparison with the Gagliardo-Nirenberg-Sobolev inequality

In (3) it is not known whether symmetry holds or if there is symmetry breaking, while in (4), that is, for $a_0 \leq a < a_c$, symmetry holds by the Schwarz symmetrization



One bound state Lieb-Thirring inequalities and symmetry

Joint work with Maria J. Esteban and M. Loss

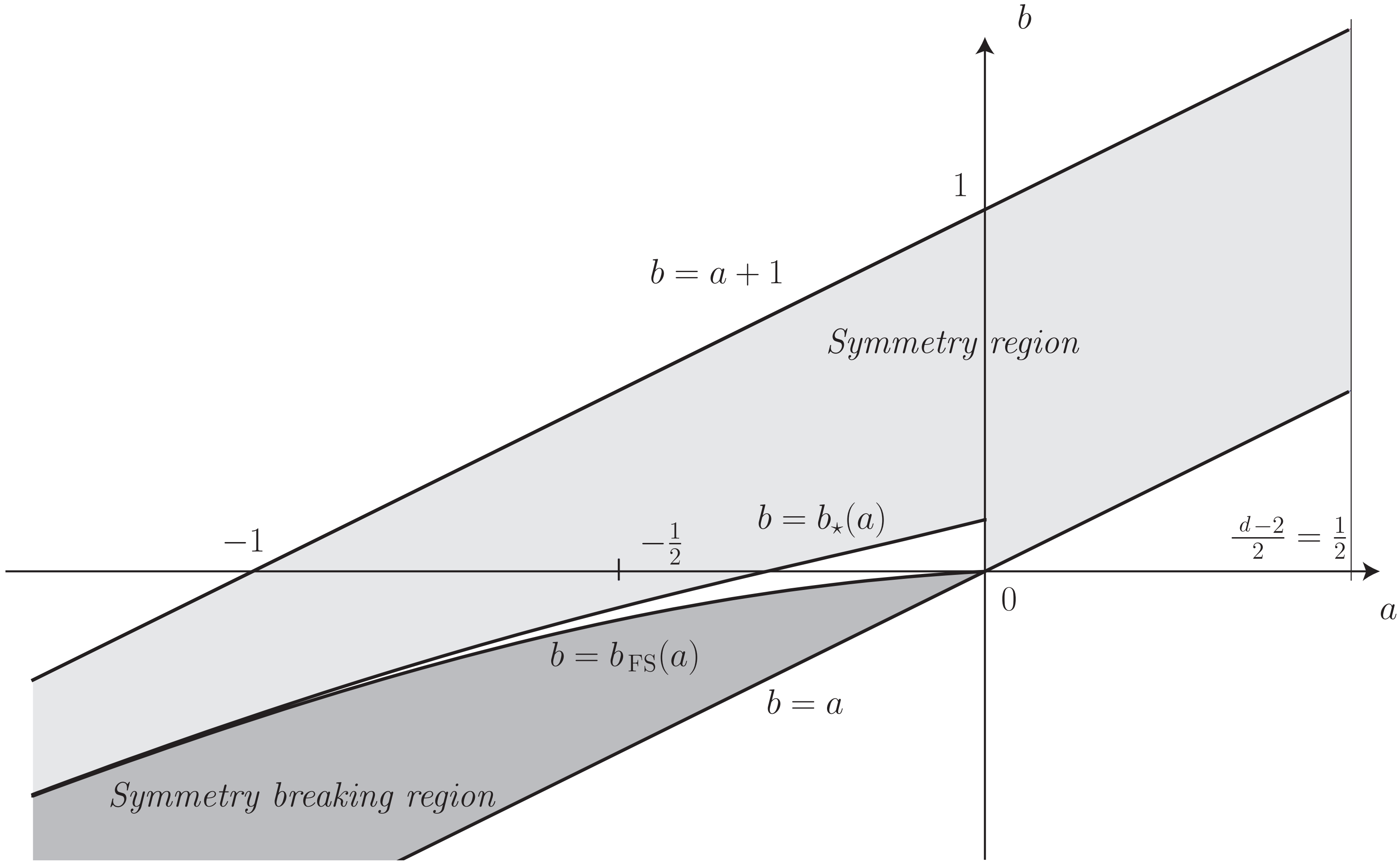
Symmetry: a new quantitative approach

$$b_\star(a) := \frac{d\,(d-1) + 4\,d\,(a-a_c)^2}{6\,(d-1) + 8\,(a-a_c)^2} + a - a_c\;.$$

Theorem 12. *Let $d \geq 2$. When $a < 0$ and $b_\star(a) \leq b < a+1$, the extremals of the Caffarelli-Kohn-Nirenberg inequality with $\theta = 1$ are radial and*

$$C^d_{a,b} = \left|\mathbb{S}^{d-1}\right|^{\frac{p-2}{p}} \left[\frac{(a-a_c)^2\,(p-2)^2}{p+2}\right]^{\frac{p-2}{2\,p}} \left[\frac{p+2}{2\,p\,(a-a_c)^2}\right] \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{2}{p-2}\right)}\right]^{\frac{p-2}{p}}$$

The symmetry region



The symmetry result on the cylinder

$$\Lambda_\star(p) := \frac{(d-1)\,(6-p)}{4\,(p-2)}$$

$d\omega$: the uniform probability measure on $\mathbb{S}^{d-1}$
L^2: the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$

Theorem 13. *Let $d \ge 2$ and let u be a non-negative function on $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{d-1}$ that satisfies*

$$-\partial_s^2 u - L^2 u + \Lambda\, u = u^{p-1}$$

and consider the symmetric solution u_*. Assume that

$$\int_{\mathcal{C}} |u(s,\omega)|^p\, ds\, d\omega \le \int_{\mathbb{R}} |u_*(s)|^p\, ds$$

for some $2 < p < 6$ satisfying $p \le \frac{2\,d}{d-2}$. If $\Lambda \le \Lambda_\star(p)$, then for a.e. $\omega \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$, we have $u(s,\omega) = u_*(s-s_0)$ for some constant s_0

The one-bound state version of the Lieb-Thirring inequality

Let $K(\Lambda,p,d) := C^d_{a,b}$ and

$$\Lambda^d_\gamma(\mu) := \inf\left\{\Lambda>0\,:\,\mu^{\frac{2\gamma}{2\gamma+1}} = 1/K(\Lambda,p,d)\right\}$$

Lemma 14. *For any $\gamma\in(2,\infty)$ if $d=1$, or for any $\gamma\in(1,\infty)$ such that $\gamma\ge\frac{d-1}{2}$ if $d\ge2$, if V is a non-negative potential in $\mathrm{L}^{\gamma+\frac12}(\mathcal{C})$, then the operator $-\partial^2-L^2-V$ has at least one negative eigenvalue, and its lowest eigenvalue, $-\lambda_1(V)$ satisfies*

$$\lambda_1(V)\le\Lambda^d_\gamma(\mu)\quad\textit{with}\quad \mu=\mu(V):=\left(\int_{\mathcal{C}} V^{\gamma+\frac12}\,ds\,d\omega\right)^{\frac1\gamma}$$

Moreover, equality is achieved if and only if the eigenfunction u corresponding to $\lambda_1(V)$ satisfies $u=V^{(2\gamma-1)/4}$ and u is optimal for (CKN)

$$\text{Symmetry}\quad\Longleftrightarrow\quad\Lambda^d_\gamma(\mu)=\Lambda^d_\gamma(1)\,\mu$$

The one-bound state Lieb-Thirring inequality (2)

Let $V = V(s)$ be a non-negative real valued potential in $\mathrm{L}^{\gamma+\frac{1}{2}}(\mathbb{R})$ for some $\gamma > 1/2$ and let $-\lambda_1(V)$ be the lowest eigenvalue of the Schrödinger operator $-\frac{d^2}{ds^2} - V$. Then

$$\lambda_1(V)^\gamma \leq c_{\mathrm{LT}}(\gamma) \int_{\mathbb{R}} V^{\gamma+1/2}(s)\, ds$$

with $c_{\mathrm{LT}}(\gamma) = \frac{\pi^{-1/2}}{\gamma-1/2} \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1/2)} \left(\frac{\gamma-1/2}{\gamma+1/2}\right)^{\gamma+1/2}$, with equality if and only if, up to scalings and translations,

$$V(s) = \frac{\gamma^2 - 1/4}{\cosh^2(s)} =: V_0(s)$$

Moreover $\lambda_1(V_0) = (\gamma - 1/2)^2$ and the corresponding ground state eigenfunction is given by

$$\psi_\gamma(s) = \pi^{-1/4} \left(\frac{\Gamma(\gamma)}{\Gamma(\gamma - 1/2)}\right)^{1/2} \left[\cosh(s)\right]^{-\gamma+1/2}$$

The generalized Poincaré inequality

Theorem 15. [Bidaut-Véron, Véron] $(\mathcal{M}, g)$ *is a compact Riemannian manifold of dimension* $d-1 \geq 2$*, without boundary,* Δ_g *is the Laplace-Beltrami operator on* $\mathcal{M}$*, the Ricci tensor* R *and the metric tensor* g *satisfy* $R \geq \frac{d-2}{d-1}\,(q-1)\,\lambda\,g$ *in the sense of quadratic forms, with* $q > 1$*,* $\lambda > 0$ *and* $q \leq \frac{d+1}{d-3}$*. Moreover, one of these two inequalities is strict if* $(\mathcal{M}, g)$ *is* $\mathbb{S}^{d-1}$ *with the standard metric.*

If u *is a positive solution of*

$$\Delta_g\, u - \lambda\, u + u^q = 0$$

then u *is constant with value* $\lambda^{1/(q-1)}$ *Moreover, if* $\mathrm{vol}(\mathcal{M}) = 1$ *and* $\mathsf{D}(\mathcal{M}, q) := \max\{\lambda > 0 \,:\, R \geq \frac{N-2}{N-1}\,(q-1)\,\lambda\,g\}$ *is positive, then*

$$\frac{1}{\mathsf{D}(\mathcal{M}, q)} \int_{\mathcal{M}} |\nabla v|^2 + \int_{\mathcal{M}} |v|^2 \geq \left(\int_{\mathcal{M}} |v|^{q+1} \right)^{\frac{2}{q+1}} \quad \forall\, v \in W^{1,1}(\mathcal{M})$$

Applied to $\mathcal{M} = \mathbb{S}^{d-1}$: $\mathsf{D}(\mathbb{S}^{d-1}, q) = \frac{q-1}{d-1}$

The case: $\theta < 1$

$$\mathfrak{C}(p,\theta) := \frac{(p+2)^{\frac{p+2}{(2\,\theta-1)\,p+2}}}{(2\,\theta-1)\,p+2}\left(\frac{2-p\,(1-\theta)}{2}\right)^{2\,\frac{2-p\,(1-\theta)}{(2\,\theta-1)\,p+2}}$$
$$\cdot\left(\frac{\Gamma(\frac{p}{p-2})}{\Gamma(\frac{\theta\,p}{p-2})}\right)^{\frac{4\,(p-2)}{(2\,\theta-1)\,p+2}}\left(\frac{\Gamma(\frac{2\,\theta\,p}{p-2})}{\Gamma(\frac{2\,p}{p-2})}\right)^{\frac{2\,(p-2)}{(2\,\theta-1)\,p+2}}$$

Notice that $\mathfrak{C}(p,\theta) \geq 1$ and $\mathfrak{C}(p,\theta) = 1$ if and only if $\theta = 1$

Theorem 16. *With the above notations, for any $d \geq 3$, any $p \in (2, 2^*)$ and any $\theta \in [\vartheta(p,d), 1)$, we have the estimate*

$$\mathsf{C}^*_{\rm CKN}(\theta,a,p) \leq \mathsf{C}_{\rm CKN}(\theta,a,p) \leq \mathsf{C}^*_{\rm CKN}(\theta,a,p)\,\mathfrak{C}(p,\theta)^{\frac{(2\,\theta-1)\,p+2}{2\,p}}$$

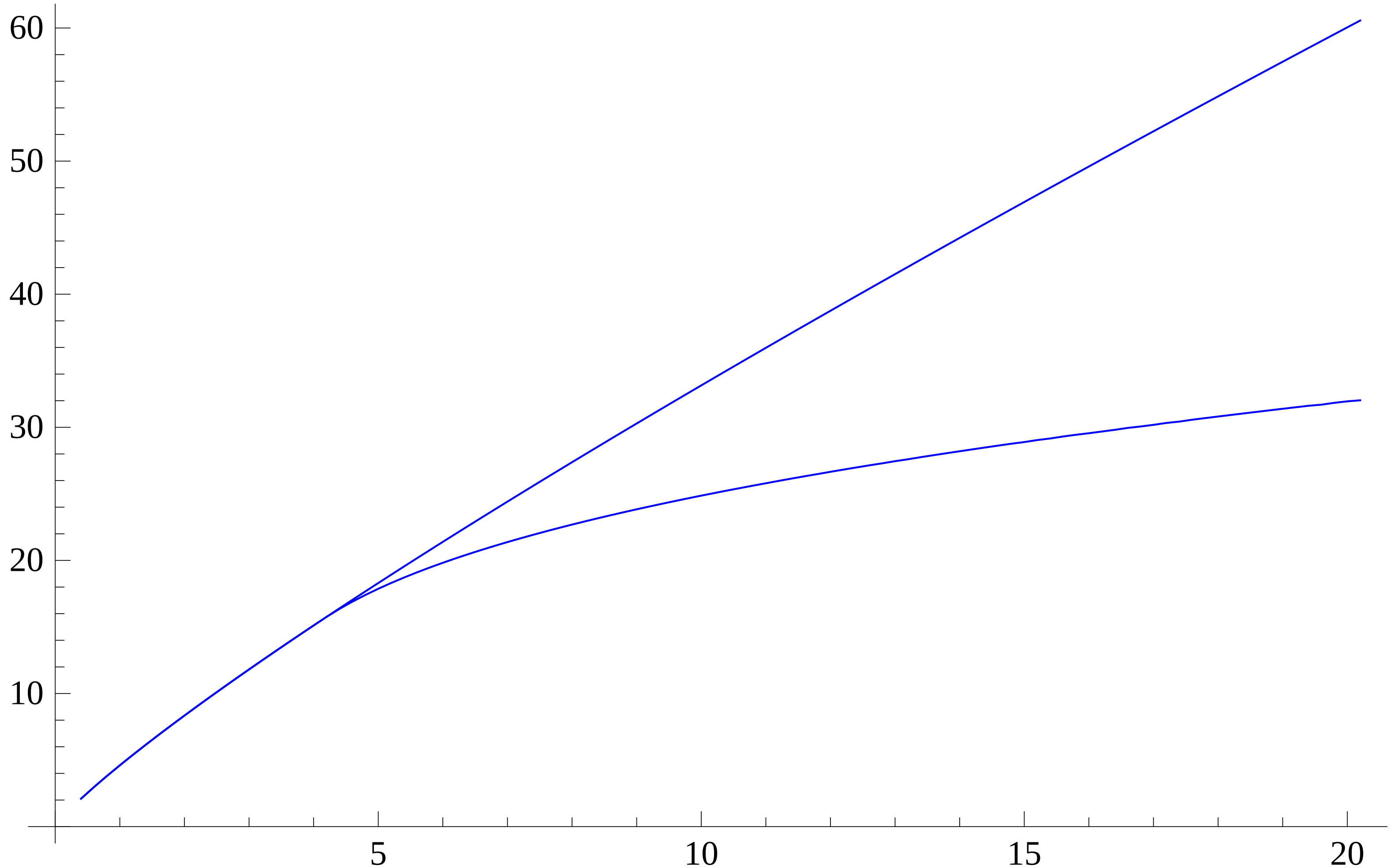
under the condition

$$(a-a_c)^2 \leq \frac{(d-1)}{\mathfrak{C}(p,\theta)}\,\frac{(2\,\theta-3)\,p+6}{4\,(p-2)}$$

Numerical results

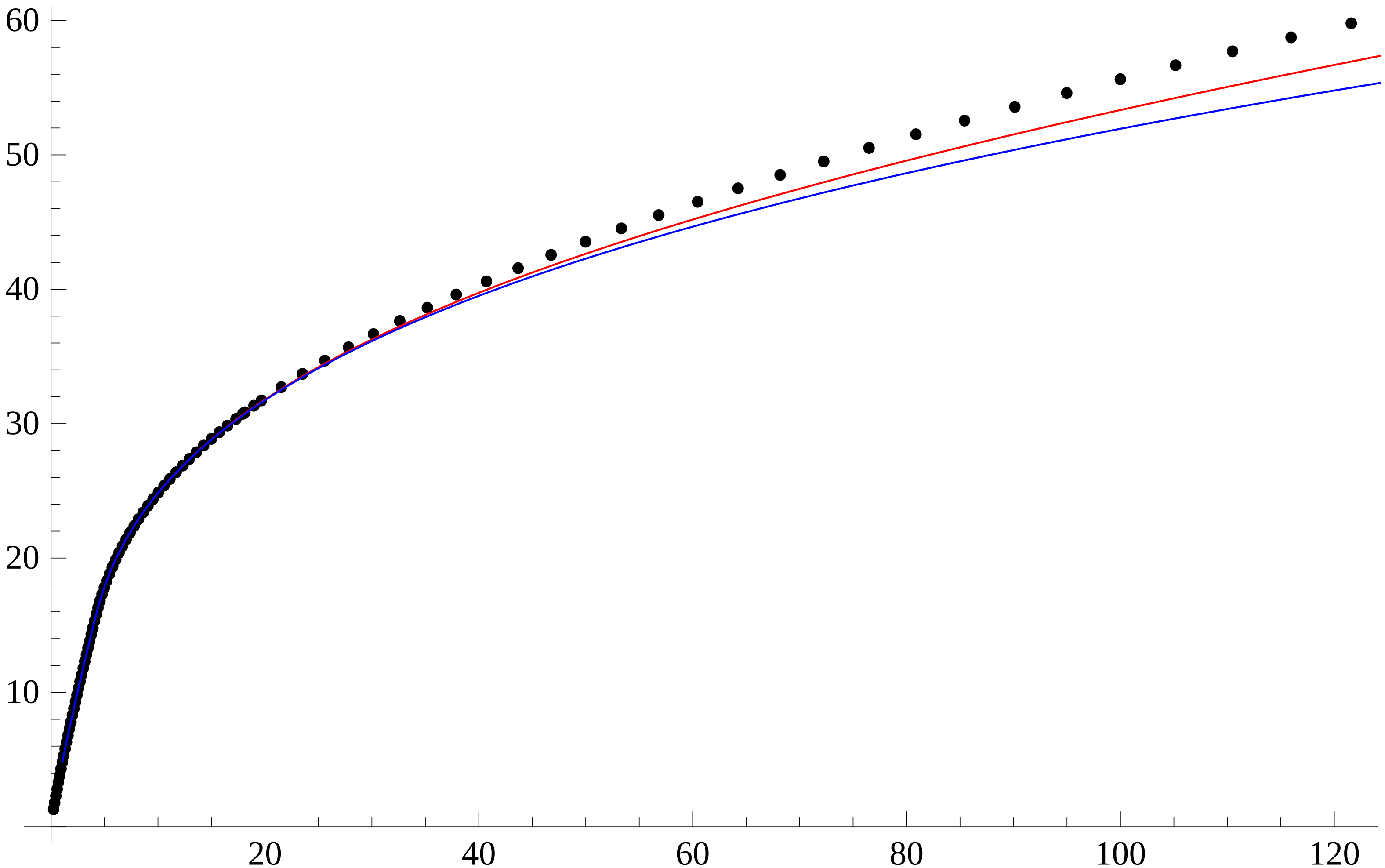
Ongoing work with Maria J. Esteban

Energy: symmetric / non symmetric optimal functions



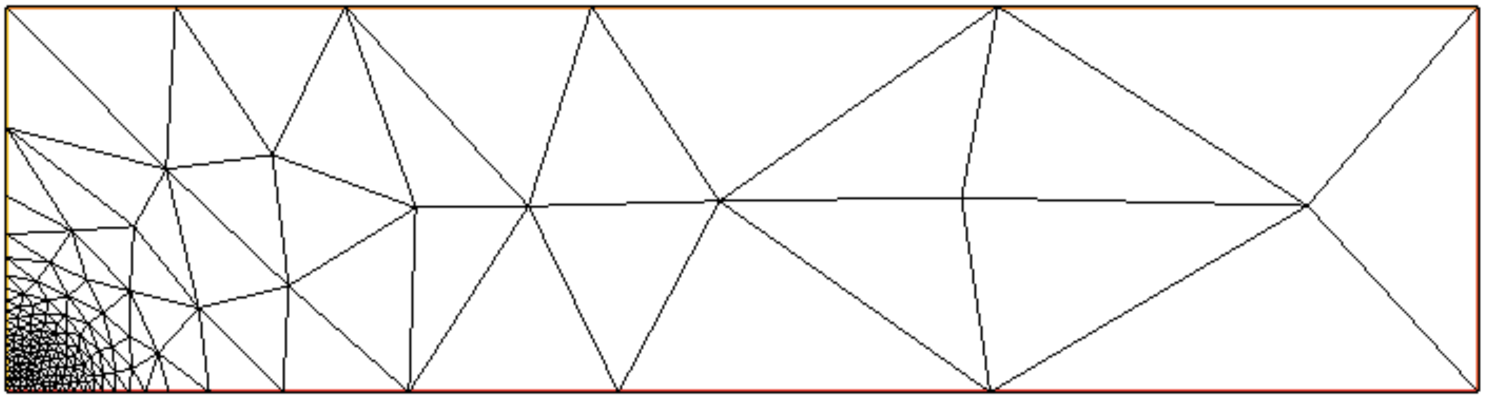
$$\Lambda \mapsto \min\{\|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\,\|w\|^2_{L^2(\mathcal{C})}\ :\ \|w\|_{L^p(\mathcal{C})} = 1\}$$

Non symmetric optimal functions: grid issues

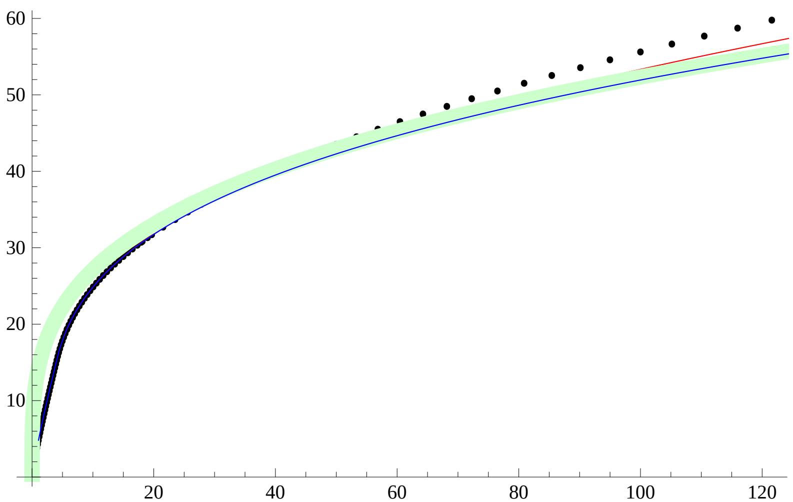


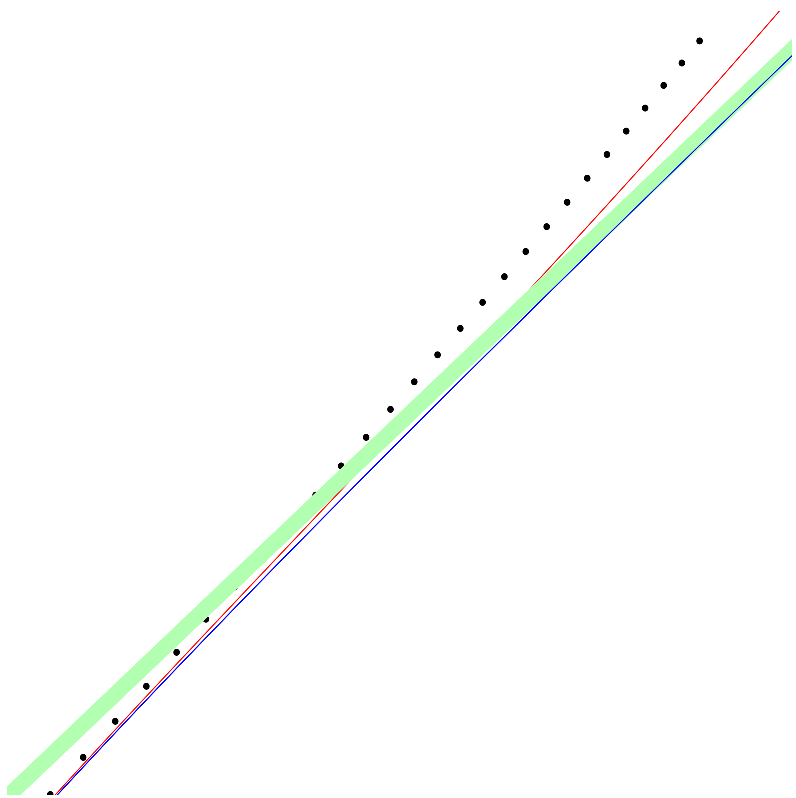
Coarse / refined / self-adaptive grids

A self-adaptive grid

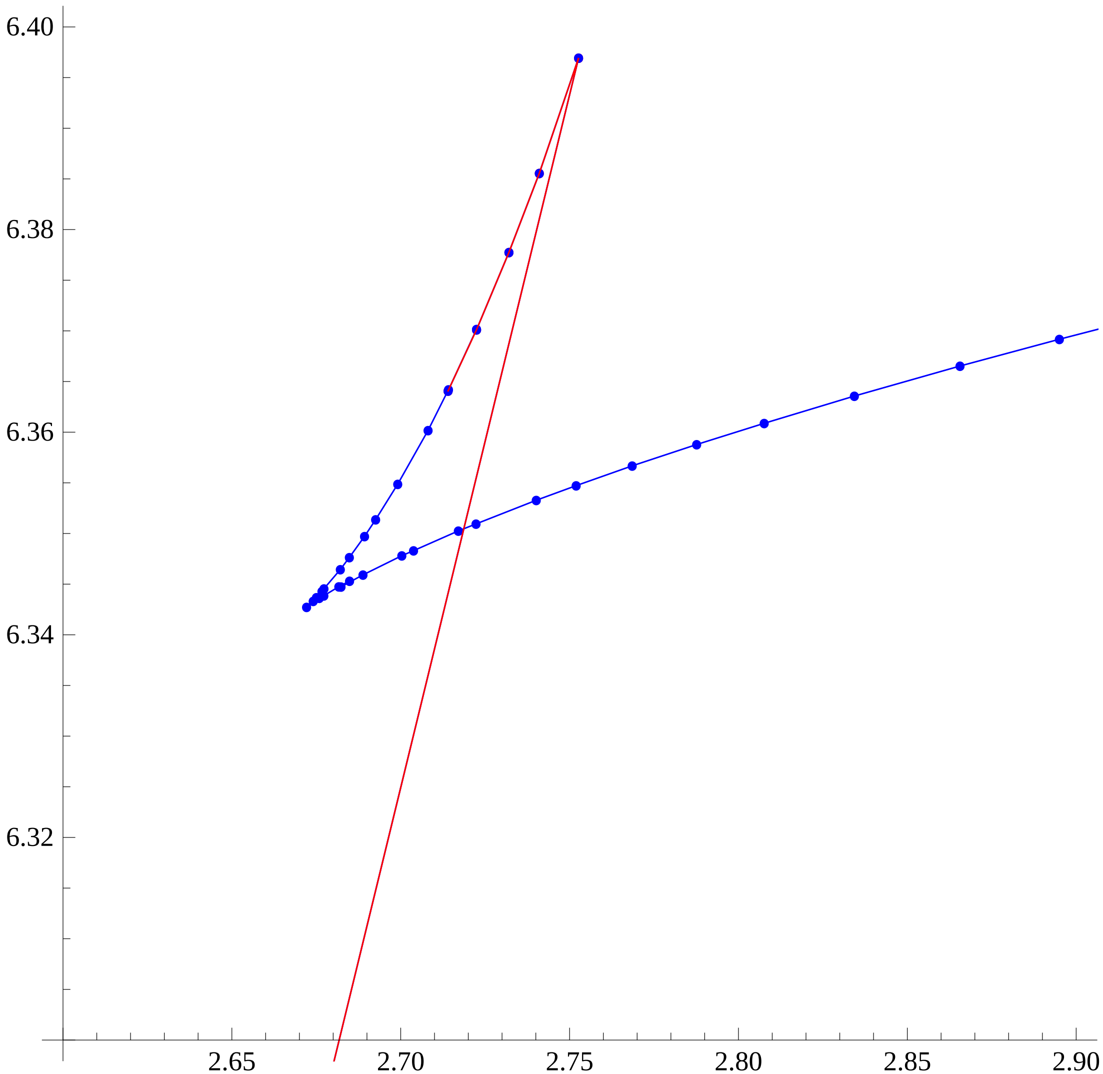


Comparison with the asymptotic regime





An explanation for the case $\theta < 1$



Thank you !