Interpolation inequalities: rigidity results, nonlinear flows and improved inequalities

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Scope (1/3): rigidity results

**Rigidity results** for semilinear elliptic PDEs on manifolds...

Let $(\mathcal{M}, g)$ be a smooth compact Riemannian manifold of dimension $d \geq 2$, no boundary, $\Delta_g$ is the Laplace-Beltrami operator, the Ricci tensor $\mathcal{R}$ has good properties (which ones?)

Let $p \in (2, 2^*)$, with $2^* = \frac{2d}{d-2}$ if $d \geq 3$, $2^* = \infty$ if $d = 2$

For which values of $\lambda > 0$ the equation

$$- \Delta_g v + \lambda v = v^{p-1}$$

has a unique positive solution $v \in C^2(\mathcal{M})$: $v \equiv \lambda^{\frac{1}{p-2}}$ ?

A typical rigidity result is: there exists $\lambda_0 > 0$ such that $v \equiv \lambda^{\frac{2}{p-2}}$ if $\lambda \in (0, \lambda_0]$

Assumptions ?

Optimal $\lambda_0$ ?
Still on a smooth compact Riemannian manifold \((\mathcal{M}, g)\)
we assume that \(\text{vol}_g(\mathcal{M}) = 1\)

For any \(p \in (1, 2) \cup (2, 2^*)\) or \(p = 2^*\) if \(d \geq 3\), consider the interpolation inequality

\[
\| \nabla v \|^2_{L^2(\mathcal{M})} \geq \frac{\lambda}{p - 2} \left[ \| v \|^2_{L^p(\mathcal{M})} - \| v \|^2_{L^2(\mathcal{M})} \right] \quad \forall \ v \in H^1(\mathcal{M})
\]

What is the largest possible value of \(\lambda\)?

- using \(u = 1 + \varepsilon \varphi\) as a test function proves that \(\lambda \leq \lambda_1\)
- the minimum of \(v \mapsto \| \nabla v \|^2_{L^2(\mathcal{M})} - \frac{\lambda}{p - 2} \left[ \| v \|^2_{L^p(\mathcal{M})} - \| v \|^2_{L^2(\mathcal{M})} \right]\)
under the constraint \(\| v \|_{L^p(\mathcal{M})} = 1\) is negative if \(\lambda\) is above the rigidity threshold
- the threshold case \(p = 2\) is the logarithmic Sobolev inequality

\[
\| \nabla u \|^2_{L^2(\mathcal{M})} \geq \lambda \int_{\mathcal{M}} u^2 \log \left( \frac{u^2}{\| u \|^2_{L^2(\mathcal{M})}} \right) \, dv_g \quad \forall \ u \in H^1(\mathcal{M})
\]
We shall consider a flow of porous media / fast diffusion type

\[ u_t = u^{2-2\beta} \left( \Delta_g u + \kappa \frac{\nabla u^2}{u} \right), \quad \kappa = 1 + \beta (p - 2) \]

If \( v = u^\beta \), then \( \frac{d}{dt} \|v\|_{L^p(\mathcal{M})} = 0 \) and the functional

\[ F[u] := \int_{\mathcal{M}} |\nabla u^\beta|^2 \, d\nu_g + \frac{\lambda}{p-2} \left[ \int_{\mathcal{M}} u^{2\beta} \, d\nu_g - \left( \int_{\mathcal{M}} u^{\beta p} \, d\nu_g \right)^{2/p} \right] \]

is monotone decaying as long as \( \lambda \) is not too big. Hence, if the limit as \( t \to \infty \) is 0 (convergence to the constants), we know that \( F[u] \geq 0 \)

*Structure ? Link with computations in the rigidity approach*
Some references (incomplete) and goals

- rigidity results and elliptic PDEs: [Gidas-Spruck 1981], [Bidaut-Véron & Véron 1991], [Licois & Véron 1995]
  → systematize and clarify the strategy

- semi-group approach and $\Gamma_2$ or carré du champ method:
  → emphasize the role of the flow, get various improvements
  → get rid of pointwise constraints on the curvature, discuss optimality

- harmonic analysis, duality and spectral theory: [Lieb 1983], [Beckner 1993]
  → apply results to get new spectral estimates
Outline

1. The case of the sphere
   - Inequalities on the sphere
   - Flows on the sphere
   - Spectral consequences
   - Improved inequalities

2. The case of Riemannian manifolds
   - Flows
   - Spectral consequences

3. Inequalities on the line
   - Variational approaches
   - Mass transportation
   - Flows

4. The Moser-Trudinger-Onofri inequality... + another flow

Joint work with:

M.J. Esteban, G. Jankowiak, M. Kowalczyk, A. Laptev and M. Loss
The sphere

The case of the sphere as a simple example
Inequalities on the sphere
A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere:

\[
\frac{p - 2}{d} \int_{\mathbb{S}^d} |\nabla u|^2 \, d\nu_g + \int_{\mathbb{S}^d} |u|^2 \, d\nu_g \geq \left( \int_{\mathbb{S}^d} |u|^p \, d\nu_g \right)^{2/p} \quad \forall \, u \in H^1(\mathbb{S}^d, d\nu_g)
\]

- for any \( p \in (2, 2^*] \) with \( 2^* = \frac{2d}{d-2} \) if \( d \geq 3 \)
- for any \( p \in (2, \infty) \) if \( d = 2 \)

Here \( d\nu_g \) is the uniform probability measure: \( \nu_g(\mathbb{S}^d) = 1 \)

- 1 is the optimal constant, equality achieved by constants
- \( p = 2^* \) corresponds to Sobolev’s inequality...
Stereographic projection
Sobolev inequality

The stereographic projection of $S^d \subset \mathbb{R}^d \times \mathbb{R}$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1, 1]$, $\rho \geq 0$, $\phi \in S^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2 - 1}{r^2 + 1} = 1 - \frac{2}{r^2 + 1}, \quad \rho = \frac{2 r}{r^2 + 1}$$

and transform any function $u$ on $S^d$ into a function $v$ on $\mathbb{R}^d$ using

$$u(y) = \left(\frac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\frac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1 - z)^{-\frac{d-2}{2}} v(x)$$

$p = 2^*$, $S_d = \frac{1}{4} d (d - 2) |S^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, dx \geq S_d \left[ \int_{\mathbb{R}^d} |v|^{\frac{2d}{d-2}} \, dx \right]^{\frac{d-2}{d}} \quad \forall v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$
Extended inequality

\[ \int_{S^d} |\nabla u|^2 \, d\nu_g \geq \frac{d}{p-2} \left[ \left( \int_{S^d} |u|^p \, d\nu_g \right)^{2/p} - \int_{S^d} |u|^2 \, d\nu_g \right] \quad \forall \, u \in H^1(S^d, d\mu) \]

is valid

- for any \( p \in (1, 2) \cup (2, \infty) \) if \( d = 1, 2 \)
- for any \( p \in (1, 2) \cup (2, 2^*] \) if \( d \geq 3 \)

- **Case** \( p = 2 \): Logarithmic Sobolev inequality

\[ \int_{S^d} |\nabla u|^2 \, d\nu_g \geq \frac{d}{2} \int_{S^d} |u|^2 \log \left( \frac{|u|^2}{\int_{S^d} |u|^2 \, d\nu_g} \right) \, d\nu_g \quad \forall \, u \in H^1(S^d, d\mu) \]

- **Case** \( p = 1 \): Poincaré inequality

\[ \int_{S^d} |\nabla u|^2 \, d\nu_g \geq d \int_{S^d} |u - \bar{u}|^2 \, d\nu_g \quad \text{with} \quad \bar{u} := \int_{S^d} u \, d\nu_g \quad \forall \, u \in H^1(S^d, d\mu) \]
A spectral approach when $p \in (1, 2) - 1^{st}$ step


**Nelson’s hypercontractivity result.** Consider the heat equation

$$\frac{\partial f}{\partial t} = \Delta_g f$$

with initial datum $f(t = 0, \cdot) = u \in L^{2/p}(S^d)$, for some $p \in (1, 2]$, and let $F(t) := \|f(t, \cdot)\|_{L^{p(t)}(S^d)}$. The key computation goes as follows.

$$\frac{F'}{F} = \frac{p'}{p^2 F^p} \left[ \int_{S^d} v^2 \log \left( \frac{v^2}{\int_{S^d} v^2 \, d v_g} \right) \, d v_g + 4 \frac{p - 1}{p'} \int_{S^d} |\nabla v|^2 \, d v_g \right]$$

with $v := |f|^{p(t)/2}$. With $4 \frac{p - 1}{p'} = \frac{2}{d}$ and $t^* > 0$ such that $p(t^*) = 2$, we have

$$\|f(t^*, \cdot)\|_{L^2(S^d)} \leq \|u\|_{L^{2/p}(S^d)} \quad \text{if} \quad \frac{1}{p - 1} = e^{2d t^*}$$
A spectral approach when $p \in (1, 2)$ – 2nd step

**Spectral decomposition.** Let $u = \sum_{k \in \mathbb{N}} u_k$ be a spherical harmonics decomposition, $\lambda_k = k (d + k - 1)$, $a_k = \| u_k \|_{L^2(S^d)}^2$ so that

$$\| u \|_{L^2(S^d)}^2 = \sum_{k \in \mathbb{N}} a_k \text{ and } \| \nabla u \|_{L^2(S^d)}^2 = \sum_{k \in \mathbb{N}} \lambda_k a_k$$

$$\| f(t_*, \cdot) \|_{L^2(S^d)}^2 = \sum_{k \in \mathbb{N}} a_k e^{-2 \lambda_k t_*}$$

$$\frac{\| u \|_{L^2(S^d)}^2 - \| u \|_{L^p(S^d)}^2}{2 - p} \leq \frac{\| u \|_{L^2(S^d)}^2 - \| f(t_*, \cdot) \|_{L^2(S^d)}^2}{2 - p}$$

$$= \frac{1}{2 - p} \sum_{k \in \mathbb{N}^*} \lambda_k a_k \frac{1 - e^{-2 \lambda_k t_*}}{\lambda_k}$$

$$\leq \frac{1 - e^{-2 \lambda_1 t_*}}{(2 - p) \lambda_1} \sum_{k \in \mathbb{N}^*} \lambda_k a_k = \frac{1 - e^{-2 \lambda_1 t_*}}{(2 - p) \lambda_1} \| \nabla u \|_{L^2(S^d)}^2$$

The conclusion easily follows if we notice that $\lambda_1 = d$, and

$$e^{-2 \lambda_1 t_*} = p - 1 \text{ so that } \frac{1 - e^{-2 \lambda_1 t_*}}{(2 - p) \lambda_1} = \frac{1}{d}$$
Optimality: a perturbation argument

The optimality of the constant can be checked by a Taylor expansion of \( u = 1 + \varepsilon \nu \) at order two in terms of \( \varepsilon > 0 \), small.

For any \( p \in (1, 2^*] \) if \( d \geq 3 \), any \( p > 1 \) if \( d = 1 \) or \( 2 \), it is remarkable that

\[
Q[u] := \frac{(p - 2) \| \nabla u \|^2_{L^2(S^d)}}{\| u \|^2_{L^p(S^d)} - \| u \|^2_{L^2(S^d)}} \geq \inf_{u \in H^1(S^d, d\mu)} Q[u] = \frac{1}{d}
\]

is achieved by \( Q[1 + \varepsilon \nu] \) as \( \varepsilon \to 0 \) and \( \nu \) is an eigenfunction associated with the first nonzero eigenvalue of \( \Delta_g \).

\( p > 2 \): no simple proof based on spectral analysis is available: [Beckner], an approach based on Lieb’s duality, the Funk-Hecke formula and some (non-trivial) computations.

Elliptic methods / \( \Gamma_2 \) formalism of Bakry-Emery / flow... they are the same (main contribution) and can be simplified (!) As a side result, you can go beyond these approaches and discuss optimality.
Schwarz symmetry and the ultraspherical setting

\[(\xi_0, \xi_1, \ldots \xi_d) \in S^d, \xi_d = z, \sum_{i=0}^{d} |\xi_i|^2 = 1 \] [Smets-Willem]

**Lemma**

*Up to a rotation, any minimizer of \(Q\) depends only on \(\xi_d = z\)*

- Let \(d\sigma(\theta) := \frac{(\sin \theta)^{d-1}}{Z_d} d\theta\), \(Z_d := \sqrt{\pi} \frac{\Gamma(d)}{\Gamma(d+1/2)}\): \(\forall \nu \in H^1([0, \pi], d\sigma)\)

\[
\frac{p-2}{d} \int_0^\pi |\nu'(\theta)|^2 \, d\sigma + \int_0^\pi |\nu(\theta)|^2 \, d\sigma \geq \left( \int_0^\pi |\nu(\theta)|^p \, d\sigma \right)^{\frac{2}{p}}
\]

- Change of variables \(z = \cos \theta, \nu(\theta) = f(z)\)

\[
\frac{p-2}{d} \int_{-1}^1 |f'|^2 \nu \, d\nu_d + \int_{-1}^1 |f|^2 \, d\nu_d \geq \left( \int_{-1}^1 |f|^p \, d\nu_d \right)^{\frac{2}{p}}
\]

where \(\nu_d(z) \, dz = d\nu_d(z) := Z_d^{-1} \nu^{d-1} \, dz\), \(\nu(z) := 1 - z^2\)
The ultraspherical operator

With $d\nu_d = Z_d^{-1} \nu^\frac{d}{2} - 1 \, dz$, $\nu(z) := 1 - z^2$, consider the space $L^2((-1, 1), d\nu_d)$ with scalar product

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1 f_2 \, d\nu_d, \quad \|f\|_p = \left( \int_{-1}^{1} f^p \, d\nu_d \right)^{\frac{1}{p}}$$

The self-adjoint ultraspherical operator is

$$\mathcal{L} f := (1 - z^2) f'' - dz f' = \nu f'' + \frac{d}{2} \nu f'$$

which satisfies $\langle f_1, \mathcal{L} f_2 \rangle = -\int_{-1}^{1} f_1' f_2' \nu \, d\nu_d$

**Proposition**

*Let* $p \in [1, 2) \cup (2, 2^*)$, $d \geq 1$

$$-\langle f, \mathcal{L} f \rangle = \int_{-1}^{1} |f'|^2 \nu \, d\nu_d \geq d \frac{\|f\|_p^2 - \|f\|_2^2}{p - 2} \quad \forall f \in H^1([-1, 1], d\nu_d)$$
Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents
Heat flow and the Bakry-Emery method

With $g = f^p$, i.e. $f = g^\alpha$ with $\alpha = 1/p$

\[
\langle f, \mathcal{L} f \rangle = -\langle g^\alpha, \mathcal{L} g^\alpha \rangle =: \mathcal{I}[g] \geq d \frac{\|g\|_1^{2\alpha} - \|g^2\|_1^{2\alpha}}{p-2} =: \mathcal{F}[g]
\]

Heat flow

\[
\frac{\partial g}{\partial t} = \mathcal{L} g
\]

\[
\frac{d}{dt} \|g\|_1 = 0, \quad \frac{d}{dt} \|g^{2\alpha}\|_1 = -2 (p-2) \langle f, \mathcal{L} f \rangle = 2 (p-2) \int_{-1}^{1} |f'|^2 \nu \, d\nu_d
\]

which finally gives

\[
\frac{d}{dt} \mathcal{F}[g(t, \cdot)] = - \frac{d}{p-2} \frac{d}{dt} \|g^{2\alpha}\|_1 = -2 d \mathcal{I}[g(t, \cdot)]
\]

Ineq. $\iff \frac{d}{dt} \mathcal{F}[g(t, \cdot)] \leq -2 d \mathcal{F}[g(t, \cdot)] \iff \frac{d}{dt} \mathcal{I}[g(t, \cdot)] \leq -2 d \mathcal{I}[g(t, \cdot)]$
The equation for $g = f^p$ can be rewritten in terms of $f$ as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + (p - 1) \frac{|f'|^2}{f} \nu$$

$$- \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \, d\nu = \frac{1}{2} \frac{d}{dt} \langle f, \mathcal{L} f \rangle = \langle \mathcal{L} f, \mathcal{L} f \rangle + (p - 1) \langle \frac{|f'|^2}{f} \nu, \mathcal{L} f \rangle$$

$$\frac{d}{dt} \mathcal{I}[g(t, \cdot)] + 2d \mathcal{I}[g(t, \cdot)] = \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \, d\nu + 2d \int_{-1}^{1} |f'|^2 \nu \, d\nu$$

$$= -2 \int_{-1}^{1} \left( |f''|^2 + (p - 1) \frac{d}{d+2} \frac{|f'|^4}{f^2} - 2(p - 1) \frac{d-1}{d+2} \frac{|f'|^2 f''}{f} \right) \nu^2 \, d\nu$$

is nonpositive if

$$|f''|^2 + (p - 1) \frac{d}{d+2} \frac{|f'|^4}{f^2} - 2(p - 1) \frac{d-1}{d+2} \frac{|f'|^2 f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[ (p - 1) \frac{d-1}{d+2} \right]^2 \leq (p - 1) \frac{d}{d+2} \iff p \leq \frac{2d^2 + 1}{(d - 1)^2} < \frac{2d}{d - 2} = 2^*$$
The sphere
Riemannian manifolds
The line
The Moser-Trudinger-Onofri inequality
Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

... up to the critical exponent: a proof on two slides

\[
\left[ \frac{d}{dz}, \mathcal{L} \right] u = (\mathcal{L} u)' - \mathcal{L} u' = -2z u'' - du'
\]

\[
\int_{-1}^{1} (\mathcal{L} u)^2 d\nu_d = \int_{-1}^{1} |u''|^2 \nu^2 d\nu_d + d \int_{-1}^{1} |u'|^2 \nu d\nu_d
\]

\[
\int_{-1}^{1} (\mathcal{L} u) \frac{|u'|^2}{u} \nu d\nu_d = \frac{d}{d+2} \int_{-1}^{1} \frac{|u'|^4}{u^2} \nu^2 d\nu_d - 2 \frac{d-1}{d+2} \int_{-1}^{1} \frac{|u'|^2 u''}{u} \nu^2 d\nu_d
\]

On \((-1, 1)\), let us consider the porous medium (fast diffusion) flow

\[
u_t = u^{2-2\beta} \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right)
\]

If \(\kappa = \beta (p - 2) + 1\), the \(L^p\) norm is conserved

\[
\frac{d}{dt} \int_{-1}^{1} u^\beta p d\nu_d = \beta p (\kappa - \beta (p - 2) - 1) \int_{-1}^{1} u^{\beta(p-2)} |u'|^2 \nu d\nu_d = 0
\]
\[ f = u^\beta, \quad \|f\|_{L^2(S^d)}^2 + \frac{d}{p-2} \left( \|f\|_{L^2(S^d)}^2 - \|f\|_{L^p(S^d)}^2 \right) \geq 0? \]

\[ \mathcal{A} := -\frac{1}{2 \beta^2} \frac{d}{dt} \int_{-1}^{1} \left( (u^\beta)' \right)^2 \nu + \frac{d}{p-2} (u^{2\beta} - \bar{u}^{2\beta}) \right) d\nu \\
= \int_{-1}^{1} \left( \mathcal{L} u + (\beta - 1) \frac{|u'|^2}{u} \nu \right) \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right) d\nu \\
+ \frac{d}{p-2} \frac{\kappa - 1}{\beta} \int_{-1}^{1} |u'|^2 \nu d\nu \\
= \int_{-1}^{1} |u''|^2 \nu^2 d\nu - 2 \frac{d-1}{d+2} (\kappa + \beta - 1) \int_{-1}^{1} u'' \frac{|u'|^2}{u} \nu^2 d\nu \\
+ \left[ \kappa (\beta - 1) + \frac{d}{d+2} (\kappa + \beta - 1) \right] \int_{-1}^{1} \frac{|u'|^4}{u^2} \nu^2 d\nu \\
= \int_{-1}^{1} \left| u'' - \frac{p + 2}{6-p} \frac{|u'|^2}{u} \right|^2 \nu^2 d\nu \geq 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p} \]

\( \mathcal{A} \) is nonnegative for some \( \beta \) if \( \frac{8 d^2}{(d + 2)^2} (p - 1)(2^* - p) \geq 0 \)
Which computation have we done? \( u_t = u^{2-2\beta} \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right) \)

\[-\mathcal{L} u - (\beta - 1) \frac{|u'|^2}{u} \nu + \frac{\lambda}{p-2} u = \frac{\lambda}{p-2} u^\kappa\]

Multiply by \( \mathcal{L} u \) and integrate

\[\ldots \int_{-1}^{1} \mathcal{L} u u^\kappa \, d\nu_d = -\kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \, d\nu_d\]

Multiply by \( \kappa \frac{|u'|^2}{u} \) and integrate

\[\ldots = + \kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \, d\nu_d\]

The two terms cancel and we are left only with the two-homogenous terms
Spectral consequences

A quantitative deviation with respect to the semi-classical regime
Some references (2/2)

Consider the Schrödinger operator $H = -\Delta - V$ on $\mathbb{R}^d$ and denote by $(\lambda_k)_{k \geq 1}$ its eigenvalues.

- **Euclidean case** [Keller, 1961]

  $$|\lambda_1|^{\gamma} \leq L_{\gamma,d}^1 \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}$$

  [Lieb-Thirring, 1976]

  $$\sum_{k \geq 1} |\lambda_k|^{\gamma} \leq L_{\gamma,d} \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}$$

$\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ if $d \geq 3$ [Weidl], [Cwikel], [Rosenbljum], [Aizenman], [Laptev-Weidl], [Helffer], [Robert], [Dolbeault-Felmer-Loss-Paturel]... [Dolbeault-Laptev-Loss 2008]

- **Compact manifolds: log Sobolev case** [Federbusch], [Rothaus]; case $\gamma = 0$ (Rozenbljum-Lieb-Cwikel inequality): [Levin-Solomyak]; [Lieb], [Levin], [Ouabaz-Poupaud]... [Ilyin]
Lemma (Dolbeault-Esteban-Laptev)

Let \( q \in (2, 2^*) \). Then there exists a concave increasing function \( \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) with the following properties

\[
\mu(\alpha) = \alpha \quad \forall \alpha \in [0, \frac{d}{q-2}] \quad \text{and} \quad \mu(\alpha) < \alpha \quad \forall \alpha \in \left(\frac{d}{q-2}, +\infty\right)
\]

\[
\mu(\alpha) = \mu_{\text{asymp}}(\alpha) \left(1 + o(1)\right) \quad \text{as} \quad \alpha \to +\infty, \quad \mu_{\text{asymp}}(\alpha) := \frac{K_{q,d}}{\kappa_{q,d}} \alpha^{1-\vartheta}
\]

such that

\[
\| \nabla u \|_{L^2(S^d)}^2 + \alpha \| u \|_{L^2(S^d)}^2 \geq \mu(\alpha) \| u \|_{L^q(S^d)}^2 \quad \forall u \in H^1(S^d)
\]

If \( d \geq 3 \) and \( q = 2^* \), the inequality holds with \( \mu(\alpha) = \min \{ \alpha, \alpha_* \} \), \( \alpha_* := \frac{1}{4} d (d - 2) \)
\( \mu_{\text{asym}}(\alpha) := \frac{K_{q,d}}{\kappa_{q,d}} \alpha^{1-\vartheta}, \vartheta := d \frac{q-2}{2q} \) corresponds to the semi-classical regime and \( K_{q,d} \) is the optimal constant in the Euclidean Gagliardo-Nirenberg-Sobolev inequality

\[
K_{q,d} \| \nabla v \|_{L^q(\mathbb{R}^d)}^2 \leq \| \nabla v \|_{L^2(\mathbb{R}^d)}^2 + \| v \|_{L^2(\mathbb{R}^d)}^2 \quad \forall v \in H^1(\mathbb{R}^d)
\]

Let \( \varphi \) be a non-trivial eigenfunction of the Laplace-Beltrami operator corresponding the first nonzero eigenvalue

\[
-\Delta \varphi = d \varphi
\]

Consider \( u = 1 + \varepsilon \varphi \) as \( \varepsilon \to 0 \) Taylor expand \( Q_\alpha \) around \( u = 1 \)

\[
\mu(\alpha) \leq Q_\alpha [1 + \varepsilon \varphi] = \alpha + [d + \alpha (2 - q)] \varepsilon^2 \int_{S^d} |\varphi|^2 \, d\nu_g + o(\varepsilon^2)
\]

By taking \( \varepsilon \) small enough, we get \( \mu(\alpha) < \alpha \) for all \( \alpha > d/(q - 2) \)

Optimizing on the value of \( \varepsilon > 0 \) (not necessarily small) provides an interesting test function...
The sphere
Riemannian manifolds
The line
The Moser-Trudinger-Onofri inequality
Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

\[ \mu = \mu(\alpha) \]

\[ \mu = \mu_{\text{asymp}}(\alpha) \]

\[ \mu = \mu_{\pm}(\alpha) \]

\[ \mu = \alpha \]
Consider the Schrödinger operator $-\Delta - V$ and the energy

$$
E[u] := \int_{S^d} |\nabla u|^2 - \int_{S^d} V |u|^2
\geq \int_{S^d} |\nabla u|^2 - \mu \|u\|_{L^q(S^d)}^2 \geq -\alpha(\mu) \|u\|_{L^2(S^d)}^2 \quad \text{if } \mu = \|V_+\|_{L^p(S^d)}
$$

**Theorem (Dolbeault-Esteban-Laptev)**

Let $d \geq 1$, $p \in \left( \max\{1, d/2\}, +\infty \right)$. Then there exists a convex increasing function $\alpha$ s.t. $\alpha(\mu) = \mu$ if $\mu \in \left[0, \frac{d}{2} (p-1)\right]$ and $\alpha(\mu) > \mu$ if $\mu \in \left( \frac{d}{2} (p-1), +\infty \right)$

$$
|\lambda_1(-\Delta - V)| \leq \alpha\left(\|V\|_{L^p(S^d)}\right) \quad \forall V \in L^p(S^d)
$$

For large values of $\mu$, we have $\alpha(\mu)^{p- \frac{d}{2}} = L^1_{p- \frac{d}{2},d} (\kappa_{q,d, \mu})^p \left(1 + o(1)\right)$ and the above estimate is optimal.

If $p = d/2$ and $d \geq 3$, the inequality holds with $\alpha(\mu) = \mu$ iff $\mu \in [0, \alpha_\ast]$.
A Keller-Lieb-Thirring inequality

Corollary (Dolbeault-Esteban-Laptev)

Let $d \geq 1, \gamma = p - d/2$

$$|\lambda_1(-\Delta - V)|^\gamma \lesssim L_{\gamma,d}^1 \int_{\mathbb{S}^d} V^{\gamma + \frac{d}{2}}$$

as $\mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{S}^d)} \to \infty$

if either $\gamma > \max\{0, 1 - d/2\}$ or $\gamma = 1/2$ and $d = 1$

However, if $\mu = \|V\|_{L^{\gamma + \frac{d}{2}}(\mathbb{S}^d)} \leq \frac{1}{4} d (2\gamma + d - 2)$, then we have

$$|\lambda_1(-\Delta - V)|^{\gamma + \frac{d}{2}} \leq \int_{\mathbb{S}^d} V^{\gamma + \frac{d}{2}}$$

for any $\gamma \geq \max\{0, 1 - d/2\}$ and this estimate is optimal

$L_{\gamma,d}^1$ is the optimal constant in the Euclidean one bound state ineq.

$$|\lambda_1(-\Delta - \phi)|^\gamma \leq L_{\gamma,d}^1 \int_{\mathbb{R}^d} \phi^{\gamma + \frac{d}{2}} \, dx$$
Another interpolation inequality (II)

Let $d \geq 1$ and $\gamma > d/2$ and assume that $L^1_{-\gamma,d}$ is the optimal constant in

$$\lambda_1(-\Delta + \phi)^{-\gamma} \leq L^1_{-\gamma,d} \int_{\mathbb{R}^d} \phi^{\frac{d}{2} - \gamma} \, dx$$

$$q = 2 \frac{2\gamma - d}{2\gamma - d + 2} \quad \text{and} \quad p = \frac{q}{2 - q} = \gamma - \frac{d}{2}$$

**Theorem (Dolbeault-Esteban-Laptev)**

$$\left(\lambda_1(-\Delta + W)\right)^{-\gamma} \lesssim L^1_{-\gamma,d} \int_{\mathbb{S}^d} W^{\frac{d}{2} - \gamma} \quad \text{as} \quad \beta = \|W^{-1}\|_{L^\gamma - \frac{d}{2} (\mathbb{S}^d)} \rightarrow \infty$$

However, if $\gamma \geq \frac{d}{2} + 1$ and $\beta = \|W^{-1}\|_{L^\gamma - \frac{d}{2} (\mathbb{S}^d)} \leq \frac{1}{4} d (2\gamma - d + 2)$

$$(\lambda_1(-\Delta + W))^{\frac{d}{2} - \gamma} \leq \int_{\mathbb{S}^d} W^{\frac{d}{2} - \gamma}$$

*and this estimate is optimal*
\( K_{q,d}^* \) is the optimal constant in the Gagliardo-Nirenberg-Sobolev inequality

\[
K_{q,d}^* \| v \|_{L^2(\mathbb{R}^d)}^2 \leq \| \nabla v \|_{L^2(\mathbb{R}^d)}^2 + \| v \|_{L^q(\mathbb{R}^d)}^2 \quad \forall \, v \in H^1(\mathbb{R}^d)
\]

and \( \mathcal{L}^1_{-\gamma,d} := \left( K_{q,d}^* \right)^{-\gamma} \) with \( q = 2 \frac{2 \gamma - d}{2 \gamma - d + 2} \), \( \delta := \frac{2q}{2d - q(d-2)} \),

**Lemma (Dolbeault-Esteban-Laptev)**

Let \( q \in (0, 2) \) and \( d \geq 1 \). There exists a concave increasing function \( \nu \)

\[
\nu(\beta) \leq \beta \quad \forall \, \beta > 0 \quad \text{and} \quad \nu(\beta) < \beta \quad \forall \, \beta \in \left( \frac{d}{2-q}, +\infty \right)
\]

\[
\nu(\beta) = \beta \quad \forall \, \beta \in \left[ 0, \frac{d}{2-q} \right] \quad \text{if} \quad q \in [1, 2)
\]

\[
\nu(\beta) = K_{q,d}^* (\kappa_{q,d} \beta)^\delta (1 + o(1)) \quad \text{as} \quad \beta \to +\infty
\]

such that

\[
\| \nabla u \|_{L^2(S^d)}^2 + \beta \| u \|_{L^q(S^d)}^2 \geq \nu(\beta) \| u \|_{L^2(S^d)}^2 \quad \forall \, u \in H^1(S^d)
\]
The threshold case: \( q = 2 \)

**Lemma (Dolbeault-Esteban-Laptev)**

Let \( p > \max\{1, d/2\} \). There exists a concave nondecreasing function \( \xi \)

\[
\xi(\alpha) = \alpha \quad \forall \alpha \in (0, \alpha_0) \quad \text{and} \quad \xi(\alpha) < \alpha \quad \forall \alpha > \alpha_0
\]

for some \( \alpha_0 \in \left[\frac{d}{2} (p - 1), \frac{d}{2} p\right] \), and \( \xi(\alpha) \sim \alpha^{1 - \frac{d}{2p}} \) as \( \alpha \to +\infty \)

such that, for any \( u \in H^1(S^d) \) with \( \|u\|_{L^2(S^d)} = 1 \)

\[
\int_{S^d} |u|^2 \log |u|^2 \, dv_g + p \log \left(\frac{\xi(\alpha)}{\alpha}\right) \leq p \log \left(1 + \frac{1}{\alpha} \|\nabla u\|_{L^2(S^d)}^2\right)
\]

**Corollary (Dolbeault-Esteban-Laptev)**

\[
e^{-\frac{\lambda_1(-\Delta - W)}{\alpha}} \leq \frac{\alpha}{\xi(\alpha)} \left(\int_{S^d} e^{-\frac{p W}{\alpha}} \, dv_g\right)^{1/p}
\]
Improvements of the inequalities (subcritical range)

as long as the exponent is either in the range $(1, 2)$ or in the range $(2, 2^*)$, one can establish improved inequalities

[Dolbeault-Esteban-Kowalczyk-Loss]
What does “improvement” mean?

An improved inequality is

\[
d \|u\|_{L^2(S^d)}^2 \Phi \left( \frac{e}{\|u\|_{L^2(S^d)}^2} \right) \leq i \quad \forall u \in H^1(S^d)
\]

for some function \( \Phi \) such that \( \Phi(0) = 0, \Phi'(0) = 1, \Phi' > 0 \) and \( \Phi(s) > s \) for any \( s \). With \( \Psi(s) := s - \Phi^{-1}(s) \)

\[
i - de \geq d \|u\|_{L^2(S^d)}^2 (\Psi \circ \Phi) \left( \frac{e}{\|u\|_{L^2(S^d)}^2} \right) \quad \forall u \in H^1(S^d)
\]

Lemma (Generalized Csiszár-Kullback inequalities)

\[
\|\nabla u\|_{L^2(S^d)}^2 - \frac{d}{p-2} \left[ \|u\|_{L^p(S^d)}^2 - \|u\|_{L^2(S^d)}^2 \right] \\
\geq d \|u\|_{L^2(S^d)}^2 (\Psi \circ \Phi) \left( C \frac{\|u\|_{L^2(S^d)}^2}{\|u\|_{L^2(S^d)}^2} \|u^r - \bar{u}^r\|_{L^q(S^d)}^2 \right) \quad \forall u \in H^1(S^d)
\]

\( s(p) := \max\{2, p\} \) and \( p \in (1, 2): q(p) := 2/p, r(p) := p; p \in (2, 4): q = p/2, r = 2; p \geq 4: q = p/(p-2), r = p-2 \)
Linear flow: improved Bakry-Emery method

Cf. [Arnold, JD]

\[ w_t = \mathcal{L} w + \kappa \frac{|w'|^2}{w} \nu \]

With \( 2^\# := \frac{2d^2+1}{(d-1)^2} \)

\[ \gamma_1 := \left( \frac{d-1}{d+2} \right)^2 (p-1) (2^\# - p) \quad \text{if} \quad d > 1, \quad \gamma_1 := \frac{p-1}{3} \quad \text{if} \quad d = 1 \]

If \( p \in [1, 2) \cup (2, 2^\#] \) and \( w \) is a solution, then

\[ \frac{d}{dt} (i - d e) \leq -\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2} \, d\nu_d \leq -\gamma_1 \frac{|e'|^2}{1 - (p - 2)e} \]

Recalling that \( e' = -i \), we get a differential inequality

\[ e'' + d e' \geq \gamma_1 \frac{|e'|^2}{1 - (p - 2)e} \]

After integration: \( d \Phi(e(0)) \leq i(0) \)
Nonlinear flow: the Hölder estimate

\[ w_t = w^{2-2\beta} \left( \mathcal{L} w + \kappa \frac{|w'|^2}{w} \right) \]

For all \( p \in [1, 2^*] \), \( \kappa = \beta (p - 2) + 1 \), \( \frac{d}{dt} \int_{-1}^{1} w^{\beta p} \, d\nu_d = 0 \)

\[ -\frac{1}{2\beta^2} \frac{d}{dt} \int_{-1}^{1} \left( (w^{\beta})'|^2 \nu + \frac{d}{p-2} (w^{2\beta} - \overline{w^{2\beta}}) \right) \, d\nu_d \geq \gamma \int_{-1}^{1} \frac{|w'|^4}{w^2} \nu^2 \, d\nu_d \]

**Lemma**

*For all \( w \in H^1((-1, 1), d\nu_d) \), such that \( \int_{-1}^{1} w^{\beta p} \, d\nu_d = 1 \)*

\[ \int_{-1}^{1} \frac{|w'|^4}{w^2} \nu^2 \, d\nu_d \geq \frac{1}{\beta^2} \int_{-1}^{1} (w^{\beta})'|^2 \nu \, d\nu_d \int_{-1}^{1} |w'|^2 \nu \, d\nu_d \frac{1}{\left( \int_{-1}^{1} w^{2\beta} \, d\nu_d \right)^{\delta}} \]

*... but there are conditions on \( \beta \)*
Admissible \((p, \beta)\) for \(d = 1, 2\)
Admissible \((p, \beta)\) for \(d = 3, 4\)
Admissible \((p, \beta)\) for \(d = 5, 10\)
no sign is required on the Ricci tensor and an improved integral criterion is established

the flow explores the energy landscape... and shows the non-optimality of the improved criterion
Riemannian manifolds with positive curvature

$(\mathcal{M}, g)$ is a smooth compact connected Riemannian manifold dimension $d$, no boundary, $\Delta_g$ is the Laplace-Beltrami operator $\text{vol} (\mathcal{M}) = 1$, $\mathcal{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1 (-\Delta_g)$

$$\rho := \inf_{\mathcal{M}} \inf_{\xi \in S^{d-1}} \mathcal{R}(\xi, \xi)$$

**Theorem (Licois-Véron, Bakry-Ledoux)**

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1 - \theta) \lambda_1 + \theta \frac{d \rho}{d - 1} \quad \text{where} \quad \theta = \frac{(d - 1)^2 (p - 1)}{d (d + 2) + p - 1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$- \Delta_g v + \frac{\lambda}{p - 2} (v - v^{p-1}) = 0$$

has a unique positive solution $v \in C^2(\mathcal{M})$: $v \equiv 1$
Riemannian manifolds: first improvement

**Theorem (Dolbeault-Esteban-Loss)**

For any \( p \in (1, 2) \cup (2, 2^*) \)

\[
0 < \lambda < \lambda_* = \inf_{u \in H^2(\mathcal{M})} \frac{\int_\mathcal{M} \left[ (1 - \theta)(\Delta_g u)^2 + \frac{\theta d}{d - 1} \mathfrak{R}(\nabla u, \nabla u) \right] d v_g}{\int_\mathcal{M} |\nabla u|^2 d v_g}
\]

there is a unique positive solution in \( C^2(\mathcal{M}) \): \( u \equiv 1 \)

\[
\lim_{p \to 1^+} \theta(p) = 0 \implies \lim_{p \to 1^+} \lambda_*(p) = \lambda_1 \text{ if } \rho \text{ is bounded}
\]

\[
\lambda_* = \lambda_1 = d \frac{\rho}{(d - 1)} = d \text{ if } \mathcal{M} = S^d \text{ since } \rho = d - 1
\]

\[
(1 - \theta) \lambda_1 + \theta \frac{d \rho}{d - 1} \leq \lambda_* \leq \lambda_1
\]
Riemannian manifolds: second improvement

H\_g u denotes Hessian of u and \( \theta = \frac{(d - 1)^2 (p - 1)}{d (d + 2) + p - 1} \)

\[
Q\_g u := H\_g u - \frac{g}{d} \Delta\_g u - \frac{(d - 1) (p - 1)}{\theta (d + 3 - p)} \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right]
\]

\[
\Lambda^* := \inf_{u \in H^2(M) \setminus \{0\}} \frac{(1 - \theta) \int_M (\Delta u)^2 \, dv_g + \frac{\theta d}{d - 1} \int_M \left[ \|Q_g u\|^2 + \mathcal{K}(\nabla u, \nabla u) \right]}{\int_M |\nabla u|^2 \, dv_g}
\]

Theorem (Dolbeault-Esteban-Loss)

Assume that \( \Lambda^* > 0 \). For any \( p \in (1, 2) \cup (2, 2^*) \), the equation has a unique positive solution in \( C^2(M) \) if \( \lambda \in (0, \Lambda^*) \): \( u \equiv 1 \)
Optimal interpolation inequality

For any $p \in (1, 2) \cup (2, 2^*)$ or $p = 2^*$ if $d \geq 3$

\[
\| \nabla v \|_{L^2(M)}^2 \geq \frac{\lambda}{p-2} \left[ \| v \|_{L^p(M)}^2 - \| v \|_{L^2(M)}^2 \right] \quad \forall v \in H^1(M)
\]

**Theorem (Dolbeault-Esteban-Loss)**

Assume $\Lambda_* > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_*, \lambda_1]$

If $\Lambda_* < \lambda_1$, then the optimal constant $\Lambda$ is such that

\[\Lambda_* < \Lambda \leq \lambda_1\]

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \varepsilon \varphi$ as a test function where $\varphi$ we get $\lambda \leq \lambda_1$

A minimum of

\[
v \mapsto \| \nabla v \|_{L^2(M)}^2 - \frac{\lambda}{p-2} \left[ \| v \|_{L^p(M)}^2 - \| v \|_{L^2(M)}^2 \right]
\]

under the constraint $\| v \|_{L^p(M)} = 1$ is negative if $\lambda > \lambda_1$
The sphere
Riemannian manifolds
The line
The Moser-Trudinger-Onofri inequality
Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

The flow

The key tools the flow

\[ u_t = u^{2-2\beta} \left( \Delta_g u + \kappa \frac{\nabla u}{u} \right), \quad \kappa = 1 + \beta (p - 2) \]

If \( v = u^\beta \), then \( \frac{d}{dt} \| v \|_{L^p(M)} = 0 \) and the functional

\[ F[u] := \int_M |\nabla (u^\beta)|^2 d\nu_g + \frac{\lambda}{p-2} \left[ \int_M u^{2\beta} d\nu_g - \left( \int_M u^{\beta p} d\nu_g \right)^{2/p} \right] \]

is monotone decaying

Elementary observations (1/2)

Let \( d \geq 2 \), \( u \in C^2(\mathcal{M}) \), and consider the trace free Hessian

\[
L_g u := H_g u - \frac{g}{d} \Delta_g u
\]

Lemma

\[
\int_{\mathcal{M}} (\Delta_g u)^2 \, dv_g = \frac{d}{d-1} \int_{\mathcal{M}} \| L_g u \|^2 \, dv_g + \frac{d}{d-1} \int_{\mathcal{M}} \mathfrak{R}(\nabla u, \nabla u) \, dv_g
\]

Based on the Bochner-Lichnerovicz-Weitzenböck formula

\[
\frac{1}{2} \Delta |\nabla u|^2 = \| H_g u \|^2 + \nabla (\Delta_g u) \cdot \nabla u + \mathfrak{R}(\nabla u, \nabla u)
\]
Elementary observations (2/2)

Lemma

\[
\int_{\mathcal{M}} \Delta_g u \frac{\left| \nabla u \right|^2}{u} \, d\nu_g = \frac{d}{d + 2} \int_{\mathcal{M}} \frac{\left| \nabla u \right|^4}{u^2} \, d\nu_g - \frac{2d}{d + 2} \int_{\mathcal{M}} \left[ L_g u : \left[ \frac{\nabla u \otimes \nabla u}{u} \right] \right] \, d\nu_g
\]

Lemma

\[
\int_{\mathcal{M}} (\Delta_g u)^2 \, d\nu_g \geq \lambda_1 \int_{\mathcal{M}} \left| \nabla u \right|^2 \, d\nu_g \quad \forall \, u \in H^2(\mathcal{M})
\]

and \( \lambda_1 \) is the optimal constant in the above inequality
The sphere
Riemannian manifolds
The line
The Moser-Trudinger-Onofri inequality
Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

The key estimates

\[ G[u] := \int_{\mathcal{M}} \left[ \theta (\Delta_g u)^2 + (\kappa + \beta - 1) \Delta_g u \frac{|\nabla u|^2}{u} + \kappa (\beta - 1) \frac{|\nabla u|^4}{u^2} \right] d v_g \]

**Lemma**

\[ \frac{1}{2 \beta^2} \frac{d}{dt} \mathcal{F}[u] = -(1 - \theta) \int_{\mathcal{M}} (\Delta_g u)^2 d v_g - G[u] + \lambda \int_{\mathcal{M}} |\nabla u|^2 d v_g \]

\[ Q_{\theta}^u := L_g u - \frac{1}{\theta} \frac{d-1}{d+2} (\kappa + \beta - 1) \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right] \]

**Lemma**

\[ G[u] = \frac{\theta d}{d-1} \left[ \int_{\mathcal{M}} \|Q_{\theta}^u\|^2 d v_g + \int_{\mathcal{M}} \mathcal{K}(\nabla u, \nabla u) d v_g \right] - \mu \int_{\mathcal{M}} \frac{|\nabla u|^4}{u^2} d v_g \]

with \( \mu := \frac{1}{\theta} \left( \frac{d-1}{d+2} \right)^2 (\kappa + \beta - 1)^2 - \kappa (\beta - 1) - (\kappa + \beta - 1) \frac{d}{d+2} \)
Assume that $d \geq 2$. If $\theta = 1$, then $\mu$ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall \, p \in (1, 2^*)$$

where $\beta_\pm := \frac{b\pm\sqrt{b^2-a}}{2a}$ with $a = 2 - p + \left[\frac{(d-1)(p-1)}{d+2}\right]^2$ and $b = \frac{d+3-p}{d+2}$

Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2(p-1)}{d(d+2)+p-1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

**Proposition**

Let $d \geq 2$, $p \in (1, 2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2\beta^2} \frac{d}{dt} F[u] \leq (\lambda - \Lambda_*) \int_M |\nabla u|^2 \, dv_g$$
The line
One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

\[ \| f \|_{L^p(\mathbb{R})} \leq C_{GN}(p) \| f' \|_{L^2(\mathbb{R})}^{\theta} \| f \|_{L^2(\mathbb{R})}^{1-\theta} \quad \text{if } p \in (2, \infty) \]

\[ \| f \|_{L^2(\mathbb{R})} \leq C_{GN}(p) \| f' \|_{L^2(\mathbb{R})}^{\eta} \| f \|_{L^p(\mathbb{R})}^{1-\eta} \quad \text{if } p \in (1, 2) \]

with \( \theta = \frac{p-2}{2p} \) and \( \eta = \frac{2-p}{2+p} \)

The threshold case corresponding to the limit as \( p \to 2 \) is the logarithmic Sobolev inequality

\[ \int_{\mathbb{R}} u^2 \log \left( \frac{u^2}{\| u \|_{L^2(\mathbb{R})}^2} \right) \, dx \leq \frac{1}{2} \| u \|_{L^2(\mathbb{R})}^2 \log \left( \frac{2}{\pi e} \frac{\| u' \|_{L^2(\mathbb{R})}^2}{\| u \|_{L^2(\mathbb{R})}^2} \right) \]

If \( p > 2 \), \( u_\star(x) = (\cosh x)^{-\frac{2}{p-2}} \) solves

\[ - (p-2)^2 u'' + 4 u - 2 p |u|^{p-2} u = 0 \]

If \( p \in (1, 2) \) consider \( u_\star(x) = (\cos x)^{\frac{2}{2-p}} \), \( x \in (-\pi/2, \pi/2) \)
Mass transportation

**Theorem (Dolbeault-Esteban-Laptev-Loss)**

*If* $p \in (2, \infty)$, *we have*

$$
\sup_G \frac{\int \nabla G \cdot \nabla^{\frac{p+2}{3p-2}} \, dy}{\left( \int \nabla G \cdot \nabla^{\frac{p-2}{3p-2}} \, dy \right)^{\frac{2(p-2)}{3p-2}} \left( \int \nabla G \, dy \right)^{\frac{4}{3p-2}}} = c_p \inf_f \frac{\| f' \|_{L^2(\mathbb{R})} \cdot \| f \|_{L^2(\mathbb{R})}^{\frac{2(p+2)}{3p-2}}}{\| f \|_{L^p(\mathbb{R})}^{\frac{2(p-2)}{3p-2}}}
$$

*and if* $p \in (1, 2)$, *we obtain*

$$
\sup_G \frac{\int \nabla G \cdot \nabla^{\frac{2}{4-p}} \, dy}{\left( \int \nabla G \cdot \nabla^{\frac{2-p}{2(4-p)}} \, dy \right)^{\frac{2-p}{2(4-p)}} \left( \int \nabla G \, dy \right)^{\frac{p+2}{2(4-p)}}} = c_p \inf_f \frac{\| f' \|_{L^2(\mathbb{R})} \cdot \| f \|_{L^p(\mathbb{R})}^{\frac{2}{4-p}}}{\| f \|_{L^p(\mathbb{R})}^{\frac{p+2}{4-p}}}
$$

*for some explicit numerical constant* $c_p$. 

---

*J. Dolbeault*  
Interpolation inequalities: rigidity results, nonlinear flows and improved inequalities
Flow

Let us define on $H^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|_{L^2(\mathbb{R})}^2 + \frac{4}{(p-2)^2} \|v\|_{L^2(\mathbb{R})}^2 - C \|v\|_{L^p(\mathbb{R})}^2 \quad \text{s.t. } \mathcal{F}[u_\star] = 0$$

With $z(x) := \tanh x$, consider the flow

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1 - z^2}} \left[ v'' + \frac{2p}{p-2} z v' + \frac{p}{2} \frac{|v'|^2}{v} + \frac{2}{p-2} v \right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2, \infty)$. Then

$$\frac{d}{dt} \mathcal{F}[v(t)] \leq 0 \quad \text{and} \quad \lim_{t \to \infty} \mathcal{F}[v(t)] = 0$$

$$\frac{d}{dt} \mathcal{F}[v(t)] = 0 \quad \iff \quad v_0(x) = u_\star(x - x_0)$$

Similar result for $p \in (1, 2)$

J. Dolbeault

Interpolation inequalities: rigidity results, nonlinear flows and improved inequalities
The inequality \((p > 2)\) and the ultraspherical operator

---

The problem on the line is equivalent to the critical problem for the ultraspherical operator

\[
\int_{\mathbb{R}} |v'|^2 \, dx + \frac{4}{(p-2)^2} \int_{\mathbb{R}} |v|^2 \, dx \geq C \left( \int_{\mathbb{R}} |v|^p \, dx \right)^{\frac{2}{p}}
\]

With

\[z(x) = \tanh x, \quad v_\star = (1 - z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\star(x) f(z(x))\]

equality is achieved for \(f = 1\) and, if we let \(\nu(z) := 1 - z^2\), then

\[
\int_{-1}^{1} |f'|^2 \nu \, d\nu_d + \frac{2p}{(p-2)^2} \int_{-1}^{1} |f|^2 \, d\nu_d \geq \frac{2p}{(p-2)^2} \left( \int_{-1}^{1} |f|^p \, d\nu_d \right)^{\frac{2}{p}}
\]

where \(d\nu_p\) denotes the probability measure \(d\nu_p(z) := \frac{1}{\zeta_p} \nu^{\frac{2}{p-2}} \, dz\)

\[
d = \frac{2p}{p-2} \quad \iff \quad p = \frac{2d}{d-2}
\]

Change of variables = stereographic projection + Emden-Fowler
The Moser-Trudinger-Onofri inequality

Joint work with Maria J. Esteban and G. Jankowiak
Three equivalent forms

- The Euclidean (Moser-Trudinger-)Onofri inequality:
  \[
  \frac{1}{16 \pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \log \left( \int_{\mathbb{R}^2} e^u \, d\mu \right) - \int_{\mathbb{R}^2} u \, d\mu
  \]
  \[d\mu = \mu(x) \, dx, \mu(x) = \frac{1}{\pi} \left( 1 + |x|^2 \right)^{-2}, x \in \mathbb{R}^2\]

- The Onofri inequality on the two-dimensional sphere \( S^2 \):
  \[
  \frac{1}{4} \int_{S^2} |\nabla v|^2 \, d\sigma \geq \log \left( \int_{S^2} e^v \, d\sigma \right) - \int_{S^2} v \, d\sigma
  \]
  \[d\sigma \text{ is the uniform probability measure}\]

- The Onofri inequality on the two-dimensional cylinder \( C = S^1 \times \mathbb{R} \):
  \[
  \frac{1}{16 \pi} \int_{C} |\nabla w|^2 \, dy \geq \log \left( \int_{C} e^w \, \nu \, dy \right) - \int_{C} w \, \nu \, dy
  \]
  \[y = (\theta, s) \in C = S^1 \times \mathbb{R}, \nu(y) = \frac{1}{4\pi} (\cosh s)^{-2}\]

[Moser (1971)], [Onofri (1982)]
The inequality seen as a limit case of the Gagliardo-Nirenberg inequalities

**Proposition**

[JD] Assume that \( u \in \mathcal{D}(\mathbb{R}^2) \) is such that \( \int_{\mathbb{R}^2} u \, d\mu = 0 \) and let

\[
f_p := F_p \left( 1 + \frac{u}{2p} \right), \quad F_p(x) = (1 + |x|^2)^{-\frac{1}{p-1}} \quad \forall \, x \in \mathbb{R}^2
\]

Then we have

\[
1 \leq \lim_{p \to \infty} C_{p,2} \frac{\| \nabla f_p \|_{L^2(\mathbb{R}^2)}^{\theta(p)} \| f_p \|_{L^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\| f_p \|_{L^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\pi}} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx}{\int_{\mathbb{R}^2} e^u \, d\mu}
\]
Rigidity method in the symmetric case

Under an appropriate normalization, a critical point of

\[ G_\lambda[f] := \frac{1}{8} \int_{-1}^{1} |f'|^2 \, \nu \, dz + \frac{\lambda}{2} \int_{-1}^{1} f \, dz \geq \log \left( \frac{1}{2} \int_{-1}^{1} e^f \, dz \right) \]

solves the Euler-Lagrange equation

\[ -\frac{1}{2} \mathcal{L}f + \lambda = e^f \]

**Theorem**

For any \( \lambda \in (0, 1) \), the EL equation has a unique smooth solution \( f = \log \lambda \). If \( \lambda = 1 \), \( f \) has to satisfy the differential equation \( f'' = \frac{1}{2} |f'|^2 \)

and is either a constant or

\[ f(z) = C_1 - 2 \log(C_2 - z) \]

\[ \frac{1}{8} \int_{-1}^{1} \nu^2 \left| f'' - \frac{1}{2} |f'|^2 \right|^2 e^{-f/2} \nu \, dz + \frac{1 - \lambda}{4} \int_{-1}^{1} \nu \left| f' \right|^2 e^{-f/2} \nu \, dz = 0 \]
Rigidity method in the symmetric case: proof

Multiply by $\mathcal{L}(e^{-f/2})$ and integrate by parts

$$0 = \int_{-1}^{1} \left( \frac{1}{2} \mathcal{L}f + \lambda - e^f \right) \mathcal{L}(e^{-f/2}) \, \nu \, dz$$

$$= \frac{1}{4} \int_{-1}^{1} \nu^2 |f''|^2 e^{-f/2} \, \nu \, dz - \frac{1}{8} \int_{-1}^{1} \nu^2 |f'|^2 f'' e^{-f/2} \, \nu \, dz$$

$$+ \frac{1}{2} \int_{-1}^{1} \nu |f'|^2 e^{-f/2} \, \nu \, dz - \frac{1}{2} \int_{-1}^{1} \nu |f'|^2 e^{f/2} \, \nu \, dz$$

Multiply by $\frac{\nu}{2} |f'|^2 e^{-f/2}$ and integrate by parts

$$0 = \int_{-1}^{1} \left( \frac{1}{2} \mathcal{L}f + \lambda - e^f \right) \left( \frac{\nu}{2} |f'|^2 e^{-f/2} \right) \, \nu \, dz$$

$$= \frac{1}{8} \int_{-1}^{1} \nu^2 |f'|^2 f'' e^{-f/2} \, \nu \, dz - \frac{1}{16} \int_{-1}^{1} \nu^2 |f'|^4 e^{-f/2} \, \nu \, dz$$

$$+ \frac{\lambda}{2} \int_{-1}^{1} \nu |f'|^2 e^{-f/2} \, \nu \, dz - \frac{1}{2} \int_{-1}^{1} \nu |f'|^2 e^{f/2} \, \nu \, dz$$
A nonlinear flow method in the general case

On $\mathbb{S}^2$ let us consider the nonlinear evolution equation

$$\frac{\partial f}{\partial t} = \Delta_{\mathbb{S}^2} (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2}$$

where $\Delta_{\mathbb{S}^2}$ denotes the Laplace-Beltrami operator. Let us define

$$\mathcal{R}_\lambda[f] := \frac{1}{2} \int_{\mathbb{S}^2} \| L_{\mathbb{S}^2} f - \frac{1}{2} M_{\mathbb{S}^2} f \|^2 e^{-f/2} \ d\sigma + \frac{1}{2} (1 - \lambda) \int_{\mathbb{S}^2} |\nabla f|^2 e^{-f/2} \ d\sigma$$

where

$$L_{\mathbb{S}^2} f := \text{Hess}_{\mathbb{S}^2} f - \frac{1}{2} \Delta_{\mathbb{S}^2} f \text{ Id} \quad \text{and} \quad M_{\mathbb{S}^2} f := \nabla f \otimes \nabla f - \frac{1}{2} |\nabla f|^2 \text{ Id}$$

**Theorem**

Assume that $f$ is a solution to with initial datum $v - \log \left( \int_{\mathbb{S}^2} e^v \ d\sigma \right)$, where $v \in L^1(\mathbb{S}^2)$ is such that $\nabla v \in L^2(\mathbb{S}^2)$. Then for any $\lambda \in (0, 1]$ we have

$$\mathcal{G}_\lambda[v] \geq \int_0^\infty \mathcal{R}_\lambda[f(t, \cdot)] \ dt$$
The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban
We shall also denote by $\mathcal{R}$ the Ricci tensor, by $H_g u$ the Hessian of $u$ and by

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

the trace free Hessian. Let us denote by $M_g u$ the trace free tensor

$$M_g u := \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2$$

We define

$$\lambda_* := \inf_{u \in H^2(M) \setminus \{0\}} \frac{\int_M \left[ \| L_g u - \frac{1}{2} M_g u \|^2 + \mathcal{R}(\nabla u, \nabla u) \right] e^{-u/2} d\nu_g}{\int_M |\nabla u|^2 e^{-u/2} d\nu_g}$$
Theorem

Assume that $d = 2$ and $\lambda_* > 0$. If $u$ is a smooth solution to

$$-\frac{1}{2} \Delta_g u + \lambda = e^u$$

then $u$ is a constant function if $\lambda \in (0, \lambda_*)$

The Moser-Trudinger-Onofri inequality on $\mathcal{M}$

$$\frac{1}{4} \| \nabla u \|_{L^2(\mathcal{M})}^2 + \lambda \int_{\mathcal{M}} u \, d\nu_g \geq \lambda \log \left( \int_{\mathcal{M}} e^u \, d\nu_g \right) \quad \forall u \in H^1(\mathcal{M})$$

for some constant $\lambda > 0$. Let us denote by $\lambda_1$ the first positive eigenvalue of $-\Delta_g$

Corollary

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4 \pi, \lambda_*\}$. Moreover, if $\Lambda$ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than $\Lambda$
The flow

\[
\frac{\partial f}{\partial t} = \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2}
\]

\[
G_\lambda[f] := \int_M \| L_g f - \frac{1}{2} M_g f \|^2 e^{-f/2} d\nu_g + \int_M \mathcal{H}(\nabla f, \nabla f) e^{-f/2} d\nu_g - \lambda \int_M |\nabla f|^2 e^{-f/2} d\nu_g
\]

Then for any \( \lambda \leq \lambda_* \) we have

\[
\frac{d}{dt} F_\lambda[f(t, \cdot)] = \int_M \left( -\frac{1}{2} \Delta_g f + \lambda \right) \left( \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2} \right) d\nu_g = -G_\lambda[f(t, \cdot)]
\]

Since \( F_\lambda \) is nonnegative and \( \lim_{t \to \infty} F_\lambda[f(t, \cdot)] = 0 \), we obtain that

\[
F_\lambda[u] \geq \int_0^\infty G_\lambda[f(t, \cdot)] dt
\]
Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure $\mu$ does the inequality

$$\frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \lambda \left[ \log \left( \int_{\mathbb{R}^2} e^u \, d\mu \right) - \int_{\mathbb{R}^2} u \, d\mu \right]$$

hold for some $\lambda > 0$? Let

$$\Lambda_* := \inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8\pi\mu}$$

**Theorem**

Assume that $\mu$ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_*$ and the inequality holds with $\lambda = \Lambda_*$ if equality is achieved among radial functions.
Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Joint work with G. Jankowiak
Preliminary observations
Legendre duality: Onofri and log HLS

Legendre's duality: \( F^*[v] := \sup \left( \int_{\mathbb{R}^d} u \, v \, dx - F[u] \right) \)

\[ F_1[u] := \log \left( \int_{\mathbb{R}^2} e^u \, d\mu \right), \quad F_2[u] := \frac{1}{16 \pi} \int_0^{\infty} |\nabla u|^2 \, r^{d-1} \, dr + \int_0^{\infty} u \, \mu \, r^{d-1} \, dr \]

Onofri's inequality amounts to \( F_1[u] \leq F_2[u] \) with \( d\mu(x) := \mu(x) \, dx \), \( \mu(x) := \frac{1}{\pi (1+|x|^2)^2} \)

Proposition

*For any \( v \in L^1_+(\mathbb{R}^2) \) with \( \int_0^{\infty} v \, r^{d-1} \, dr = 1 \), such that \( v \log v \) and \( (1 + \log |x|^2) \, v \in L^1(\mathbb{R}^2) \), we have

\[
F_1^*[v] - F_2^*[v] = \int_0^{\infty} v \log \left( \frac{v}{\mu} \right) \, r^{d-1} \, dr - 4 \pi \int_0^{\infty} (v - \mu)(-\Delta)^{-1}(v - \mu) \, r^{d-1} \, dr \geq 0
\]

[E. Carlen, M. Loss] [W. Beckner] [V. Calvez, L. Corrias]
A puzzling result of E. Carlen, J.A. Carrillo and M. Loss

[E. Carlen, J.A. Carrillo and M. Loss] The fast diffusion equation
\[ \frac{\partial v}{\partial t} = \Delta v^m \quad t > 0 , \quad x \in \mathbb{R}^d \]
with exponent \( m = d/(d + 2) \), when \( d \geq 3 \), is such that
\[ H_d[v] := \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \| v \|_{L^{2d/(d+2)}(S^d)}^2 \]
obesys to
\[ \frac{1}{2} \frac{d}{dt} H_d[v(t, \cdot)] = \frac{1}{2} \frac{d}{dt} \left[ \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \| v \|_{L^{2d/(d+2)}(S^d)}^2 \right] \]
\[ = \frac{d(d-2)}{(d-1)^2} S_d \| u \|_{L^{4/(d-1)}(S^d)}^{4/(d-1)} \| \nabla u \|_{L^2(S^d)}^2 - \| u \|_{L^{2q}(S^d)}^{2q} \]
with \( u = v^{(d-1)/(d+2)} \) and \( q = \frac{d+1}{d-1} \). If \( \frac{d(d-2)}{(d-1)^2} S_d = (Cq, d)^{2q} \), the r.h.s. is nonnegative. Optimality is achieved simultaneously in both functionals (Barenblatt regime): the Hardy-Littlewood-Sobolev inequalities can be improved by an integral remainder term.
... and the two-dimensional case

Recall that \((-\Delta)^{-1} v = G_d * v\) with
- \(G_d(x) = \frac{1}{d-2} |S^{d-1}|^{-1} |x|^{2-d}\) if \(d \geq 3\)
- \(G_2(x) = \frac{1}{2\pi} \log |x|\) if \(d = 2\)

Same computation in dimension \(d = 2\) with \(m = 1/2\) gives

\[
\frac{\|v\|_{L^1(\mathbb{R}^2)}}{8} \frac{d}{dt} \left[ \frac{4\pi}{\|v\|_{L^1(\mathbb{R}^2)}} \int_0^\infty v (-\Delta)^{-1} v r^{d-1} dr - \int_0^\infty v \log v r^{d-1} dr \right]
= \|u\|_{L^4(\mathbb{R}^2)}^4 \|\nabla u\|_{L^2(\mathbb{R}^2)}^2 - \pi \|v\|_{L^6(\mathbb{R}^2)}^6
\]

The r.h.s. is one of the Gagliardo-Nirenberg inequalities \((d = 2, q = 3): \pi (C3, 2)^6 = 1\)

The l.h.s. is bounded from below by the logarithmic Hardy-Littlewood-Sobolev inequality and achieves its minimum if \(v = \mu\) with

\[
\mu(x) := \frac{1}{\pi (1 + |x|^2)^2} \quad \forall x \in \mathbb{R}^2
\]
As it has been noticed by E. Lieb, Sobolev’s inequality in $\mathbb{R}^d$, $d \geq 3$,
\[
\|u\|_{L^{2^*}(S^d)}^2 \leq S_d \|\nabla u\|_{L^2(S^d)}^2 \quad \forall \ u \in D^{1,2}(\mathbb{R}^d)
\] (1)

and the Hardy-Littlewood-Sobolev inequality
\[
S_d \|v\|_{L^{\frac{2d}{d+2}}(S^d)}^2 \geq \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx \quad \forall \ v \in L^{\frac{2d}{d+2}}(\mathbb{R}^d)
\] (2)

are dual of each other. Here $S_d$ is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$. Can we recover this using a nonlinear flow approach? Can we improve it?

Using the Yamabe / Ricci flow
Using a nonlinear flow to relate Sobolev and HLS

Consider the fast diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0, \quad x \in \mathbb{R}^d$$

(3)

If we define $H(t) := H_d[v(t, \cdot)]$, with

$$H_d[v] := \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \| v \|_{L^{\frac{2d}{d+2}}(S^d)}^2$$

then we observe that

$$\frac{1}{2} H' = - \int_{\mathbb{R}^d} v^{m+1} \, dx + S_d \left( \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}} \, dx \right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}} \, dx$$

where $v = v(t, \cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m + 1 = \frac{2d}{d+2}$
A first statement

Proposition

[JD] Assume that \( d \geq 3 \) and \( m = \frac{d-2}{d+2} \). If \( v \) is a solution of (3) with nonnegative initial datum in \( L^{2d/(d+2)}(\mathbb{R}^d) \), then

\[
\frac{1}{2} \frac{d}{dt} \left[ \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx - S_d \| v \|_{L^{2d/(d+2)}(\mathbb{S}^d)}^2 \right] = \left( \int_{\mathbb{R}^d} v^{m+1} \, dx \right)^{\frac{2}{d}} \left[ S_d \| \nabla u \|_{L^2(\mathbb{S}^d)}^2 - \| u \|_{L^{2^*}(\mathbb{S}^d)}^2 \right] \geq 0
\]

The HLS inequality amounts to \( H \leq 0 \) and appears as a consequence of Sobolev, that is \( H' \geq 0 \) if we show that \( \limsup_{t \to 0} H(t) = 0 \). Notice that \( u = v^m \) is an optimal function for (1) if \( v \) is optimal for (2).
Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons.

**Theorem**

[JD] Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $C \leq (1 + \frac{2}{d}) \left(1 - e^{-d/2}\right) S_d$ such that

$$S_d \| w^q \|^{2}_{L^{\frac{2d}{d+2}}(S^d)} - \int_{\mathbb{R}^d} w^q (-\Delta)^{-1} w^q \, dx$$

$$\leq C \| w \|_{L^{\frac{8}{d-2}}(S^d)} \left[ \| \nabla w \|^{2}_{L^2(S^d)} - S_d \| w \|^{2}_{L^{2*}(S^d)} \right]$$

for any $w \in D^{1,2}(\mathbb{R}^d)$.
Solutions with separation of variables

Consider the solution of \( \frac{\partial v}{\partial t} = \Delta v^m \) vanishing at \( t = T \):
\[
\bar{v}_T(t, x) = c (T - t)^\alpha (F(x))^{\frac{d+2}{d-2}}
\]
where \( F \) is the Aubin-Talenti solution of
\[
-\Delta F = d (d - 2) F^{(d+2)/(d-2)}
\]
Let \( \|v\|_* := \sup_{x \in \mathbb{R}^d} (1 + |x|^2)^{d+2} |v(x)| \)

Lemma

[M. del Pino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodriguez]
For any solution \( v \) with initial datum \( v_0 \in L^{2d/(d+2)}(\mathbb{R}^d) \), \( v_0 > 0 \), there exists \( T > 0 \), \( \lambda > 0 \) and \( x_0 \in \mathbb{R}^d \) such that
\[
\lim_{t \to T^-} (T - t)^{-\frac{1}{1-m}} \|v(t, \cdot)/\bar{v}(t, \cdot) - 1\|_* = 0
\]
with \( \bar{v}(t, x) = \lambda^{(d+2)/2} \bar{v}_T(t, (x - x_0)/\lambda) \)
Improved inequality: proof (1/2)

The function $J(t) := \int_{\mathbb{R}^d} v(t, x)^{m+1} \, dx$ satisfies

$$J' = -(m+1) \| \nabla v^m \|_{L^2(S^d)}^2 \leq -\frac{m+1}{S_d} J^{1 - \frac{2}{d}}$$

If $d \geq 5$, then we also have

$$J'' = 2m(m+1) \int_{\mathbb{R}^d} v^{m-1} (\Delta v^m)^2 \, dx \geq 0$$

Notice that

$$\frac{J'}{J} \leq -\frac{m+1}{S_d} J^{-\frac{2}{d}} \leq -\kappa \quad \text{with} \quad \kappa T = \frac{2d}{d+2} \frac{T}{S_d} \left( \int_{\mathbb{R}^d} v_0^{m+1} \, dx \right)^{-\frac{2}{d}} \leq \frac{d}{2}$$
Improved inequality: proof (2/2)

By the Cauchy-Schwarz inequality, we have

\[
\frac{J'}{2} = \left\| \nabla v^m \right\|_{L^2(S^d)}^4 = \left( \int_{\mathbb{R}^d} v^{(m-1)/2} \Delta v^m \cdot v^{(m+1)/2} \, dx \right)^2 \leq \int_{\mathbb{R}^d} v^{m-1} (\Delta v^m)^2 \, dx \int_{\mathbb{R}^d} v^{m+1} \, dx = \text{Cst} \, J'' \, J
\]

so that \( Q(t) := \left\| \nabla v^m(t, \cdot) \right\|_{L^2(S^d)}^2 \left( \int_{\mathbb{R}^d} v^{m+1}(t, x) \, dx \right)^{-(d-2)/d} \) is monotone decreasing, and

\[
H' = 2 J (S_d Q - 1) , \quad H'' = \frac{J'}{J} H' + 2 J S_d Q' \leq \frac{J'}{J} H' \leq 0
\]

\[
H'' \leq -\kappa H' \quad \text{with} \quad \kappa = \frac{2 d}{d + 2} \frac{1}{S_d} \left( \int_{\mathbb{R}^d} v_0^{m+1} \, dx \right)^{-2/d}
\]

By writing that \(-H(0) = H(T) - H(0) \leq H'(0) (1 - e^{-\kappa T}) / \kappa \) and using the estimate \( \kappa T \leq d/2 \), the proof is completed \( \square \)
\[d = 2: \text{Onofri's and log HLS inequalities}\]

\[H_2[v] := \int_0^\infty (v - \mu) (-\Delta)^{-1}(v - \mu) \, r^{d-1} \, dr - \frac{1}{4 \pi} \int_0^\infty v \log \left(\frac{v}{\mu}\right) r^{d-1} \, dr\]

With \(\mu(x) := \frac{1}{\pi} (1 + |x|^2)^{-2}\). Assume that \(v\) is a positive solution of

\[\frac{\partial v}{\partial t} = \Delta \log \left(\frac{v}{\mu}\right) \quad t > 0, \quad x \in \mathbb{R}^2\]

**Proposition**

If \(v = \mu e^{u/2}\) is a solution with nonnegative initial datum \(v_0\) in \(L^1(\mathbb{R}^2)\) such that \(\int_0^\infty v_0 \, r^{d-1} \, dr = 1, v_0 \log v_0 \in L^1(\mathbb{R}^2)\) and \(v_0 \log \mu \in L^1(\mathbb{R}^2)\), then

\[\frac{d}{dt} H_2[v(t, \cdot)] = \frac{1}{16 \pi} \int_0^\infty |\nabla u|^2 \, r^{d-1} \, dr - \int_{\mathbb{R}^2} \left(\frac{e^u}{2} - 1\right) u \, d\mu\]

\[\geq \frac{1}{16 \pi} \int_0^\infty |\nabla u|^2 \, r^{d-1} \, dr + \int_{\mathbb{R}^2} u \, d\mu - \log \left(\int_{\mathbb{R}^2} e^u \, d\mu\right) \geq 0\]
Improvements
Theorem

[JD, G. Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $C \leq 1$ such that

$$S_d \| w^q \|_{L^{\frac{2d}{d+2}}(\mathbb{S}^d)}^2 - \int_{\mathbb{R}^d} w^q (-\Delta)^{-1} w^q \, dx$$

$$\leq C S_d \| w \|_{L^{\frac{8d}{d-2}}(\mathbb{S}^d)} \left[ \| \nabla w \|_{L^2(\mathbb{S}^d)}^2 - S_d \| w \|_{L^{2^*}(\mathbb{S}^d)}^2 \right]$$

for any $w \in D^{1,2}(\mathbb{R}^d)$
Proof: the completion of a square

Integrations by parts show that
\[ \int_{\mathbb{R}^d} |\nabla (-\Delta)^{-1} v|^2 \, dx = \int_{\mathbb{R}^d} v (-\Delta)^{-1} v \, dx \]
and, if \( v = u^q \) with \( q = \frac{d+2}{d-2} \),
\[ \int_{\mathbb{R}^d} \nabla u \cdot \nabla (-\Delta)^{-1} v \, dx = \int_{\mathbb{R}^d} u v \, dx = \int_{\mathbb{R}^d} u^{2*} \, dx \]
Hence the expansion of the square
\[ 0 \leq \int_{\mathbb{R}^d} \left| S_d \left\| u^{\frac{4}{d-2}}_{L^2(\mathbb{S}^d)} \right\| \nabla u - \nabla (-\Delta)^{-1} v \right|^2 \, dx \]
shows that
\[ 0 \leq S_d \left\| u^{\frac{8}{d-2}}_{L^2(\mathbb{S}^d)} \right\| S_d \left\| \nabla u \right\|_{L^2(\mathbb{S}^d)}^2 - \left\| u \right\|_{L^{2*}(\mathbb{S}^d)}^2 \]
\[ - \left[ S_d \left\| u^q \right\|_{L^{\frac{2d}{d+2}}(\mathbb{S}^d)}^2 - \int_{\mathbb{R}^d} u^q (-\Delta)^{-1} u^q \, dx \right] \]
The equality case

Equality is achieved if and only if

$$S_d \|u\|_{L_{2*}(\mathbb{S}^d)}^{\frac{4}{d-2}} u = (-\Delta)^{-1} v = (-\Delta)^{-1} u^q$$

that is, if and only if $u$ solves

$$-\Delta u = \frac{1}{S_d} \|u\|_{L_{2*}(\mathbb{S}^d)}^{-\frac{4}{d-2}} u^q$$

which means that $u$ is an Aubin-Talenti extremal function

$$u_*(x) := (1 + |x|^2)^{-\frac{d-2}{2}} \quad \forall x \in \mathbb{R}^d$$
An identity

\[ 0 = S_d \left\| u \right\|_{L^2(S^d)}^{\frac{8}{d-2}} \left[ S_d \left( \left\| \nabla u \right\|_{L^2(S^d)}^2 - \left\| u \right\|_{L^2(S^d)}^2 \right) \right] \\
\quad - \left[ S_d \left\| u^q \right\|_{L^{\frac{2d}{d+2}}(S^d)}^2 - \int_{\mathbb{R}^d} u^q (-\Delta)^{-1} u^q \, dx \right] \\
\quad - \int_{\mathbb{R}^d} \left( S_d \left\| u \right\|_{L^2(S^d)}^{\frac{4}{d-2}} \nabla u - \nabla (-\Delta)^{-1} u^q \right)^2 \, dx \]
Another improvement

\[ J_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}} \, dx \quad \text{and} \quad H_d[v] := \int_{\mathbb{R}^d} v (-\Delta)^{-1} \, dx - S_d \|v\|_{L^2(S^d)}^{\frac{2d}{d+2}} \]

**Theorem**

*Assume that* \( d \geq 3 \). *Then we have*

\[
0 \leq H_d[v] + S_d J_d[v]^{\frac{2}{d+2}} \varphi \left( J_d[v]^{\frac{2}{d+1}} \left[ S_d \|\nabla u\|_2^2(S^d) - \|u\|_{L^2(S^d)}^2 \right] \right) \\
\forall u \in \mathcal{D}^{1,2}(\mathbb{R}^d), \quad v = u^{\frac{d+2}{2}}
\]

*where* \( \varphi(x) := \sqrt{C^2 + 2Cx} - C \) *for any* \( x \geq 0 \)

**Proof:** \( H(t) = -Y(J(t)) \quad \forall \, t \in [0, T), \quad \kappa_0 := \frac{H_0}{J_0} \) *and consider the differential inequality*

\[
Y' \left( C S_d s^{1+\frac{2}{d}} + Y \right) \leq \frac{d+2}{2} C \kappa_0 S_d^2 s^{1+\frac{4}{d}}, \quad Y(0) = 0, \quad Y(J_0) = -H_0
\]
... but $C = 1$ is not optimal

**Theorem**

[JD, G. Jankowiak] *In the inequality*

$$S_d \|w^q\|_{L^{\frac{2d}{d+2}}(S^d)}^2 - \int_{\mathbb{R}^d} w^q (-\Delta)^{-1} w^q \, dx$$

$$\leq C S_d \|w\|_{L^{\frac{8}{d-2}}(S^d)}^\frac{8}{d-2} \left[ \|\nabla w\|_{L^2(S^d)}^2 - S_d \|w\|_{L^{2^*}(S^d)}^2 \right]$$

we have

$$\frac{d}{d+4} \leq C_d < 1$$

based on a (painful) linearization like the one used by Bianchi and Egnell

- **Extensions:** magnetic Laplacian [JD, Esteban, Laptev] or fractional Laplacian operator [Jankowiak, Nguyen]
Theorem

Assume that $d = 2$. The inequality

$$
\int_{\mathbb{R}^2} g \log \left( \frac{g}{M} \right) \, dx - \frac{4 \pi}{M} \int_{\mathbb{R}^2} g \, (-\Delta)^{-1} g \, dx + M \left( 1 + \log \pi \right)
$$

$$
\leq M \left[ \frac{1}{16 \pi} \| \nabla f \|_{L^2(S^d)}^2 + \int_{\mathbb{R}^2} f \, d\mu - \log M \right]
$$

holds for any function $f \in \mathcal{D}(\mathbb{R}^2)$ such that $M = \int_{\mathbb{R}^2} e^f \, d\mu$ and $g = \pi \, e^f \, \mu$

Recall that

$$
\mu(x) := \frac{1}{\pi (1 + |x|^2)^2} \quad \forall \, x \in \mathbb{R}^2
$$
A summary
the sphere: the flow tells us what to do, and provides a simple proof (choice of the exponents / of the nonlinearity) once the problem is reduced to the ultraspherical setting

the spectral point of view on the inequality: how to measure the deviation with respect to the semi-classical estimates, a nice example of bifurcation (and symmetry breaking)

Riemannian manifolds: no sign is required on the Ricci tensor and an improved integral criterion is established. We extend the theory from pointwise criteria to a non-local Schrödinger type estimate (Rayleigh quotient). The flow explores the energy landscape... and generically shows the non-optimality of the improved criterion

the flow is a nice way of exploring an energy space. Rigidity result tell you that a local result is actually global because otherwise the flow would relate (far away) extremal points while keeping the energy minimal
http://www.ceremade.dauphine.fr/~dolbeaul
▷ Preprints (or arxiv, or HAL)


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- J.D., Maria J. Esteban, Gaspard Jankowiak. The Moser-Trudinger-Onofri inequality, Preprint, 2014
- J.D., Maria J. Esteban, Gaspard Jankowiak. Rigidity results for semilinear elliptic equation with exponential nonlinearities and Moser-Trudinger-Onofri inequalities on two-dimensional Riemannian manifolds, Preprint, 2014
These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures
Thank you for your attention!