Onofri type inequalities and diffusions

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Outline

1. Onofri's inequality as an endpoint of Gagliardo-Nirenberg inequalities [M. del Pino, JD]
2. Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows [JD]
3. Keller-Segel model, a functional analysis approach [J. Campos, JD]

A – Onofri's inequality as an endpoint of Gagliardo-Nirenberg inequalities

The fast diffusion equation

Consider the fast diffusion equation (FDE)

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

with exponent $m \in (\frac{d-1}{d}, 1)$, $d \geq 3$, or its Fokker-Planck version

$$\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot (x\, u) \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

with $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d))$ such that $u_0^m \in \mathrm{L}^1_+(\mathbb{R}^d))$ and $|x|^2\, u_0 \in \mathrm{L}^1_+(\mathbb{R}^d))$

Any solution converges as $t \to \infty$ to the *Barenblatt profile*

$$u_\infty(x) = \left(C_M + \tfrac{1-m}{2m}\, |x|^2\right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

Asymptotic behaviour of the solutions of FDE

[J. Ralston, W.I. Newman] Define the relative entropy (or free energy) by

$$\mathcal{F}[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - u_\infty^m - m\, u_\infty^{m-1}(u - u_\infty) \right] \, dx$$

$$\tfrac{d}{dt}\mathcal{F}[u(t,\cdot)] = -\left(\tfrac{m}{m-1}\right)^2 \int_{\mathbb{R}^d} u \, |\nabla u^{m-1} - \nabla u_\infty^{m-1}|^2 \, dx =: -\mathcal{I}[u(t,\cdot)]$$

$$\mathcal{F}[u(t,\cdot)] \leq \tfrac{1}{2}\, \mathcal{I}[u(t,\cdot)]$$

if m is in the range $(\frac{d-1}{d}, 1)$, thus showing that

$$\mathcal{F}[u(t,\cdot)] \leq \mathcal{F}[u_0]\, e^{-2t} \quad \forall\, t \geq 0$$

With $p = \frac{1}{2m-1}$, the inequality $\mathcal{F}[u] \leq \frac{1}{2}\,\mathcal{I}[u]$ can be rewritten in terms of $f = u^{m-1/2}$ as

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \leq \mathsf{C}_{p,d}\, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\, \|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

$f_\infty = u_\infty^{m-1/2}$ is optimal [M. del Pino, JD] [F. Otto] [D. Cordero-Erausquin, B. Nazaret, C. Villani]

Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathsf{C}_{p,d}\,\|\nabla f\|^{\theta}_{\mathrm{L}^2(\mathbb{R}^d)}\,\|f\|^{1-\theta}_{\mathrm{L}^{p+1}(\mathbb{R}^d)}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$, $p = \frac{1}{2m-1}$

- $1 < p \le \frac{d}{d-2}$ if $d \ge 3$, $\frac{d-1}{d} \le m < 1$
- $1 < p < \infty$ if $d = 2$, $\frac{1}{2} < m < 1$

[M. del Pino, JD] equality holds in if $f = F_p$ with

$$F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

and that all extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling.

- If $d \ge 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality [T. Aubin, G. Talenti]
- When $p \to 1$, we recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form [F. Weissler]
- If $d = 2$ and $p \to \infty$...

$d = 2$, $m = 1/2$: the limit case

The basin of attraction of the Brenblatt self-similar profiles
[A. Blanchet, M. Bonforte, JD, G. Grillo, J.-L. Vázquez]
The fast diffusion equation (FDE)

$$\frac{\partial v}{\partial t} = \Delta \sqrt{v} \quad t > 0\,, \quad x \in \mathbb{R}^2$$

can be transformed into a Fokker-Planck version

$$\frac{\partial u}{\partial t} = \Delta \sqrt{u} + \nabla \cdot (x\, u) \quad t > 0\,, \quad x \in \mathbb{R}^2$$

with $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d))$ such that $\sqrt{u_0} \in \mathrm{L}^1_+(\mathbb{R}^d))$ and $|x|^2\, u_0 \in \mathrm{L}^1_+(\mathbb{R}^d))$

Any solution converges as $t \to \infty$ to the *Barenblatt profile*
$u_\infty(x) = \left(C_M + \frac{1}{2}\,|x|^2\right)^{-2}$ but the inequality $\mathcal{F}[u(t,\cdot)] \le C\,\mathcal{I}[u(t,\cdot)]$ holds with $C = \frac{1}{2}$ only as $t \to \infty$: $u = u_\infty\,(1 + \varepsilon\, f\, \sqrt{u_\infty})$, $\varepsilon \to 0$ gives

$$\mathcal{F}[u] \sim \frac{1}{4} \int_{\mathbb{R}^2} |f|^2\, u_\infty^{\frac{3}{2}}\, dx \le \frac{1}{2}\, \frac{1}{4} \int_{\mathbb{R}^2} |\nabla f|^2\, u_\infty\, dx \sim \frac{1}{2}\, \mathcal{I}[u]$$

$d = 2$, $m = 1/2$: the linearized equation

Consider the scalar product $\langle \cdot, \cdot \rangle$ such that $\langle f, f \rangle = \int_{\mathbb{R}^2} \frac{|f|^2}{(1+|x|^2)^3}\,dx$
The linearized fast diffusion equation (ℓFDE) takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L} f$$

where $\mathcal{L} f := (1+|x|^2)^3\, \nabla \left[\frac{\nabla f}{(1+|x|^2)^2} \right]$ defines self-adjoint on $L^2(\mathbb{R}^2, (1+|x|^2)^{-3}\,dx)$. A solution of

$$\frac{d}{dt} \langle f, f \rangle = -\langle \mathcal{L} f, f \rangle$$

has exponential decay because of the Hardy-Poincaré inequality

$$\int_{\mathbb{R}^2} \frac{|f|^2}{(1+|x|^2)^3}\,dx = \langle f, f \rangle \le \frac{1}{4} \langle \mathcal{L} f, f \rangle = \int_{\mathbb{R}^2} \frac{|\nabla f|^2}{(1+|x|^2)^3}\,dx$$

[J. Denzler, R. McCann], [A. Blanchet, M. Bonforte, JD, G. Grillo, J.-L. Vázquez]

... but not anymore in the framework of global functional inequalities

Back to global estimates: $d=2$, $m\in(1/2,1)$ or $d\ge 3$, $m\in(m_1,1)$

The fast diffusion equation (FDE) with $m\ge m_1:=(d-1)/d$

$$\frac{\partial v}{\partial t}=\frac{1}{m}\,\Delta v^m\quad t>0\,,\quad x\in\mathbb{R}^d$$

can be rewritten in terms of $w=v^{2\,p}$, $p=\frac{1}{2m-1}$ as

$$\frac{\partial}{\partial t}\left(w^{2\,p}\right)=\frac{p+1}{2\,p}\,\Delta w^{p+1}\quad t>0\,,\quad x\in\mathbb{R}^d$$

Using $\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\le \mathsf{C}_{p,d}\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^\theta\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$, we get

$$\frac{d}{dt}\int_{\mathbb{R}^2}w^{p+1}\,dx=\frac{p^2-1}{4\,p}\,(p+1)\int_{\mathbb{R}^2}|\nabla w|^2\,dx\ge C\left(\int_{\mathbb{R}^2}w^{p+1}\,dx\right)^{-\frac{1-\theta}{\theta}}$$

an estimate for which Barenblatt solutions are optimal (no need of rescaling here). What can be done for $d=2$, $m=1/2$?

Onofri's inequality as a limit case

When $d=2$, Onofri's inequality can be seen as an endpoint case of the family of the Gagliardo-Nirenberg inequalities [JD]

Proposition

[JD] *Assume that* $g \in \mathcal{D}(\mathbb{R}^d)$ *is such that* $\int_{\mathbb{R}^2} g\, d\mu = 0$ *and let*

$$f_p := F_p \left(1 + \frac{g}{2p}\right)$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$, *and* $d\mu(x) := \mu(x)\, dx$, *we have*

$$1 \le \lim_{p\to\infty} \mathsf{C}_{p,2} \frac{\|\nabla f_p\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)}\, \|f_p\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f_p\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla g|^2\, dx}}{\int_{\mathbb{R}^2} e^g\, d\mu}$$

The standard form of the euclidean version of Onofri's inequality is

$$\log\left(\int_{\mathbb{R}^2} e^g\, d\mu\right) - \int_{\mathbb{R}^2} g\, d\mu \le \frac{1}{16\,\pi} \int_{\mathbb{R}^2} |\nabla g|^2\, dx$$

Some details on the proof

$$\frac{\int_{\mathbb{R}^2}|f|^{2p}\,dx}{\int_{\mathbb{R}^2}|F_p|^{2p}\,dx}\le\left(\frac{\int_{\mathbb{R}^2}|\nabla f|^2\,dx}{\int_{\mathbb{R}^2}|\nabla F_p|^2\,dx}\right)^{\frac{p-1}{2}}\frac{\int_{\mathbb{R}^2}|f|^{p+1}\,dx}{\int_{\mathbb{R}^2}|F_p|^{p+1}\,dx}$$

$\lim_{p\to\infty}\int_{\mathbb{R}^2}|F_p|^{2p}\,dx=\int_{\mathbb{R}^2}\frac{1}{(1+|x|^2)^2}\,dx=\pi$ and with $f=F_p\,(1+g/2p)$

$$\lim_{p\to\infty}\int_{\mathbb{R}^2}|f_p|^{2p}\,dx=\int_{\mathbb{R}^2}F_p^{2p}\,(1+\tfrac{g}{2p})^{2p}\,dx=\int_{\mathbb{R}^2}\frac{e^g}{(1+|x|^2)^2}\,dx=\pi\int_{\mathbb{R}^2}e^{\,g}\,d\mu$$

$\int_{\mathbb{R}^2}|F_p|^{p+1}\,dx=(p-1)\,\pi/2$, so that $\lim_{p\to\infty}\frac{\int_{\mathbb{R}^2}|f_p|^{p+1}\,dx}{\int_{\mathbb{R}^2}|F_p|^{p+1}\,dx}=1$

Expansion of the square with $\int_{\mathbb{R}^2}g\,d\mu=0$ gives

$$\int_{\mathbb{R}^2}|\nabla f_p|^2\,dx=\frac{1}{4p^2}\int_{\mathbb{R}^2}F_p^2\,|\nabla g|^2\,dx-\int_{\mathbb{R}^2}(1+\tfrac{g}{2p})^2\,F_p\,\Delta F_p\,dx$$

$$=\frac{1}{4p^2}\int_{\mathbb{R}^2}|\nabla g|^2\,dx+\frac{2\pi}{p+1}+o(p^{-2})$$

$$\left(\frac{\int_{\mathbb{R}^2}|\nabla f_p|^2\,dx}{\int_{\mathbb{R}^2}|\nabla F_p|^2\,dx}\right)^{\frac{p-1}{2}}\sim\left(1+\frac{p+1}{8\pi\,p^2}\int_{\mathbb{R}^2}|\nabla g|^2\,dx\right)^{\frac{p-1}{2}}\sim e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx}$$

Comments (formal level)

In dimension $d = 2$, Onofri's inequality

$$\log\left(\int_{\mathbb{R}^2} e^{g}\, d\mu\right) - \int_{\mathbb{R}^2} g\, d\mu \le \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla g|^2\, dx$$

is the endpoint of a family of Gagliardo-Nirenberg inequalities in dimension $d = 2$, whose other endpoint is the logarithmic Sobolev inequality

To which evolution equation is it associated ? Consider

$$\frac{\partial}{\partial t}\,(e^{g}\,\mu) = \Delta\, g$$

With $g = \log(\frac{v}{\mu})$, $\mu(x) = \frac{1}{\pi}\,(1+|x|^2)^{-2}$, this can be rewritten as

$$\frac{\partial v}{\partial t} = \Delta \log\left(\frac{v}{\mu}\right) \quad t > 0\,, \quad x \in \mathbb{R}^2$$

Onofri's inequality in higher dimensions

[E. Carlen, M. Loss]
[W. Beckner]
[M. del Pino, JD]

Higher dimensions: Gagliardo-Nirenberg inequalities

Theorem (M. del Pino, JD)

Let $p \in (1, d]$, $a > 1$ *such that* $a \le \frac{p\,(d-1)}{d-p}$ *if* $p < d$, *and* $b = p\,\frac{a-1}{p-1}$
For any function $f \in \mathrm{L}^a(\mathbb{R}^d, dx)$ *with* $\nabla f \in \mathrm{L}^p(\mathbb{R}^d, dx)$, *if* $a > p$

$$\|f\|_{\mathrm{L}^b(\mathbb{R}^2)} \le \mathsf{C}_{p,a}\,\|\nabla f\|^{\theta}_{\mathrm{L}^p(\mathbb{R}^2)}\,\|f\|^{1-\theta}_{\mathrm{L}^a(\mathbb{R}^2)} \quad \text{with } \theta = \frac{(a-p)\,d}{(a-1)\,(d\,p-(d-p)\,a)}$$

and, if $a < p$,

$$\|f\|_{\mathrm{L}^a(\mathbb{R}^2)} \le \mathsf{C}_{p,a}\,\|\nabla f\|^{\theta}_{\mathrm{L}^p(\mathbb{R}^2)}\,\|f\|^{1-\theta}_{\mathrm{L}^b(\mathbb{R}^2)} \quad \text{with } \theta = \frac{(p-a)\,d}{a\,(d\,(p-a)+p\,(a-1))}$$

In both cases, equality holds for any function taking the form

$$f(x) = A\left(1 + B\,|x - x_0|^{\frac{p}{p-1}}\right)_+^{-\frac{p-1}{a-p}} \quad \forall\, x \in \mathbb{R}^d$$

for some $(A,\ B,\ x_0) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, *where* B *has the sign of* $a - p$

Comments

$$\|f\|_{\mathrm{L}^b(\mathbb{R}^2)} \le \mathsf{C}_{p,a} \, \|\nabla f\|^{\theta}_{\mathrm{L}^p(\mathbb{R}^2)} \, \|f\|^{1-\theta}_{\mathrm{L}^a(\mathbb{R}^2)}$$

- For $a = p$, inequality degenerates into an equality: as a limit case, we get the *optimal Euclidean* L^p*-Sobolev logarithmic inequality*
For $1 < p \le d$, and any $u \in \mathrm{W}^{1,p}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |u|^p \, dx = 1$ we have

$$\int_{\mathbb{R}^d} |u|^p \log |u|^p \, dx \le \frac{d}{p} \log \left[\beta_{p,d} \int_{\mathbb{R}^d} |\nabla u|^p \, dx \right]$$

[M. del Pino, JD, I. Gentil], [D. Cordero-Erausquin]

- When $p < d$, $a = \frac{p\,(d-1)}{d-p}$ corresponds to the Sobolev inequality
- When $p = d$, we get a *d-dimensional Onofri inequality* by passing to a limit as $a \to +\infty$

Notations

On $d \geq 2$ consider the probability measure

$$d\mu_d(x) := \frac{d}{|\mathbb{S}^{d-1}|}\,\frac{dx}{\left(1+|x|^{\frac{d}{d-1}}\right)^d}$$

and the functions

$$\mathsf{R}_d(X,Y) := |X+Y|^d - |X|^d - d\,|X|^{d-2}\,X\cdot Y\;, \quad (X,Y)\in\mathbb{R}^d\times\mathbb{R}^d$$

which is a polynomial if d is even. With

$$\mathsf{H}_d(x,p) := \mathsf{R}_d\left(-\frac{d\,|x|^{-\frac{d-2}{d-1}}}{1+|x|^{\frac{d}{d-1}}}\,x,\;\frac{d-1}{d}\,p\right) \quad (x,p)\in\mathbb{R}^d\times\mathbb{R}^d$$

we define the quotient $\mathcal{Q}_d[u]$ as

$$\mathcal{Q}_d[u] := \frac{\int_{\mathbb{R}^d}\mathsf{H}_d(x,\nabla u)\,dx}{\log\left(\int_{\mathbb{R}^d}e^u\,d\mu_d\right)-\int_{\mathbb{R}^2}u\,d\mu}$$

Higher dimensions: Onofri type inequalities

Theorem (M. del Pino, JD)

Any smooth compactly supported function u satisfies

$$\log\left(\int_{\mathbb{R}^d} e^u \, d\mu_d\right) - \int_{\mathbb{R}^2} u \, d\mu \le \alpha_d \int_{\mathbb{R}^d} \mathsf{H}_d(x,\nabla u)\, dx$$

The optimal constant is $\alpha_d = \frac{d^{1-d}\,\Gamma(d/2)}{2\,(d-1)\,\pi^{d/2}}$ *and* $\lim_{\varepsilon\to 0} \mathcal{Q}_d[\varepsilon\, v] = \frac{1}{\alpha_d}$ *with*

$$v(x) = -\,d\,\frac{x\cdot e}{|x|^{\frac{d-2}{d-1}}\left(1+|x|^{\frac{d}{d-1}}\right)}$$

Example

- If $d=2$, $\int_{\mathbb{R}^d} \mathsf{H}_2(x,\nabla u)\, dx = \frac{1}{4}\int_{\mathbb{R}^2} |\nabla u|^2\, dx$, $1/\alpha_2 = 4\,\pi$
- If $d=4$, $\mathsf{H}_d(x,p) := \mathsf{R}_4\left(-\frac{4\,|x|^{-2/3}}{1+|x|^{4/3}}\,x, 3\,p/4\right)$ with

$$\mathsf{R}_4(X,Y) = 4\,(X\cdot Y)^2 + |Y|^2(|Y|^2 + 4\,X\cdot Y + 2\,|X|^2)$$

Higher dimensions: Poincaré type inequalities

$$\mathsf{G}_d(x,p) := \mathsf{Q}_d\left(-\frac{d\,|x|^{-\frac{d-2}{d-1}}}{1+|x|^{\frac{d}{d-1}}}\,x,\,\frac{d-1}{d}\,p\right)\quad (x,p)\in\mathbb{R}^d\times\mathbb{R}^d$$

Corollary

With α_d as in Theorem 2, we have

$$\int_{\mathbb{R}^d}|v-\overline{v}|^2\,d\mu_d \le \alpha_d\int_{\mathbb{R}^d}\mathsf{G}_d(x,\nabla v)\,dx\quad \textit{with}\quad \overline{v}=\int_{\mathbb{R}^2}v\,d\mu$$

for any $v\in\mathrm{L}^1(\mathbb{R}^d,d\mu_d)$ such that $\nabla v\in\mathrm{L}^2(\mathbb{R}^d,dx)$

B – Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Preliminary observations

Legendre duality: Onofri and log HLS

Legendre's duality: $F^*[v] := \sup\left(\int_{\mathbb{R}^d} u\, v\, dx - F[u]\right)$

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u\, d\mu\right) \quad \text{and} \quad F_2[u] := \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\, dx + \int_{\mathbb{R}^2} u\,\mu\, dx$$

Onofri's inequality amounts to $F_1[u] \le F_2[u]$ with $d\mu(x) := \mu(x)\, dx$, $\mu(x) := \frac{1}{\pi\,(1+|x|^2)^2}$

Proposition

For any $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ *with* $\int_{\mathbb{R}^2} v\, dx = 1$, *such that* $v \log v$ *and* $(1+\log|x|^2)\, v \in \mathrm{L}^1(\mathbb{R}^2)$, *we have*

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v \log\left(\frac{v}{\mu}\right) dx - 4\,\pi \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\, dx \ge 0$$

[E. Carlen, M. Loss] [W. Beckner] [V. Calvez, L. Corrias]

A puzzling result of E. Carlen, J.A. Carrillo and M. Loss

[E. Carlen, J.A. Carrillo and M. Loss] The fast diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with exponent $m = d/(d+2)$, when $d \geq 3$, is such that

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

obeys to

$$\frac{1}{2}\,\frac{d}{dt}\mathsf{H}_d[v(t,\cdot)] = \frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \frac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d\,\|u\|^{4/(d-1)}_{\mathrm{L}^{q+1}(\mathbb{R}^d)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^{2q}_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

with $u = v^{(d-1)/(d+2)}$ and $q = \frac{d+1}{d-1}$. If $\frac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d = (\mathsf{C}_{q,\,d})^{2q}$, the r.h.s. is nonnegative. Optimality is achieved simultaneously in both functionals (Barenblatt regime): the Hardy-Littlewood-Sobolev inequalities can be improved by an integral remainder term

... and the two-dimensional case

Recall that $(-\Delta)^{-1} v = G_d * v$ with

- $G_d(x) = \frac{1}{d-2}\,|\mathbb{S}^{d-1}|^{-1}\,|x|^{2-d}$ if $d \geq 3$
- $G_2(x) = \frac{1}{2\,\pi}\,\log|x|$ if $d = 2$

Same computation in dimension $d = 2$ with $m = 1/2$ gives

$$\frac{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}{8}\,\frac{d}{dt}\left[\frac{4\,\pi}{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}\int_{\mathbb{R}^2} v\,(-\Delta)^{-1} v\;dx - \int_{\mathbb{R}^2} v\,\log v\;dx\right]$$
$$= \|u\|^4_{\mathrm{L}^4(\mathbb{R}^2)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} - \pi\,\|v\|^6_{\mathrm{L}^6(\mathbb{R}^2)}$$

The r.h.s. is one of the Gagliardo-Nirenberg inequalities ($d = 2$, $q = 3$): $\pi\,(\mathsf{C}_{3,2})^6 = 1$
The l.h.s. is bounded from below by the logarithmic Hardy-Littlewood-Sobolev inequality and achieves its minimum if $v = \mu$ with

$$\mu(x) := \frac{1}{\pi\,(1 + |x|^2)^2} \quad \forall\, x \in \mathbb{R}^2$$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall \, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall \, v \in \mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$. Can we recover this using a nonlinear flow approach ? Can we improve it ?

Keller-Segel model: another motivation [J.A. Carrillo, E. Carlen and M. Loss] and [A. Blanchet, E. Carlen and J.A. Carrillo]

Using the Yamabe / Ricci flow

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\; dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\; dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\; dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2d}{d+2}$

A first statement

Proposition

[JD] *Assume that* $d \ge 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \ge 0$$

The HLS inequality amounts to $\mathsf{H} \le 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \ge 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

Theorem

[JD] *Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. del Pino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]
For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Improved inequality: proof (1/2)

$\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa\,T = \frac{2\,d}{d+2}\,\frac{T}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2}$$

Improved inequality: proof (2/2)

By the **Cauchy-Schwarz inequality**, we have

$$\frac{\mathsf{J}'^2}{(m+1)^2} = \|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2$$

$$\leq \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx = Cst\,\mathsf{J}''\,\mathsf{J}$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q} - 1)\,, \quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \leq \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \leq 0$$

$$\mathsf{H}'' \leq -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \leq \mathsf{H}'(0)\,(1 - e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \leq d/2$, the proof is completed □

$d = 2$: Onofri's and log HLS inequalities

$$\mathrm{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

With $\mu(x) := \frac{1}{\pi}\,(1+|x|^2)^{-2}$. Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log\,(v/\mu) \quad t > 0\,, \quad x \in \mathbb{R}^2$$

Proposition

If $v = \mu\,e^{u/2}$ is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then

$$\begin{aligned}\frac{d}{dt}\mathrm{H}_2[v(t,\cdot)] &= \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu\\ &\ge \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \int_{\mathbb{R}^2} u\,d\mu - \log\left(\int_{\mathbb{R}^2} e^u\,d\mu\right) \ge 0\end{aligned}$$

C – Keller-Segel model, a functional analysis approach

1. Introduction to the Keller-Segel model
2. Spectral analysis in the functional framework determined by the relative entropy method
3. Collecting estimates: exponential convergence

Introduction to the Keller-Segel model

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \ t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \ t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log |x - y| \, u(t,y) \, dy$$

and observe that

$$\nabla v(t,x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \, u(t,y) \, dy$$

Mass conservation: $\dfrac{d}{dt} \displaystyle\int_{\mathbb{R}^2} u(t,x) \, dx = 0$

Blow-up

$M=\int_{\mathbb{R}^2} n_0\,dx>8\pi$ and $\int_{\mathbb{R}^2}|x|^2\,n_0\,dx<\infty$: blow-up in finite time
a solution u of

$$\frac{\partial u}{\partial t}=\Delta u-\nabla\cdot(u\,\nabla v)$$

satisfies

$$\frac{d}{dt}\int_{\mathbb{R}^2}|x|^2\,u(t,x)\,dx$$

$$=-\underbrace{\int_{\mathbb{R}^2}2\,x\cdot\nabla u\,dx}_{-4\,M}+\frac{1}{2\pi}\underbrace{\iint_{\mathbb{R}^2\times\mathbb{R}^2}\frac{2x\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}_{\frac{(x-y)\cdot(y-x)}{|x-y|^2}\,u(t,x)\,u(t,y)\,dx\,dy}$$

$$=4\,M-\frac{M^2}{2\pi}<0\quad\text{if}\quad M>8\pi$$

Existence and free energy

$M = \int_{\mathbb{R}^2} n_0\,dx \le 8\pi$: global existence [W. Jäger, S. Luckhaus], [JD, B. Pertha
[A. Blanchet, JD, B. Perthame], [A. Blanchet, J.A. Carrillo, N. Masmoudi]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u\,(\nabla\,(\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u\; dx - \frac{1}{2} \int_{\mathbb{R}^2} u\, v\; dx$$

satisfies

$$\frac{d}{dt} F[u(t,\cdot)] = - \int_{\mathbb{R}^2} u\, |\nabla\,(\log u) - \nabla v|^2\; dx$$

Log HLS inequality [E. Carlen, M. Loss]: F is bounded from below if $M < 8\pi$

The dimension $d = 2$

- In dimension d, the norm $L^{d/2}(\mathbb{R}^d)$ is critical. If $d = 2$, the mass is critical
- Scale invariance: if (u, v) is a solution in $\mathbb{R}^2$ of the parabolic-elliptic Keller and Segel system, then

$$\left(\lambda^2\, u(\lambda^2\, t, \lambda\, x)\,,\ v(\lambda^2\, t, \lambda\, x)\right)$$

  is also a solution
- For $M < 8\pi$, the solution vanishes as $t \to \infty$, but saying that *diffusion dominates* is not correct: to see this, study *intermediate asymptotics*

The existence setting

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \, \nabla v) & x \in \mathbb{R}^2 \,, \; t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2 \,, \; t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2)\, dx) \,, \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx) \,, \quad M := \int_{\mathbb{R}^2} n_0(x)\, dx < 8\,\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x,t)\, dx$ for any $t \geq 0$, see [W. Jäger, S. Luckhaus], [A. Blanchet, JD, B. Perthame]

$v = -\frac{1}{2\pi} \log |\cdot| * u$

Time-dependent rescaling

$$u(x,t)=\frac{1}{R^2(t)}\,n\left(\frac{x}{R(t)},\tau(t)\right)\quad\text{and}\quad v(x,t)=c\left(\frac{x}{R(t)},\tau(t)\right)$$

with $R(t)=\sqrt{1+2t}$ and $\tau(t)=\log R(t)$

$$\begin{cases}\dfrac{\partial n}{\partial t}=\Delta n-\nabla\cdot\left(n\,(\nabla c-x)\right) & x\in\mathbb{R}^2\,,\ t>0\\ c=-\dfrac{1}{2\pi}\,\log|\cdot|*n & x\in\mathbb{R}^2\,,\ t>0\\ n(\cdot,t=0)=n_0\geq 0 & x\in\mathbb{R}^2\end{cases}$$

[A. Blanchet, JD, B. Perthame] Convergence in self-similar variables

$$\lim_{t\to\infty}\|n(\cdot,\cdot+t)-n_\infty\|_{L^1(\mathbb{R}^2)}=0\quad\text{and}\quad\lim_{t\to\infty}\|\nabla c(\cdot,\cdot+t)-\nabla c_\infty\|_{L^2(\mathbb{R}^2)}=0$$

means *intermediate asymptotics* in original variables:

$$\left\|u(x,t)-\tfrac{1}{R^2(t)}\,n_\infty\left(\tfrac{x}{R(t)},\tau(t)\right)\right\|_{L^1(\mathbb{R}^2)}\searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\;, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

- Radial symmetry [Y. Naito]
- Uniqueness [P. Biler, G. Karch, P. Laurençot, T. Nadzieja]
- As $|x| \to +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0,1)$ [A. Blanchet, JD, B. Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M:

$$\lim_{M \to 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[D.D. Joseph, T.S. Lundgren] [JD, R. Stańczy]

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla\Big[n\,(\log n - x + \nabla c)\Big]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \; dx + \int_{\mathbb{R}^2} \frac{1}{2}\,|x|^2\, n \; dx - \frac{1}{2}\int_{\mathbb{R}^2} n\, c\; dx$$

satisfies

$$\frac{d}{dt} F[n(t,\cdot)] = -\int_{\mathbb{R}^2} n\,\left|\nabla\,(\log n) + x - \nabla c\right|^2\; dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2\, n(\lambda\, x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2)\; dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2}-1}{2}\,|x|^2\, n\; dx + \frac{1}{4\pi}\int_{\mathbb{R}^2\times\mathbb{R}^2} n(x)\, n(y)\; \log\frac{1}{\lambda}\; dx\, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi}\right)}_{>0 \text{ if } M<8\pi} \log\lambda + \frac{\lambda^{-2}-1}{2}\int_{\mathbb{R}^2} |x|^2\, n\; dx$$

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \dfrac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n\,(\nabla c - x)) & x \in \mathbb{R}^2\,,\;\; t > 0 \\ c = -\dfrac{1}{2\pi}\,\log|\cdot| * n & x \in \mathbb{R}^2\,,\;\; t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t\to\infty} \|n(\cdot,\cdot+t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t\to\infty} \|\nabla c(\cdot,\cdot+t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M\,\frac{e^{\,c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2}\,dx} = -\Delta c_\infty\,, \qquad c_\infty = -\frac{1}{2\pi}\log|\cdot| * n_\infty$$

A parametrization of the solutions and the linearized operator

[J. Campos, JD]

$$-\Delta c = M \, \frac{e^{-\frac{1}{2}\,|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}\,|x|^2+c}\,dx}$$

Solve

$$-\phi'' - \frac{1}{r}\,\phi' = e^{-\frac{1}{2}\,r^2+\phi}\,, \quad r>0$$

with initial conditions $\phi(0)=a$, $\phi'(0)=0$ and get

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}\,r^2+\phi_a}\,dx$$

$$n_a(x) = M(a)\,\frac{e^{-\frac{1}{2}\,r^2+\phi_a(r)}}{2\pi\int_{\mathbb{R}^2} r\,e^{-\frac{1}{2}\,r^2+\phi_a}\,dx} = e^{-\frac{1}{2}\,r^2+\phi_a(r)}$$

With $-\Delta\,\varphi_f = n_a\,f$, consider the operator defined by

$$\mathcal{L}\,f := \frac{1}{n_a}\nabla\cdot\left(n_a(\nabla(f-\varphi_f))\right) \quad x\in\mathbb{R}^2$$

Spectrum of $\mathcal{L}$ (lowest eigenvalues only)

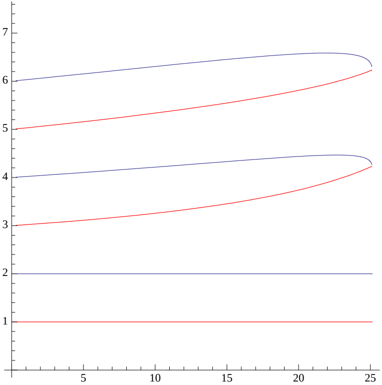

Figure: The lowest eigenvalues of $-\mathcal{L}$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

[A. Blanchet, JD, M. Escobedo, J. Fernández], [J. Campos, JD], [V. Calvez, J.A. Carrillo], [J. Bedrossian, N. Masmoudi]

Spectral analysis in the functional framework determined by the relative entropy method

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta\, f_0 = n_\infty\, f_0$$

and observe that $g_0 = f_0 / c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot \big(n_\infty \nabla (f_0 - c_\infty\, g_0) \big) =: \mathcal{L}\, f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty}\,\frac{\partial n_\infty}{\partial x_1}$ associated with $g_1 = \frac{1}{c_\infty}\,\frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of $\mathcal{L}$, such that $-\mathcal{L}\, f_1 = f_1$

With $D := x \cdot \nabla$, let $f_2 = 1 + \frac{1}{2}\, D \log n_\infty = 1 + \frac{1}{2\, n_\infty}\, D\, n_\infty$. Then

$$-\Delta\, (D\, c_\infty) + 2\, \Delta c_\infty = D\, n_\infty = 2\, (f_2 - 1)\, n_\infty$$

and so $g_2 := \frac{1}{c_\infty}\, (-\Delta)^{-1} (n_\infty\, f_2)$ is such that $-\mathcal{L}\, f_2 = 2\, f_2$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]

$$F[n] := \int_{\mathbb{R}^2} n \log\left(\frac{n}{n_\infty}\right) dx - \frac{1}{2}\int_{\mathbb{R}^2} (n - n_\infty)\,(c - c_\infty)\, dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$\mathsf{Q}_1[f] = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} F[n_\infty(1 + \varepsilon f)] \geq 0$$

if $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty\, dx)$ *such that* $\int_{\mathbb{R}^2} f\, n_\infty\, dx = 0$, *we have*

$$\int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\, n_\infty\, dx \leq \int_{\mathbb{R}^2} |f|^2\, n_\infty\, dx$$

... and eigenvalues

With g such that $-\Delta(g\,c_\infty) = f\,n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1\, f_2\, n_\infty\, dx - \int_{\mathbb{R}^2} f_1\, n_\infty\, (g_2\, c_\infty)\, dx$$

on the orthogonal to f_0 in $L^2(n_\infty\, dx)$ and with $G_2(x) := -\frac{1}{2\pi}\log|x|$

$$\mathsf{Q}_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g\, c_\infty)|^2\, n_\infty\, dx \quad \text{with} \quad g = \frac{1}{c_\infty}\, G_2 * (f\, n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator $\mathcal{L}$

$$\langle f, \mathcal{L}\, f \rangle = \mathsf{Q}_2[f] \quad \forall\, f \in \mathcal{D}(\mathsf{L}_2)$$

Lemma (J. Campos, JD)

$\mathcal{L}$ has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory

$$\mathcal{L}\,f = \frac{1}{n_\infty}\nabla\cdot\big(n_\infty\nabla(f - c_\infty\,g)\big)$$

Corollary (J. Campos, JD)

$$\langle f, f\rangle \le \langle \mathcal{L}\,f, f\rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L}\,f$$

where $\mathcal{L}$ is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle\cdot,\cdot\rangle$). A solution of

$$\frac{d}{dt}\,\langle f, f\rangle = -2\,\langle \mathcal{L}\,f, f\rangle$$

has therefore exponential decay

A new Onofri type inequality

- [J. Campos, JD]

Theorem (Onofri type inequality)

For any $M \in (0, 8\pi)$, *if* $n_\infty = M \frac{e^{c_\infty - \frac{1}{2}|x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2}|x|^2} \, dx}$ *with* $c_\infty = (-\Delta)^{-1} n_\infty$, $d\mu_M = \frac{1}{M} n_\infty \, dx$, *we have the inequality*

$$\log \left(\int_{\mathbb{R}^2} e^\phi \, d\mu_M \right) - \int_{\mathbb{R}^2} \phi \, d\mu_M \le \frac{1}{2\,M} \int_{\mathbb{R}^2} |\nabla \phi|^2 \, dx \quad \forall\, \phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} \left|\psi - \overline{\psi}\right|^2 n_M \, dx \le \int_{\mathbb{R}^2} |\nabla \psi|^2 \, dx \quad \text{where} \quad \overline{\psi} = \int_{\mathbb{R}^2} \psi \, d\mu_M$$

An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty\,dx)$ *such that* $\int_{\mathbb{R}^2} f\, f_0\, n_\infty\; dx = 0$ *holds, we have*

$$-\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{R}^2} f(x)\, n_\infty(x)\, \log|x-y|\, f(y)\, n_\infty(y)\; dx\, dy \le (1-\varepsilon)\int_{\mathbb{R}^2} |f|^2\, n_\infty\; dx$$

for some $\varepsilon > 0$, *where* $g\, c_\infty = G_2 * (f\, n_\infty)$ *and, if* $\int_{\mathbb{R}^2} f\, n_\infty\; dx = 0$ *holds,*

$$\int_{\mathbb{R}^2} |\nabla(g\, c_\infty)|^2\; dx \le (1-\varepsilon)\int_{\mathbb{R}^2} |f|^2\, n_\infty\; dx$$

Collecting estimates: exponential convergence

Back to Keller-Segel: exponential convergence for any mass $M \leq 8\pi$

If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0, assume that for any $s \geq 0$

$$\exists\, \varepsilon \in (0, 8\,\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma)\, d\sigma \leq \int_{B\left(0,\sqrt{s/\pi}\right)} n_{\infty, M+\varepsilon}(x)\, dx$$

Theorem

*Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1}\, dx)$ and $M := \int_{\mathbb{R}^2} n_0\, dx < 8\,\pi$, then any solution of (**??**) with initial datum n_0 is such that*

$$\int_{\mathbb{R}^2} |n(t,x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C\, e^{-\,2\,t} \quad \forall\, t \geq 0$$

for some positive constant C, where n_∞ is the unique stationary solution with mass M

Sketch of the proof

- [J. Campos, JD] Uniform convergence of $n(t,\cdot)$ to n_∞ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [J.I. Diaz, T. Nagai, J.M. Rakotoson]
- Uniform estimates (with no rates) easily result
- Estimates based on Duhammel formula inspired by [M. Escobedo, E. Zuazua] allow to prove uniform convergence
- Spectral estimates can be incorporated to the relative entropy approach
- Exponential convergence of the relative entropy follows

Thank you for your attention !