Entropy methods in partial differential equations: fast diffusion equations

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Abstract

Many new results on asymptotic behavior, sharp rates, optimal regularization effects, etc. have been achieved for the solutions of nonlinear diffusion equations over the last few years. By (generalized) entropy, we mean special Lyapunov functionals which have a probabilistic interpretation or a physical meaning. Such entropies also have deep connections with the (nonlinear) structure of the equation. The key underlying estimate is usually a functional inequality which relates the entropy with its time derivative. In the case of fast diffusion equations, the functional inequality is an interpolation inequality of Gagliardo-Nirenberg type. The talk will be devoted to a review of some recent results.

Contents

- Linear Diffusions
  - intermediate asymptotics
  - the entropy - entropy production method (Ornstein-Uhlenbeck)
  - (spectral approaches)
- The fast diffusion equation
  - intermediate asymptotics and interpolation
  - extensions (finite mass regine)
  - the infinite mass regime and Hardy-Poincaré inequalities
- Other entropies and algebraic rates
  - Generalized Poincaré inequalities for second and fourth order equations
  - L^q Poincaré inequalities for general measures

Intermediate asymptotics of linear diffusion equations

Consider the heat equation:

$$\begin{cases} u_t = \Delta u & x \in \mathbb{R}^d\,,\ t \in \mathbb{R}^+ \\ u(\cdot, t=0) = u_0 \geq 0 & \displaystyle\int_{\mathbb{R}^d} u_0 \, dx = 1 \end{cases} \tag{1}$$

As $t \to +\infty$, $u(x,t) \sim \mathcal{U}(x,t) = \frac{e^{-|x|^2/4t}}{(4\pi t)^{d/2}}$

What is the (optimal) rate of convergence of $\|u(\cdot,t) - \mathcal{U}(\cdot,t)\|_{L^1(\mathbb{R}^d)}$?

Time dependent rescaling: Fokker-Planck equation

$$u(x,t) = \frac{1}{R^d(t)}\, v\left(\xi = \frac{x}{R(t)}, \tau = \log R(t) + \tau(0)\right)$$

allows to transform this question into that of the convergence to the stationary solution $v_\infty(\xi) = (2\pi)^{-d/2}\, e^{-|\xi|^2/2}$.

- Ansatz: $\frac{dR}{dt} = \frac{1}{R} \quad R(0) = 1 \quad \tau(0) = 0$:

$$R(t) = \sqrt{1 + 2\,t}\,, \quad \tau(t) = \log R(t)$$

As a consequence: $v(\tau = 0) = u_0$.

- Fokker-Planck equation:

$$\begin{cases} v_\tau = \Delta v + \nabla(\xi\, v) & \xi \in \mathbb{R}^d\,,\ \tau \in \mathbb{R}^+ \\ v(\cdot, \tau = 0) = u_0 \geq 0 & \int_{\mathbb{R}^d} u_0\, dx = 1 \end{cases} \tag{2}$$

Entropy (relative to the stationary solution v_∞)

$$\Sigma[v] := \int_{\mathbb{R}^d} v \, \log\left(\frac{v}{v_\infty}\right) \, dx$$

If v is a solution of (2), then (I is the Fisher information)

$$\frac{d}{d\tau}\Sigma[v(\cdot,\tau)] = -\int_{\mathbb{R}^d} v \left|\nabla \log\left(\frac{v}{v_\infty}\right)\right|^2 \, dx =: -I[v(\cdot,\tau)]$$

- Euclidean logarithmic Sobolev inequality: If $\|v\|_{L^1} = 1$, then

$$\int_{\mathbb{R}^d} v \log v \; dx + d\left(1 + \frac{1}{2}\log(2\pi)\right) \leq \frac{1}{2}\int_{\mathbb{R}^d} \frac{|\nabla v|^2}{v} \, dx$$

$\Sigma[v(\cdot,\tau)] \leq \frac{1}{2} I[v(\cdot,\tau)]$, Equality: $v(\xi) = v_\infty(\xi) = (2\pi)^{-d/2} \, e^{-|\xi|^2/2}$

$$\Sigma[v(\cdot,\tau)] \leq e^{-2\tau}\Sigma[u_0] = e^{-2\tau}\int_{\mathbb{R}^d} u_0 \, \log\left(\frac{u_0}{v_\infty}\right) \, dx$$

Csiszár-Kullback inequality

Consider $v \geq 0$, $\bar{v} \geq 0$ such that $\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} \bar{v}\, dx =: M > 0$

$$\int_{\mathbb{R}^d} v \log\left(\frac{v}{\bar{v}}\right)\, dx \geq \frac{1}{4\,M} \|v - \bar{v}\|^2_{L^1(\mathbb{R}^d)}$$

$$\Longrightarrow \|v - v_\infty\|^2_{L^1(\mathbb{R}^d)} \leq 4\,M\,\Sigma[u_0]\,e^{-2\tau}$$

$$\tau(t) = \log\sqrt{1+2t}$$

$$\|u(\cdot,t) - u_\infty(\cdot,t)\|^2_{L^1(\mathbb{R}^d)} \leq \frac{4}{1+2t}\,\Sigma[u_0]$$

$$u_\infty(x,t) = \frac{1}{R^d(t)}\,v_\infty\left(\frac{x}{R(t)}, \tau(t)\right)$$

The proof of the Csiszár-Kullback inequality is given by a Taylor development at second order.

Logarithmic Sobolev inequalities

1) independent of the dimension [Gross, 75]: gaussian form

$$\int_{\mathbb{R}^d} w \log w \, d\mu(x) \leq \frac{1}{2} \int_{\mathbb{R}^d} w \, |\nabla \log w|^2 \, d\mu(x)$$

with $w = \frac{v}{M\, v_\infty}$, $\|v\|_{L^1} = M$, $d\mu(x) = v_\infty(x)\, dx$

2) invariant under scaling [Weissler, 78]

$$\int_{\mathbb{R}^d} v^2 \log v^2 \, dx \leq \frac{d}{2} \log \left(\frac{2}{\pi\, d\, e} \int_{\mathbb{R}^d} |\nabla v|^2 \, dx \right)$$

for any $v \in H^1(\mathbb{R}^d)$ such that $\int v^2 \, dx = 1$

Proof: optimize for $v_\lambda(x) = \lambda^{d/2} v(\lambda\, x)$ w.r.t. $\lambda > 0$

Entropy-entropy production / Bakry-Emery method

... a proof of the Euclidean logarithmic Sobolev inequality:

$$\frac{d}{d\tau}\Big(I[v(\cdot,\tau)] - 2\,\Sigma[v(\cdot,\tau)]\Big) = -C \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} \left| w_{ij} + a\,\frac{w_i w_j}{w} + b\,w\,\delta_{ij} \right|^2 dx < 0$$

for some $C > 0$, a, $b \in \mathbb{R}$ and $w = \sqrt{v}$

$$I[v(\cdot,\tau)] - 2\,\Sigma[v(\cdot,\tau)] \searrow I[v_\infty] - 2\,\Sigma[v_\infty] = 0$$

$$\Longrightarrow \quad \forall\, u_0\,, \quad I[u_0] - 2\,\Sigma[u_0] \geq I[v(\cdot,\tau)] - 2\,\Sigma[v(\cdot,\tau)] \geq 0 \quad \forall\, \tau > 0$$

Entropy-entropy production method

Goal: large time behavior of parabolic equations:

$$\begin{cases} v_t = \text{div}_x[D(x)\,(\nabla_x v + v\nabla_x A(x))] \;=\; \text{div}[e^{-A}\nabla(ve^A)] \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t>0,\ x\in\mathbb{R}^d \\ v(x,t=0) = v_0(x) \in L^1_+(\mathbb{R}^d) \end{cases} \tag{3}$$

$A(x)\dots$ given ‘potential’

$v_\infty(x) = e^{-A(x)} \in L^1 \dots$ (unique) steady state

mass conservation: $\int_{\mathbb{R}^d} v(t)\,dx = \int_{\mathbb{R}^d} v_\infty\,dx = 1$

questions: exponential rate ? connection to logarithmic Sobolev inequalities ? [Bakry-Emery ’84, Gross ’75, Toscani ’96, ...]

Entropy-entropy production method

[Bakry, Emery, 84]
[Arnold, Markowich, Toscani, Unterreiter, 01]
Relative entropy of v w.r.t. v_∞:

$$\Sigma[v|v_\infty] := \int_{\mathbb{R}^d} \psi\left(\frac{v}{v_\infty}\right) v_\infty \, dx \geq 0$$

with

$$\begin{aligned} \psi(w) &\geq 0 \text{ for } w \geq 0, \quad \text{convex} \\ \psi(1) &= \psi'(1) = 0 \\ 2\,(\psi''')^2 &\leq \psi''\,\psi^{IV} \end{aligned}$$

Examples:

$\psi_1 = w \ln w - w + 1$, $\Sigma_1(v|v_\infty) = \int v \ln\left(\frac{v}{v_\infty}\right) dx \ldots$ physical entropy

$\psi_p = \frac{w^p - p(w-1) - 1}{p-1}$, $1 < p \leq 2$, $\quad \Sigma_2(v|v_\infty) = \int_{\mathbb{R}^d} (v - v_\infty)^2 v_\infty^{-1} dx$

Exponential decay of entropy production

$$I(v(t)|v_\infty) := \frac{d}{dt}\Sigma[v(t)|v_\infty] = -\int \psi''(\frac{v}{v_\infty})|\underbrace{\nabla\frac{v}{v_\infty}}_{=:u}|^2 v_\infty dx \leq 0$$

Assume: $D \equiv 1$, $\frac{\partial^2 A}{\partial x^2} \geq \lambda_1 \mathrm{Id}$, $\lambda_1 > 0$ $\quad$ ($A(x) \ldots$ unif. convex)
entropy production rate:

$$I' = 2\int \psi''(\frac{v}{v_\infty})u^T \cdot \frac{\partial^2 A}{\partial x^2} \cdot u v_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) v_\infty dx}_{\geq 0} \geq -2\,\lambda_1 I$$

$$X = \begin{pmatrix} \psi''(\frac{v}{v_\infty}) & \psi'''(\frac{v}{v_\infty}) \\ \psi'''(\frac{v}{v_\infty}) & \frac{1}{2}\psi^{IV}(\frac{v}{v_\infty}) \end{pmatrix} \geq 0\,, \quad Y = \begin{pmatrix} \sum_{ij}(\frac{\partial u_i}{\partial x_j})^2 & u^T \cdot \frac{\partial u}{\partial x} \cdot u \\ u^T \cdot \frac{\partial u}{\partial x} \cdot u & |u|^4 \end{pmatrix} \geq 0$$

Exponential decay of relative entropy: [Arnold, Markowich, Toscani, Unterreiter]

Convex Sobolev inequalities

[Arnold, Markowich, Toscani, Unterreiter]: Entropy–entropy production estimate for $A(x) = -\ln v_\infty$ uniformly convex:

$$\Sigma[v|v_\infty] \leq \frac{1}{2\lambda_1} |I(v|v_\infty)|$$

Example 1: logarithmic entropy $\psi_1(w) = w \ln w - w + 1$

$$\int v \ln(\frac{v}{v_\infty}) dx \leq \frac{1}{2\lambda_1} \int v |\nabla \ln(\frac{v}{v_\infty})|^2 dx$$

$$\forall v,\ v_\infty \in L^1_+(\mathbb{R}^d),\ \int v\, dx = \int v_\infty\, dx = 1$$

Set $f^2 = \frac{v}{v_\infty} \Rightarrow$

$$\int f^2 \ln f^2 dv_\infty \leq \frac{2}{\lambda_1} \int |\nabla f|^2 dv_\infty$$

$$\forall f \in L^2(\mathbb{R}^d, dv_\infty),\ \int f^2 dv_\infty = 1$$

logarithmic Sobolev inequality–dv_∞ measure version [Gross '75]

Convex Sobolev inequalities (continued)

Example 2: non-logarithmic entropies:

$\psi_p(w) = \frac{w^p - p(w-1) - 1}{p-1}, \qquad 1 < p \leq 2$

$$(B_p) \quad \frac{p}{p-1}\left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^p\right] \leq \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty$$

With $\frac{v}{v_\infty} = \frac{|f|^{\frac{2}{p}}}{\int |f|^{\frac{2}{p}} dv_\infty}$ $\qquad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^d, v_\infty dx)$

Poincaré-type inequality [Beckner '89], $(B_p) \Rightarrow (B_2), \quad 1 < p \leq 2$

Refined convex Sobolev inequalities

Estimate of entropy production rate / entropy production:

$$
\begin{aligned}
I' &= 2\int \psi''\left(\frac{v}{v_\infty}\right)u^T \cdot \frac{\partial^2 A}{\partial x^2}\cdot u v_\infty dx + \underbrace{2\int \mathrm{Tr}\,(XY) v_\infty dx}_{\geq 0} \\
&\geq -2\lambda_1 I
\end{aligned}
$$

[Arnold, JD]: Observe that for $\psi_p(w) = \frac{w^p - p(w-1)-1}{p-1}$, $1 < p < 2$:

$$
X = \begin{pmatrix} \psi''(\frac{v}{v_\infty}) & \psi'''(\frac{v}{v_\infty}) \\ \psi'''(\frac{v}{v_\infty}) & \frac{1}{2}\psi^{IV}(\frac{v}{v_\infty}) \end{pmatrix} > 0
$$

Refined Beckner / generalized Poincaré inequalities

- Assume $\frac{\partial A^2}{\partial x^2} \geq \lambda_1 \mathrm{Id} \Rightarrow \Sigma'' \geq -2\lambda_1 \Sigma' + \kappa \frac{|\Sigma'|^2}{1+\Sigma}$, $\qquad \kappa = \frac{2-p}{p} < 1$

$\Rightarrow k(\Sigma[v|v_\infty]) \leq \frac{1}{2\lambda_1}|\Sigma'| = \frac{1}{2\lambda_1}\int \psi''(\frac{v}{v_\infty})|\nabla \frac{v}{v_\infty}|^2 dv_\infty$

"refined convex Sobolev inequality" with $x \leq k(x) = \frac{1+x-(1+x)^\kappa}{1-\kappa}$

- Set $v/v_\infty = |f|^{\frac{2}{p}} / \int |f|^{\frac{2}{p}} dv_\infty$

Theorem 1 (Arnold, JD)

$$\frac{1}{2}\left(\frac{p}{p-1}\right)^2 \left[\int f^2 dv_\infty - \left(\int |f|^{\frac{2}{p}} dv_\infty\right)^{2(p-1)} \left(\int f^2 dv_\infty\right)^{\frac{2-p}{p}}\right]$$

$$\leq \frac{2}{\lambda_1}\int |\nabla f|^2 dv_\infty \quad \forall f \in L^{\frac{2}{p}}(\mathbb{R}^d, dv_\infty)$$

$(rB_p) \Rightarrow (rB_2) = (B_2), \quad 1 < p \leq 2$

The Bakry-Emery method revisited

[Gianazza, Savaré, Toscani]

[J.D., Nazaret, Savaré]

Consider on a domain $\Omega \subset \mathbb{R}^d$ and $d\gamma = g\,dx$, $g = e^{-F}$
Generalized Ornstein-Uhlenbeck operator: $\Delta_g v := \Delta v - \mathrm{D}F \cdot \mathrm{D}v$

$$\int_\Omega |\mathrm{D}v|^2 \, d\gamma = -\int_\Omega v \, \Delta_g v \, d\gamma \quad \forall \, v \in H^1_0(\Omega, d\gamma)$$

Let $s := v^{p/2}$ and $\alpha := (2-p)/p$, $p \in (1,2]$

$$v_t = \Delta_g v \quad x \in \Omega\,, \ t \in \mathbb{R}^+$$

$$\nabla v \cdot n = 0 \quad x \in \partial\Omega\,, \ t \in \mathbb{R}^+$$

$$\mathcal{E}_p(t) := \frac{1}{p-1} \int_\Omega \Big[v^p - 1 - p\,(v-1) \Big] \, d\gamma$$

$$\mathcal{I}_p(t) := \frac{4}{p} \int_\Omega |\mathrm{D}s|^2 \, d\gamma$$

$$\mathcal{K}_p(t) := \int_\Omega |\Delta_g s|^2 \, d\gamma + \alpha \int_\Omega \Delta_g s \, \frac{|\mathrm{D}s|^2}{s} \, d\gamma$$

Written in terms of $s = v^{p/2}$, the entropy is

$$\mathcal{E}_p = \frac{1}{p-1} \int_\Omega \left[s^2 - 1 - p\,(s^{2/p} - 1) \right] d\gamma$$

and the evolution is governed by

$$s_t = \Delta_g s + \alpha\,\frac{|\mathrm{D}s|^2}{s}$$

A simple computation shows that

$$\begin{aligned} \frac{d}{dt}\mathcal{E}_p(t) &:= -\mathcal{I}_p(t) \\ \frac{d}{dt}\mathcal{I}_p(t) &:= -\frac{8}{p}\,\mathcal{K}_p(t) \end{aligned}$$

Using the commutation relation $[\mathrm{D},\Delta_g]\,s=-\mathrm{D}^2F\,\mathrm{D}s$, we get

$$\int_\Omega (\Delta_g s)^2\,d\gamma=\int_\Omega |\mathrm{D}^2 s|^2\,d\gamma+\int_\Omega \mathrm{D}^2F\,\mathrm{D}s\cdot\mathrm{D}s\,d\gamma\underbrace{-\sum_{i,j=1}^d\int_{\partial\Omega}\partial^2_{ij}s\,\partial_i s\,n_j\;g\,d\mathcal{H}^{d-1}}_{\geq 0 \text{ if } \Omega \text{ is convex}}$$

Let $z:=\sqrt{s}$. Using : $2\,\mathrm{D}^2 s\cdot\mathrm{D}z\otimes\mathrm{D}z=\mathrm{D}\left(|\mathrm{D}z|^2\right):\mathrm{D}z$ and i.p.p., we get

$$\begin{aligned}
\mathcal{K}_p &= \int_\Omega |\Delta_g s|^2\,d\gamma+4\,\alpha\int_\Omega \Delta_g s\,|\mathrm{D}z|^2\,d\gamma\\
&\geq \int_\Omega |\mathrm{D}^2 s|^2\,d\gamma+\int_\Omega \mathrm{D}^2F\,\mathrm{D}s\cdot\mathrm{D}s\,d\gamma\\
&\qquad +4^2\,\alpha\int_\Omega |Dz|^4\,d\gamma-2\cdot 4\,\alpha\int_\Omega \mathrm{D}^2 s\;:\;\mathrm{D}z\otimes\mathrm{D}z\,d\gamma\\
&\geq (1-\alpha)\int_\Omega |\mathrm{D}^2 s|^2\,d\gamma+\int_\Omega \mathrm{D}^2F\,\mathrm{D}s\cdot\mathrm{D}s\,d\gamma
\end{aligned}$$

An extension of the criterion of Bakry-Emery

Let $V(x) := \inf_{\xi \in S^{d-1}} \left(\mathrm{D}^2 F(x)\,\xi, \xi\right)$ and define

$$\lambda_1(p) := \inf_{w \in \mathcal{D}(\Omega) \setminus \{0\}} \frac{\int_\Omega \left(2\,\frac{p-1}{p}\,|\mathrm{D}w|^2 + V\,|w|^2\right) d\gamma}{\int_\Omega |w|^2\, d\gamma}$$

Theorem 1 *Let $F \in C^2(\Omega)$, $\gamma = e^{-F} \in L^1(\Omega)$, and Ω be a convex domain in $\mathbb{R}^d$. If $\lambda_1(p)$ is positive, then*

$$\mathcal{I}_p(t) \le \mathcal{I}_p(0)\, e^{-2\,\lambda_1(p)\,t}$$

$$\mathcal{E}_p(t) \le \mathcal{E}_p(0)\, e^{-2\,\lambda_1(p)\,t}$$

Fast diffusion equations: entropy methods and functional inequalities

$$u_t = \Delta u^m \quad x \in \mathbb{R}^d\,,\ t > 0$$

- Entropy methods for fast diffusion and porous media equations: intermediate asymptotics
- Entropy methods and functional inequalities

Porous media / fast diffusion equations

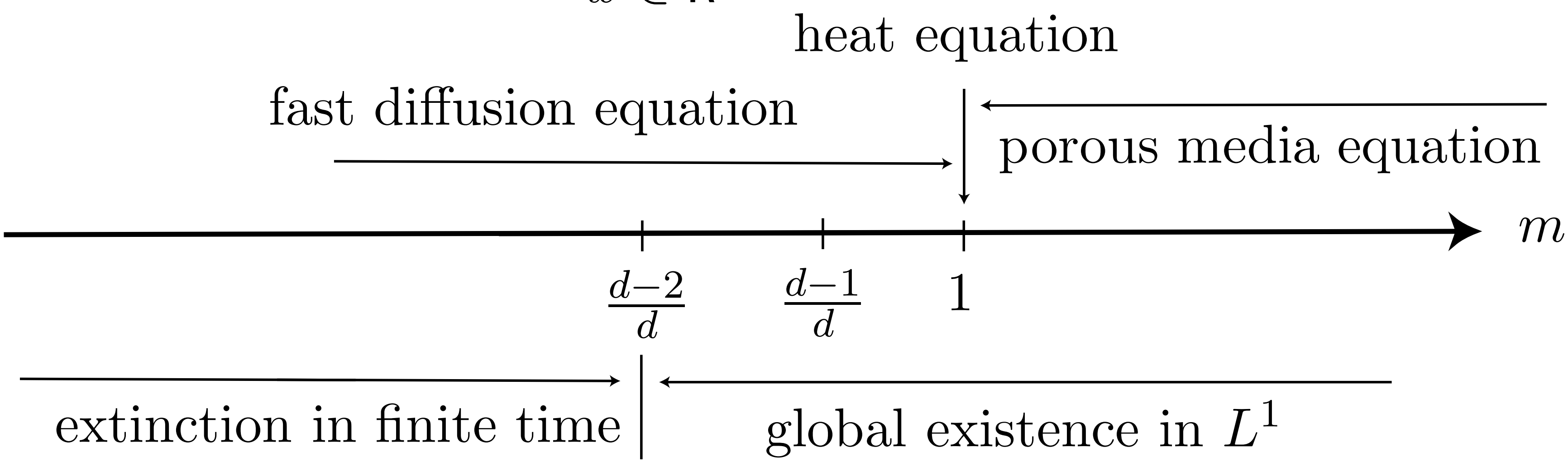
Generalized entropies and nonlinear diffusions (EDP, uncomplete):
[Del Pino, J.D.], [Carrillo, Toscani], [Otto], [Juengel, Markowich, Toscani], [Carrillo, Juengel, Markowich, Toscani, Unterreiter], [Biler, J.D., Esteban], [Markowich, Lederman], [Carrillo, Vázquez], [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub],...
[del Pino, Sáez], [Daskalopulos, Sesum]...

1) [J.D., del Pino] relate entropy and entropy-production by Gagliardo-Nirenberg inequalities

Various approaches:
2) “entropy – entropy-production method”
3) mass transport techniques
4) hypercontractivity for appropriate semi-groups

Heat equation, porous media & fast diffusion equation

$$u_t = \Delta u^m$$
$$x \in \mathbb{R}^d$$



Existence theory, critical values of the parameter m

Intermediate asymptotics for fast diffusion & porous media

$$u_t = \Delta u^m \quad \text{in } \mathbb{R}^d$$
$$u_{|t=0} = u_0 \geq 0$$
$$u_0(1+|x|^2) \in L^1\ , \quad u_0^m \in L^1$$

Intermediate asymptotics: $u_0 \in L^\infty$, $\int u_0\, dx = M > 0$

Self-similar (Barenblatt) function: $\mathcal{U}(t) = O(t^{-d/(2-d(1-m))})$
As $t \to +\infty$, [Friedmann, Kamin, 1980]

$$\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^\infty} = o(t^{-d/(2-d(1-m))})$$

$\Longrightarrow$ What about $\|u(t,\cdot) - \mathcal{U}(t,\cdot)\|_{L^1}$?

Time-dependent rescaling

Take $u(t,x) = R^{-d}(t)\, v\,(\tau(t), x/R(t))$ where

$$\dot{R} = R^{d(1-m)-1}\ , \quad R(0) = 1\ , \quad \tau = \log R$$

$$v_\tau = \Delta v^m + \nabla \cdot (x\, v)\ , \quad v_{|\tau=0} = u_0$$

[Ralston, Newman, 1984] Lyapunov functional: Entropy or Free energy

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right)\, dx - \Sigma_0$$

$$\frac{d}{d\tau}\Sigma[v] = -I[v]\ , \quad I[v] = \int v \left| \frac{\nabla v^m}{v} + x \right|^2\, dx$$

Entropy and entropy production

Stationary solution: choose C such that $\|v_\infty\|_{L^1} = \|u\|_{L^1} = M > 0$

$$v_\infty(x) = \left(C + \frac{1-m}{2m}\,|x|^2\right)_+^{-1/(1-m)}$$

Fix Σ_0 so that $\Sigma[v_\infty] = 0$. The entropy can be put in an m-homogeneous form

$$\Sigma[v] = \int \psi\left(\frac{v}{v_\infty}\right)\, v_\infty^m\, dx \quad \textit{with}\ \psi(t) = \frac{t^m - 1 - m\,(t-1)}{m-1}$$

Theorem 1 $d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$I[v] \geq 2\,\Sigma[v]$$

An equivalent formulation

$$\Sigma[v] = \int \left(\frac{v^m}{m-1} + \frac{1}{2}|x|^2 v \right) dx - \Sigma_0 \leq \frac{1}{2} \int v \left| \frac{\nabla v^m}{v} + x \right|^2 dx = \frac{1}{2} I[v]$$

$p = \frac{1}{2m-1}$, $v = w^{2p}$, $v^m = w^{p+1}$

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int |w|^{1+p} dx + K \geq 0$$

$K < 0$ if $m < 1$, $K > 0$ if $m > 1$ and, for some γ, K can be written as

$$K = K_0 \left(\int v \, dx = \int w^{2p} \, dx \right)^\gamma$$

$w = w_\infty = v_\infty^{1/2p}$ is optimal

$m = \frac{d-1}{d}$: Sobolev, $m \to 1$: logarithmic Sobolev

Gagliardo-Nirenberg inequalities

Theorem 2 [Del Pino, J.D.] *Assume that* $1 < p \leq \frac{d}{d-2}$ *and* $d \geq 3$

$$\|w\|_{2p} \leq A \, \|\nabla w\|_2^\theta \, \|w\|_{p+1}^{1-\theta}$$

$$A = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}}$$

$$\theta = \frac{d(p-1)}{p(d+2-(d-2)p)} \, , \quad y = \frac{p+1}{p-1}$$

Similar results for $0 < p < 1$

Uses [Serrin-Pucci], [Serrin-Tang]

$1 < p = \frac{1}{2m-1} \leq \frac{d}{d-2} \Longleftrightarrow$ Fast diffusion case: $\frac{d-1}{d} \leq m < 1$

$0 < p < 1 \Longleftrightarrow$ Porous medium case: $m > 1$

Intermediate asymptotics

$\Sigma[v] \leq \Sigma[u_0]\, e^{-2\tau}$+ Csiszár-Kullback inequalities

Theorem 3 [Del Pino, J.D.]
(i) $\frac{d-1}{d} < m < 1$ *if* $d \geq 3$

$$\limsup_{t \to +\infty} t^{\frac{1-d(1-m)}{2-d(1-m)}} \|u^m - u_\infty^m\|_{L^1} < +\infty$$

(ii) $1 < m < 2$

$$\limsup_{t \to +\infty} t^{\frac{1+d(m-1)}{2+d(m-1)}} \|\, [u - u_\infty]\, u_\infty^{m-1}\, \|_{L^1} < +\infty$$

$$u_\infty(t,x) = R^{-d}(t)\, v_\infty\,(x/R(t))$$

Fast diffusion equations: the finite mass regime

- If $m \geq 1$: porous medium regime or $m_1 := \frac{d-1}{d} \leq m < 1$, the decay of the entropy is governed by Gagliardo-Nirenberg inequalities, and to the limiting case $m = 1$ corresponds the logarithmic Sobolev inequality
- If $m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass.

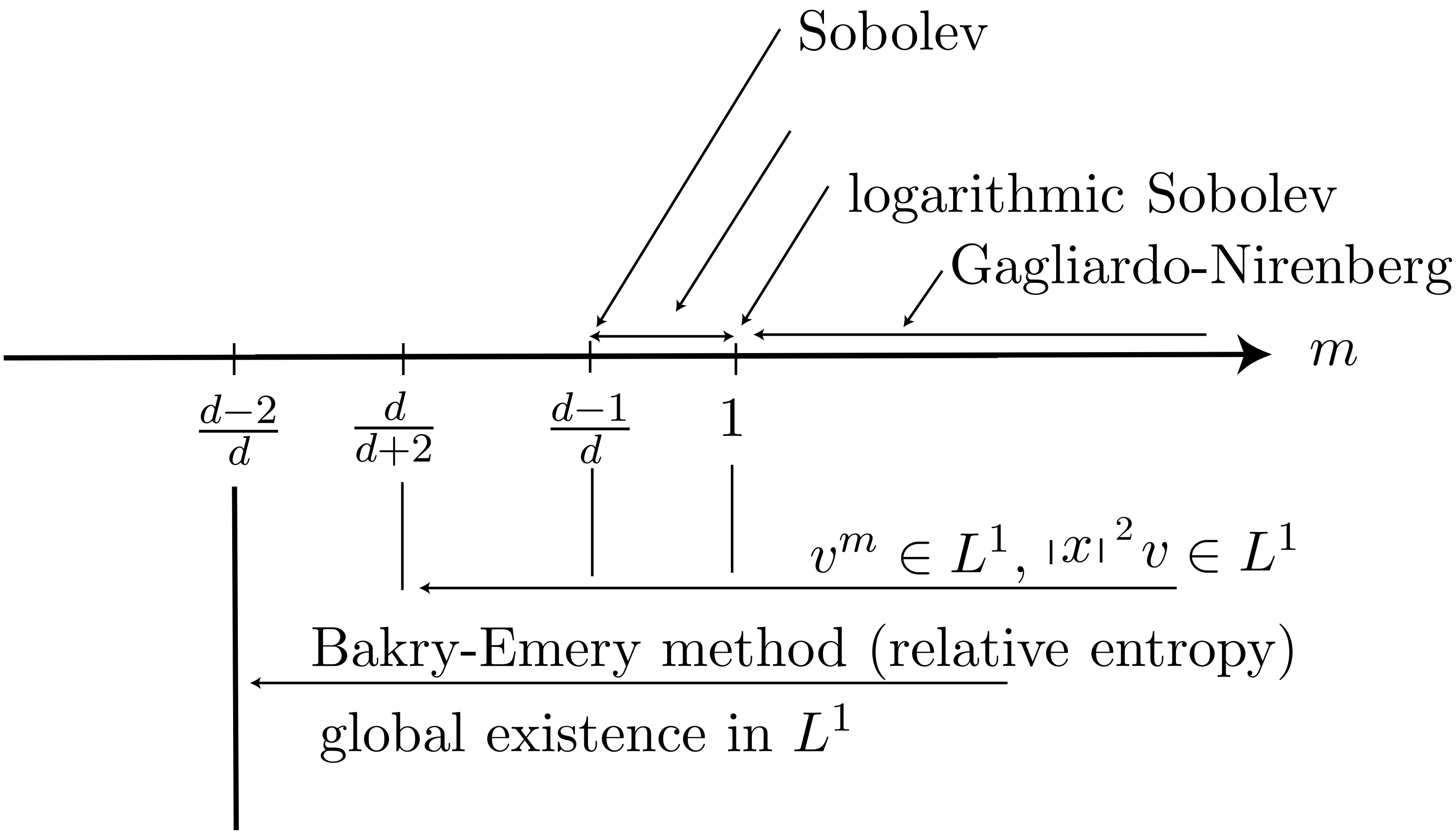
A remark on the mass transport approach

- The fast diffusion equation can be seen as the gradient flow of the generalized entropy with respect to the Wasserstein distance
- Displacement convexity holds in the same range of exponents, $m \in ((d-1)/d, 1)$, as for the Gagliardo-Nirenberg inequalities

$\Rightarrow$ How to extend to $m_c < m < m_1$ what has been done for $m \geq m_1$?

Fast diffusion: finite mass regime

Inequalities...



... existence of solutions of $u_t = \Delta u^m$

Extensions and related results

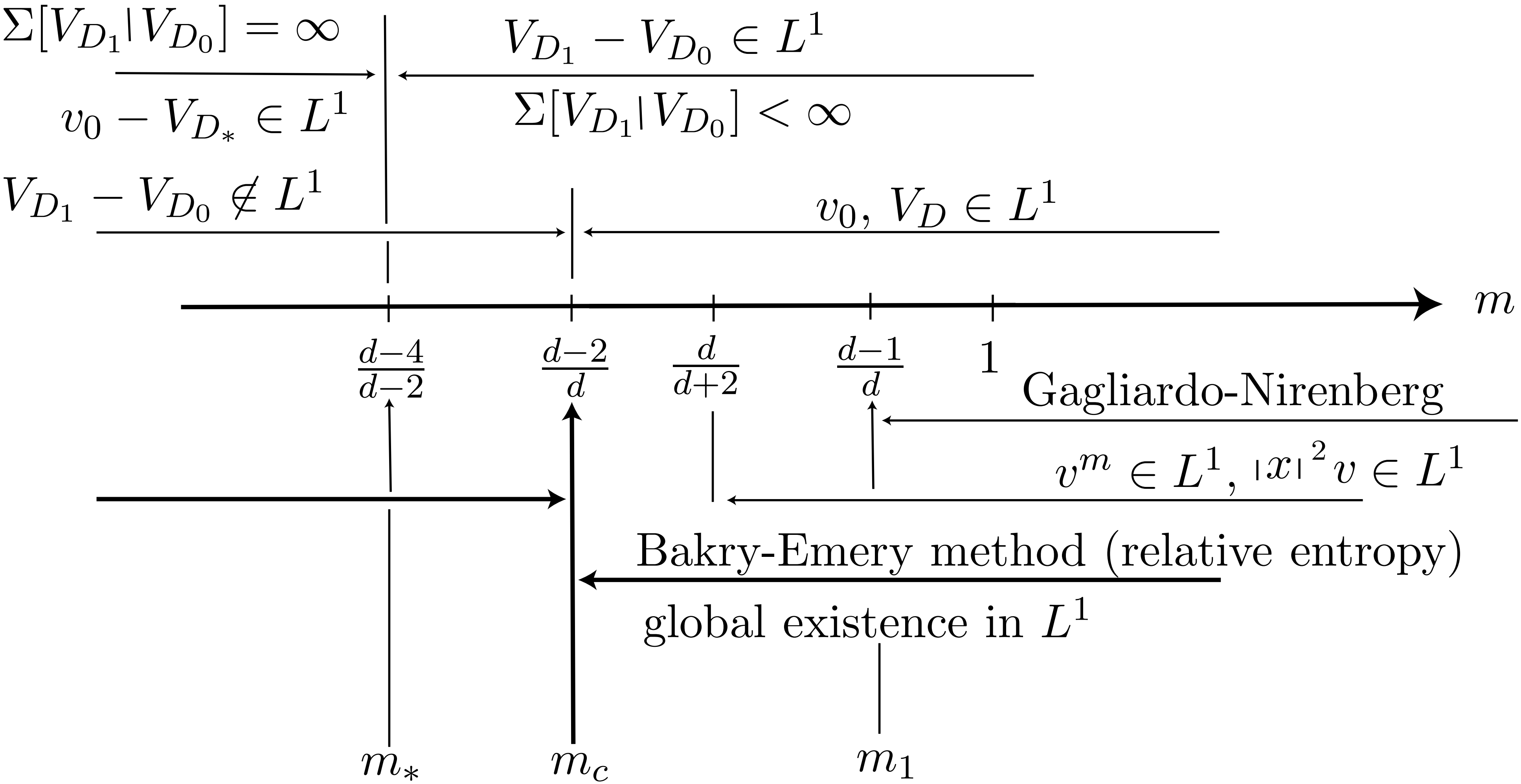
- Mass transport methods: inequalities / rates [Cordero-Erausquin, Gangbo, Houdré], [Cordero-Erausquin, Nazaret, Villani], [Agueh, Ghoussoub, Kang]
- General nonlinearities [Biler, J.D., Esteban], [Carrillo-DiFrancesco], [Carrillo-Juengel-Markowich-Toscani-Unterreiter] and gradient flows [Jordan-Kinderlehrer-Otto], [Ambrosio-Savaré-Gigli], [Otto-Westdickenberg], etc + [J.D.-Nazaret-Savaré, in progress]
- Non-homogeneous nonlinear diffusion equations [Biler, J.D., Esteban], [Carrillo, DiFrancesco]
- Extension to systems and connection with Lieb-Thirring inequalities [J.D.-Felmer-Loss-Paturel, 2006], [J.D.-Felmer-Mayorga]
- Drift-diffusion problems with mean-field terms. An example: the Keller-Segel model [J.D-Perthame, 2004], [Blanchet-J.D-Perthame, 2006], [Biler-Karch-Laurençot-Nadzieja, 2006], [Blanchet-Carrillo-Masmoudi, 2007], etc
- ... connection with linearized problems [Markowich-Lederman], [Carrillo-Vázquez], [Denzler-McCann], [McCann, Slepčev]

Fast diffusion equations: the infinite mass regime

- If $m > m_c := \frac{d-2}{d} \leq m < m_1$, solutions globally exist in L^1 and the Barenblatt self-similar solution has finite mass.
- For $m \leq m_c$, the Barenblatt self-similar solution has finite mass

$\Rightarrow$ How to extend to $m \leq m_c$ what has been done for $m > m_c$? Work in relative variables !

Fast diffusion: infinite mass regime



Entropy methods and linearization...

... intermediate asymptotics, vanishing

A. Blanchet, M. Bonforte, J.D., G. Grillo, J.L. Vázquez

- use the properties of the flow
- write everything as relative quantities (to the Barenblatt profile)
- compare the functionals (entropy, Fisher information) to their linearized counterparts

$\Longrightarrow$ Extend the domain of validity of the method to the price of a restriction

of the set of admissible solutions

Setting of the problem

We consider the solutions $u(\tau, y)$ of

$$\begin{cases} \partial_\tau u = \Delta u^m \\ u(0, \cdot) = u_0 \end{cases}$$

where $m \in (0, 1)$ (fast diffusion) and $(\tau, y) \in Q_T = (0, T) \times \mathbb{R}^d$
Two parameter ranges: $m_c < m < 1$ and $0 < m < m_c$, where

$$m_c := \frac{d-2}{d}$$

- $m_c < m < 1$, $T = +\infty$: intermediate asymptotics, $\tau \to +\infty$
- $0 < m < m_c$, $T < +\infty$: vanishing in finite time

$$\lim_{\tau \nearrow T} u(\tau, y) = 0$$

Barenblatt solutions

$$U_{D,T}(\tau,y) := \frac{1}{R(\tau)^d}\left(D + \frac{1-m}{2\,m}\left|\frac{y}{R(\tau)}\right|^2\right)^{-\frac{1}{1-m}}$$

with

- $R(\tau) := \left[d\,(m-m_c)\,(\tau+T)\right]^{\frac{1}{d\,(m-m_c)}}$ if $m_c < m < 1$
- (vanishing in finite time) if $0 < m < m_c$

$$R(\tau) := \left[d\,(m_c-m)\,(T-\tau)\right]^{-\frac{1}{d\,(m_c-m)}}$$

Time-dependent rescaling: $t := \log\left(\frac{R(\tau)}{R(0)}\right)$ and $x := \frac{y}{R(\tau)}$. The function $v(t,x) := R(\tau)^d\,u(\tau,y)$ solves a nonlinear *Fokker-Planck type equation*

$$\begin{cases} \partial_t v(t,x) = \Delta v^m(t,x) + \nabla\cdot(x\,v(t,x)) & (t,x)\in(0,+\infty)\times\mathbb{R}^d \\ v(0,x) = v_0(x) = R(0)^d\,u_0(R(0)\,x) & x\in\mathbb{R}^d \end{cases}$$

Assumptions

(H1) u_0 is a non-negative function in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$ and that there exist positive constants T and $D_0 > D_1$ such that

$$U_{D_0,T}(0,y) \leq u_0(y) \leq U_{D_1,T}(0,y) \quad \forall\, y \in \mathbb{R}^d$$

(H2) If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$u_0(y) = U_{D_*,T}(0,y) + f(y) \quad \forall\, y \in \mathbb{R}^d$$

(H1') v_0 is a non-negative function in $\mathrm{L}^1_{loc}(\mathbb{R}^d)$ and there exist positive constants $D_0 > D_1$ such that

$$V_{D_0}(x) \leq v_0(x) \leq V_{D_1}(x) \quad \forall\, x \in \mathbb{R}^d$$

(H2') If $m \in (0, m_*]$, there exist $D_* \in [D_1, D_0]$ and $f \in L^1(\mathbb{R}^d)$ such that

$$v_0(x) = V_{D_*}(x) + f(x) \quad \forall\, x \in \mathbb{R}^d$$

Convergence to the asymptotic profile (without rate)

$$m_* := \frac{d-4}{d-2} < m_c := \frac{d-2}{2}\,, \quad p(m) := \frac{d\,(1-m)}{2\,(2-m)}$$

Theorem 1 *Let* $d \geq 3$*,* $m \in (0,1)$*. Consider a solution* v *with initial data satisfying (H1')-(H2')*

(i) For any $m > m_*$*, there exists a unique* D_* *such that* $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,dx = 0$ *for any* $t > 0$*. Moreover, for any* $p \in (p(m), \infty]$*,* $\lim_{t\to\infty} \int_{\mathbb{R}^d} |v(t) - V_{D_*}|^p\,dx = 0$

(ii) For $m \leq m_*$*,* $v(t) - V_{D_*}$ *is integrable,* $\int_{\mathbb{R}^d} (v(t) - V_{D_*})\,dx = \int_{\mathbb{R}^d} f\,dx$ *and* $v(t)$ *converges to* V_{D_*} *in* $\mathrm{L}^p(\mathbb{R}^d)$ *as* $t \to \infty$*, for any* $p \in (1, \infty]$

(iii) (Convergence in Relative Error) For any $p \in (d/2, \infty]$*,*

$$\lim_{t\to\infty} \| \, v(t)/\, V_{D_*} - 1 \, \|_p = \; 0$$

[Daskalopoulos-Sesum, 06], [Blanchet-Bonforte-Grillo-Vázquez, 06-07]

Convergence with rate

$$q_* := \frac{2\,d\,(1-m)}{2\,(2-m)+d\,(1-m)}$$

Theorem 2 *If $m \neq m_*$, there exist $t_0 \geq 0$ and $\lambda_{m,d} > 0$ such that*

(i) *For any $q \in (q_*, \infty]$, there exists a positive constant C_q such that*

$$\|v(t) - V_{D_*}\|_q \leq C_q\, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0$$

(ii) *For any $\vartheta \in [0, (2-m)/(1-m))$, there exists a positive constant C_ϑ such that*

$$\big\|\, |x|^\vartheta (v(t) - V_{D_*}) \big\|_2 \leq C_\vartheta\, \mathrm{e}^{-\lambda_{m,d}\, t} \quad \forall\, t \geq t_0$$

(iii) *For any $j \in \mathbb{N}$, there exists a positive constant H_j such that*

$$\|v(t) - V_{D_*}\|_{C^j(\mathbb{R}^d)} \leq H_j\, \mathrm{e}^{-\frac{\lambda_{m,d}}{d+2(j+1)}\, t} \quad \forall\, t \geq t_0$$

Intermediate asymptotics

Corollary 3 *Let* $d \geq 3$*,* $m \in (0,1)$*,* $m \neq m_*$*. Consider a solution* u *with initial data satisfying* (H1)-(H2)*. For* τ *large enough, for any* $q \in (q_*, \infty]$*, there exists a positive constant* C *such that*

$$\|u(\tau) - U_{D_*}(\tau)\|_q \leq C\, R(\tau)^{-\alpha}$$

where $\alpha = \lambda_{m,d} + d(q-1)/q$ *and large means* $T - \tau > 0$*, small, if* $m < m_c$*, and* $\tau \to \infty$ *if* $m \geq m_c$

For any $p \in (d/2, \infty]$*, there exists a positive constant* $\mathcal{C}$ *and* $\gamma > 0$ *such that*

$$\left\| v(t)/\, V_{D_*} - 1 \right\|_{\mathrm{L}^p(\mathbb{R}^d)} \leq \mathcal{C}\, \mathrm{e}^{-\gamma\, t} \quad \forall\, t \geq 0$$

Rewriting the equation in relative variables

L^1-contraction, Maximum Principle, conservation of relative mass...

Passing to the quotient: the function $w(t,x) := \frac{v(t,x)}{V_{D_*}(x)}$ solves

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}} \nabla \cdot \left[w\, V_{D_*} \nabla \left(\dfrac{m}{m-1} (w^{m-1} - 1)\, V_{D_*}^{m-1} \right) \right] & \text{in } (0,+\infty) \times \mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

with

$$0 < \inf_{x \in \mathbb{R}^d} \frac{V_{D_0}}{V_{D_*}} \leq w(t,x) \leq \sup_{x \in \mathbb{R}^d} \frac{V_{D_1}}{V_{D_*}} < \infty$$

... Harnack Principle

$$\|w(t)\|_{C^k(\mathbb{R}^d)} \leq \overline{H}_k < +\infty \quad \forall\, t \geq t_0$$

$\exists\, t_0 \geq 0$ s.t. (H1) holds if $\exists\, R > 0$, $\sup_{|y|>R} u_0(y)\, |y|^{\frac{2}{1-m}} < \infty$, and $m > m_c$

Relative entropy

Relative entropy

$$\mathcal{F}[w] := \frac{1}{1-m} \int_{\mathbb{R}^d} \left[(w-1) - \frac{1}{m}(w^m - 1) \right] V_{D_*}^m \, dx$$

Relative Fisher information

$$\mathcal{J}[w] := \frac{m}{(m-1)^2} \int_{\mathbb{R}^d} \left| \nabla \left[\left(w^{m-1} - 1 \right) V_{D_*}^{m-1} \right] \right|^2 w \, V_{D_*} \, dx$$

Proposition 1 *Under assumptions (H1)-(H2),*

$$\frac{d}{dt} \mathcal{F}[w(t)] = -\,\mathcal{J}[w(t)]$$

Proposition 2 *Under assumptions (H1)-(H2), there exists a constant* $\lambda > 0$ *such that*

$$\mathcal{F}[w(t)] \leq \lambda^{-1} \, \mathcal{J}[w(t)]$$

Heuristics: linearization

Take $w(t,x) = 1 + \varepsilon\,\frac{g(t,x)}{V_{D_*}^{m-1}(x)}$ and formally consider the limit $\varepsilon \to 0$ in

$$\begin{cases} w_t = \dfrac{1}{V_{D_*}}\,\nabla\cdot\left[w\,V_{D_*}\,\nabla\left(\dfrac{m}{m-1}(w^{m-1}-1)\,V_{D_*}^{m-1}\right)\right] & \text{in } (0,+\infty)\times\mathbb{R}^d \\ w(0,\cdot) = w_0 := \dfrac{v_0}{V_{D_*}} & \text{in } \mathbb{R}^d \end{cases}$$

Then g solves

$$g_t = m\,V_{D_*}^{m-2}(x)\,\nabla\cdot\left[V_{D_*}(x)\,\nabla g(t,x)\right]$$

and the entropy and Fisher information functionals

$$\mathsf{F}[g] := \frac{1}{2}\int_{\mathbb{R}^d}|g|^2\,V_{D_*}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\int_{\mathbb{R}^d}|\nabla g|^2\,V_{D_*}\,dx$$

consistently verify $\dfrac{d}{dt}\,\mathsf{F}[g(t)] = -\,\mathsf{I}[g(t)]$

Comparison of the functionals

Lemma 3 *Let $m \in (0,1)$ and assume that u_0 satisfies* (H1)-(H2)
[Relative entropy]

$$C_1 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \, dx \le \mathcal{F}[w] \le C_2 \int_{\mathbb{R}^d} |w-1|^2 \, V_{D_*}^m \, dx$$

[Fisher information]

$$\mathsf{I}[g] \le \beta_1 \, \mathcal{J}[w] + \beta_2 \;\; \mathsf{F}[g] \quad \textit{with} \quad g := (w-1) \, V_{D_*}^{m-1}$$

Theorem 4 (Hardy-Poincaré) *There exists a positive constant $\lambda_{m,d}$ such that for any $m \neq m_* = (d-4)/(d-2)$, $m \in (0,1)$, for any $g \in \mathcal{D}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |g - \overline{g}|^2 \; V_{D_*}^{2-m} \, dx \le \mathcal{C}_{m,d} \int_{\mathbb{R}^d} |\nabla g|^2 \, V_{D_*} \, dx$$

with $\overline{g} = \int_{\mathbb{R}^d} g \, V_{D_*}^{2-m} \, dx$ if $m > m_*$, $\overline{g} = 0$ otherwise

Hardy-Poincaré inequalities

With $\alpha = \frac{1}{m-1}$, $\alpha_* = \frac{1}{m_*-1} = 1 - \frac{d}{2}$

Theorem 5 *Assume that* $d \geq 3$, $\alpha \in \mathbb{R} \setminus \{\alpha^*\}$, $d\mu_\alpha(x) := h_\alpha(x)\,dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$. *Then*

$$\int_{\mathbb{R}^d} \frac{|v|^2}{1+|x|^2}\,d\mu_\alpha \leq \mathcal{C}_{\alpha,d} \int_{\mathbb{R}^d} |\nabla v|^2\,d\mu_\alpha$$

holds for some positive constant $\mathcal{C}_{\alpha,d}$*, for any* $v \in \mathcal{D}(\mathbb{R}^d)$*, under the additional condition* $\int_{\mathbb{R}^d} v\,d\mu_{\alpha-1} = 0$ *if* $\alpha \in (-\infty, \alpha^*)$

Limit cases

Poincaré inequality: take $\alpha = -1/\varepsilon^2$ to $v_\varepsilon(x) := \varepsilon^{-d/2}\, v(x/\varepsilon)$ and let $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} |v|^2 \, d\nu_\infty \le \frac{1}{2} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\nu_\infty \quad \text{with} \quad d\nu_\infty(x) := e^{-|x|^2} dx$$

... under the additional condition $\int_{\mathbb{R}^d} v \; e^{-|x|^2} dx = 0$

Hardy's inequality: take $v_{1/\varepsilon}(x) := \varepsilon^{d/2}\, v(\varepsilon\, x)$ and let $\varepsilon \to 0$

$$\int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, d\nu_{0,\alpha} \le \frac{1}{(\alpha - \alpha_*)^2} \int_{\mathbb{R}^d} |\nabla v|^2 \, d\nu_{0,\alpha} \quad \text{with} \quad d\nu_{0,\alpha}(x) := |x|^{2\alpha} \, dx$$

... under the additional condition $\bar{v}_\alpha := \int_{\mathbb{R}^d} v \, d\nu_{0,\alpha} = 0$ if $\alpha < \alpha^*$

Some estimates of $\mathcal{C}_{\alpha,d}$

α	$-\infty < \alpha \le -d$	$-d < \alpha < \alpha^*$	$\alpha^* < \alpha \le 1$
$\mathcal{C}_{\alpha,d}$	$\frac{1}{2\,\lvert\alpha\rvert}$	$\mathcal{C}_{\alpha,d} \ge \frac{4}{(d+2\alpha-2)^2}$	$\frac{4}{(d+2\alpha-2)^2}$
Optimality	?	?	yes

α	$1 \le \alpha \le \bar{\alpha}(d)$	$\bar{\alpha}(d) \le \alpha \le d$	d	$\alpha > d$
$\mathcal{C}_{\alpha,d}$	$\frac{4}{d(d+2\alpha-2)}$	$\frac{1}{\alpha(d+\alpha-2)}$	$\frac{1}{2d(d-1)}$	$\frac{1}{d(d+\alpha-2)}$
Optimality	?	?	yes	?

$\alpha_* = -\frac{d-2}{2}$, $\bar{\alpha}(d) \in (1, d)$

Hardy's inequality: the "completing the square method"

Let $v \in \mathcal{D}(\mathbb{R}^d)$ with $\mathrm{supp}(v) \subset \mathbb{R}^d \setminus \{0\}$ if $\alpha < \alpha^*$

$$0 \le \int_{\mathbb{R}^d} \left| \nabla v + \lambda \frac{x}{|x|^2} v \right|^2 |x|^{2\alpha} \, dx$$
$$= \int_{\mathbb{R}^d} |\nabla v|^2 \, |x|^{2\alpha} \, dx + \left[\lambda^2 - \lambda \, (d + 2\alpha - 2) \right] \int_{\mathbb{R}^d} \frac{|v|^2}{|x|^2} \, |x|^{2\alpha} \, dx$$

An optimization of the right hand side with respect to λ gives $\lambda = \alpha - \alpha^*$, that is $(d + 2\alpha - 2)^2/4 = \lambda^2$. Such an inequality is optimal, with optimal constant λ^2, as follows by considering the test functions:

1) if $\alpha > \alpha^*$: $v_\varepsilon(x) = \min\{\varepsilon^{-\lambda}, (|x|^{-\lambda} - \varepsilon^\lambda)_+\}$

2) if $\alpha < \alpha^*$: $v_\varepsilon(x) = |x|^{1-\alpha-d/2+\varepsilon}$ for $|x| < 1$
$v_\varepsilon(x) = (2 - |x|)_+$ for $|x| \ge 1$

and letting $\varepsilon \to 0$ in both cases

The optimality case: Davies' method

Proposition 4 *Let $d \geq 3$, $\alpha \in (\alpha^*, \infty)$. Then the Hardy-Poincaré inequality holds for any $v \in \mathcal{D}(\mathbb{R}^d)$ with $\mathcal{C}_{\alpha,d} := 4/(d-2+2\,\alpha)^2$ if $\alpha \in (\alpha^*, 1]$ and $\mathcal{C}_{\alpha,d} := 4/[d\,(d-2+2\,\alpha)]$ if $\alpha \geq 1$. The constant $\mathcal{C}_{\alpha,d}$ is optimal for any $\alpha \in (\alpha^*, 1]$.*

Proof: $h_\alpha = (1+|x|^2)^\alpha$, $\nabla h_\alpha = 2\alpha\, x\, h_{\alpha-1}$,
$\Delta h_\alpha = 2\alpha\, h_{\alpha-2}[d + 2(\alpha - \alpha^*)\,|x|^2] > 0$
By Cauchy-Schwarz

$$\begin{aligned} \left| \int_{\mathbb{R}^d} |v|^2\, \Delta h_\alpha\, dx \right|^2 &\leq 4 \left(\int_{\mathbb{R}^d} |v|\,|\nabla v|\,|\nabla h_\alpha|\, dx \right)^2 \\ &\leq 4 \int_{\mathbb{R}^d} |v|^2\,|\Delta h_\alpha|\, dx \int_{\mathbb{R}^d} |\nabla v|^2\,|\nabla h_\alpha|^2\,|\Delta h_\alpha|^{-1}\, dx \end{aligned}$$

$|\Delta h_\alpha| \geq 2\,|\alpha|\ \min\{d, (d-2+2\,\alpha)\}\ \frac{h_\alpha(x)}{1+|x|^2}$

$\frac{|\nabla h_\alpha|^2}{|\Delta h_\alpha|} \leq \frac{2\,|\alpha|}{d-2+2\,\alpha}\, h_\alpha(x)$

Generalized Poincaré inequalities

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Higher order diffusion equations

The one dimensional porous medium/fast diffusion equation

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

The thin film equation

$$u_t = -(u^m \, u_{xxx})_x \,, \quad x \in S^1 \,, \quad t > 0$$

The Derrida-Lebowitz-Speer-Spohn (DLSS) equation

$$u_t = -(u \, (\log u)_{xx})_{xx} \,, \quad x \in S^1 \,, \quad t > 0$$

... with initial condition $u(\cdot, 0) = u_0 \geq 0$ in $S^1 \equiv [0, 1)$

Entropies and energies

Averages:

$$\mu_p[v] := \left(\int_{S^1} v^{1/p} \, dx \right)^p \quad \text{and} \quad \bar{v} := \int_{S^1} v \, dx$$

Entropies: $p \in (0, +\infty)$, $q \in \mathbb{R}$, $v \in H^1_+(S^1)$, $v \not\equiv 0 \, a.e.$

$$\Sigma_{p,q}[v] := \frac{1}{p\,q\,(p\,q-1)} \left[\int_{S^1} v^q \, dx - (\mu_p[v])^q \right] \quad \text{if } p\,q \neq 1 \text{ and } q \neq 0 \,,$$

$$\Sigma_{1/q,q}[v] := \int_{S^1} v^q \, \log \left(\frac{v^q}{\int_{S^1} v^q \, dx} \right) dx \quad \text{if } p\,q = 1 \text{ and } q \neq 0 \,,$$

$$\Sigma_{p,0}[v] := -\frac{1}{p} \int_{S^1} \log \left(\frac{v}{\mu_p[v]} \right) dx \quad \text{if } q = 0$$

Convexity

$\Sigma_{p,q}[v]$ is non-negative by convexity of

$$u \mapsto \frac{u^{p\,q} - 1 - p\,q\,(u-1)}{p\,q\,(p\,q-1)} =: \sigma_{p,q}(u)$$

By Jensen's inequality,

$$\begin{aligned} \Sigma_{p,q}[v] &= \mu_p[v]^q \int_{S^1} \sigma_{p,q}\left(\frac{v^{1/p}}{(\mu_p[v])^{1/p}}\right) dx \\ &\geq \mu_p[v]^q \sigma_{p,q}\left(\int_{S^1} \frac{v^{1/p}}{(\mu_p[v])^{1/p}}\, dx\right) = \mu_p[v]^q \sigma_{p,q}(1) = 0 \end{aligned}$$

Limit cases

$p\,q = 1$:

$$\lim_{p\to 1/q} \Sigma_{p,q}[v] = \Sigma_{1/q,q}[v] \quad \text{for } q > 0$$

$q = 0$:

$$\lim_{q\to 0} \Sigma_{p,q}[v] = \Sigma_{p,0}[v] \quad \text{for } p > 0$$

$p = q = 0$:

$$\Sigma_{0,0}[v] = -\int_{S^1} \log\left(\frac{v}{\|v\|_\infty}\right) dx$$

Some references (>2005):

[M. J. Cáceres, J. A. Carrillo, and G. Toscani]
[M. Gualdani, A. Jüngel, and G. Toscani]
[A. Jüngel and D. Matthes]
[R. Laugesen]

Global functional inequalities

Theorem 1 *For all* $p \in (0, +\infty)$ *and* $q \in (0, 2)$*, there exists a positive constant* $\kappa_{p,q}$ *such that, for any* $v \in H^1_+(S^1)$*,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{\kappa_{p,q}} J_1[v] := \frac{1}{\kappa_{p,q}} \int_{S^1} |v'|^2 \, dx$$

Corollary 1 *Let* $p \in (0, +\infty)$ *and* $q \in (0, 2)$*. Then, for any* $v \in H^1_+(S^1)$*,*

$$\Sigma_{p,q}[v]^{2/q} \leq \frac{1}{4\pi^2 \, \kappa_{p,q}} J_2[v] := \frac{1}{4\pi^2 \, \kappa_{p,q}} \int_{S^1} |v''|^2 \, dx$$

A minimizing sequence $(v_n)_{n\in\mathbb{N}}$ is bounded in $H^1(S^1)$

$$v_n \rightharpoonup v \quad \text{in } H^1(S^1) \quad \text{and} \quad \Sigma_{p,q}[v_n] \to \Sigma_{p,q}[v] \quad \text{as} \quad n \to \infty$$

If $\Sigma_{p,q}[v] = 0$, $\lim_{n\to\infty} J_1[v_n] = 0$. Let $\varepsilon_n := J_1[v_n]$, $w_n := \frac{v_n - 1}{\sqrt{\varepsilon_n}}$ and make a Taylor expansion

$$\left| (1 + \sqrt{\varepsilon}\, x)^{1/p} - 1 - \frac{\sqrt{\varepsilon}}{p}\, x \right| \leq \frac{1}{p}\, r(\varepsilon_0, p)\, \varepsilon \quad \forall\, (x, \varepsilon) \in \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \times (0, \varepsilon_0)$$

$$\varepsilon_n := J_1[v_n]\,, \quad \Sigma_{p,q}[v_n] \leq c(\varepsilon_0, p, q)\, \varepsilon_n$$

Hence, since $q < 2$,

$$\frac{J_1[v_n]}{\Sigma_{p,q}[v_n]^{2/q}} = \frac{\varepsilon_n\, J_1[w_n]}{\Sigma_{p,q}[v_n]^{2/q}} \geq [c(\varepsilon_0, p, q)]^{-2/q}\, \varepsilon_n^{1-2/q} \to \infty$$

gives a contradiction

Asymptotic functional inequalities

The regime of small entropies:

$$\mathcal{X}_\varepsilon^{p,q} := \left\{ v \in H^1_+(S^1) \ : \ \Sigma_{p,q}[v] \le \varepsilon \text{ and } \mu_p[v] = 1 \right\}$$

Theorem 2 *For any $p > 0$, $q \in \mathbb{R}$ and $\varepsilon_0 > 0$, there exists a positive constant C such that, for any $\varepsilon \in (0, \varepsilon_0]$,*

$$\Sigma_{p,q}[v] \le \frac{1 + C\sqrt{\varepsilon}}{8\,p^2\,\pi^2}\,J_1[v] \quad \forall\, v \in \mathcal{X}_\varepsilon^{p,q}$$

Without the condition $\mu_p[v] = 1$:

$$\Sigma_{p,q}[v] \le \frac{1 + C\sqrt{\varepsilon}}{8\,p^2\,\pi^2}\,(\mu_p[v])^{q-2}\,J_1[v]$$

If $J_1[v] \leq 8\,p^2\,\pi^2\,\varepsilon$, define $w := (v-1)/(\kappa_p^\infty\,\sqrt{\varepsilon})$: $J_1[w] \leq 1$.

$$\begin{aligned}
\Sigma_{p,q}[v] &= \frac{1}{pq(pq-1)}\left[\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^q dx - \left(\int_{S^1}(1+\kappa_p^\infty\sqrt{\varepsilon}w)^{1/p} dx\right)^{pq}\right] \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\left[\int_{S^1} w^2\;dx - \left(\int_{S^1} w\;dx\right)^2\right] + O(\varepsilon^{3/2}) \\
&= \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\int_{S^1}(w-\bar{w})^2\;dx + O(\varepsilon^{3/2}) \\
&\leq \varepsilon\,\frac{(\kappa_p^\infty)^2}{2\,p^2}\,\frac{J_1[w]}{(2\pi)^2} + O(\varepsilon^{3/2}) = \frac{J_1[v]}{8\,p^2\,\pi^2} + O(\varepsilon^{3/2})
\end{aligned}$$

using Poincaré's inequality

1^{st} application: Porous media

$$\frac{\partial u}{\partial t} = (u^m)_{xx} \quad x \in S^1,\ t > 0$$

A one parameter family of entropies :

$$\Sigma_k[u] := \begin{cases} \dfrac{1}{k\,(k+1)} \displaystyle\int_{S^1} \left(u^{k+1} - \bar{u}^{k+1}\right)\, dx & \text{if} \quad k \in \mathbb{R} \setminus \{-1, 0\} \\ \displaystyle\int_{S^1} u \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = 0 \\ -\displaystyle\int_{S^1} \log\left(\frac{u}{\bar{u}}\right)\, dx & \text{if} \quad k = -1 \end{cases}$$

With $v := u^p$, $p := \frac{m+k}{2}$, $q := \frac{k+1}{p} = 2\,\frac{k+1}{m+k}$, $\Sigma_k[u] = \Sigma_{p,q}[v]$

Lemma 1 *Let* $k \in \mathbb{R}$. *If* u *is a smooth positive solution*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} \left|(u^{(k+m)/2})_x\right|^2 dx = 0$$

with $\lambda := 4\,m/(m+k)^2$ *whenever* $k+m \neq 0$, *and*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \lambda \int_{S^1} |(\log u)_x|^2 \, dx = 0$$

with $\lambda := m$ *for* $k+m=0$.

Decay rates

Proposition 1 *Let* $m \in (0, +\infty)$, $k \in \mathbb{R} \setminus \{-m\}$, $q = 2\,(k+1)/(m+k)$, $p = (m+k)/2$ *and* u *be a smooth positive solution*

i) Short-time Algebraic Decay: If $m > 1$ *and* $k > -1$, *then*

$$\Sigma_k[u(\cdot,t)] \le \left[\Sigma_k[u_0]^{-(2-q)/q} + \frac{2-q}{q}\,\lambda\,\kappa_{p,q}\,t\right]^{-q/(2-q)}$$

ii) Asymptotically Exponential Decay: If $m > 0$ *and* $m + k > 0$, *there exists* $C > 0$ *and* $t_1 > 0$ *such that for* $t \ge t_1$,

$$\Sigma_k[u(\cdot,t)] \le \Sigma_k[u(\cdot,t_1)]\,\exp\left(-\frac{8\,p^2\,\pi^2\,\lambda\,\bar{u}^{p(2-q)}\,(t-t_1)}{1 + C\sqrt{\Sigma_k[u(\cdot,t_1)]}}\right)$$

2^{nd} Application: fourth order equations

$$u_t = -\Big(u^m \; \big(u_{xxx} + a\,u^{-1}\,u_x\,u_{xx} + b\,u^{-2}\,u_x^3\big)\Big)_x, \quad x \in S^1,\ t > 0$$

Example 1. The thin film equation: $a = b = 0$

$$u_t = -(u^m\,u_{xxx})_x,$$

Example 2. The DLSS equation: $m = 0$, $a = -2$, and $b = 1$

$$u_t = -\Big(u\,(\log u)_{xx}\Big)_{xx},$$

$L_\pm := \frac{1}{4}(3\,a + 5) \pm \frac{3}{4}\sqrt{(a-1)^2 - 8\,b}$

$A := (k+m+1)^2 - 9\,(k+m-1)^2 + 12\,a\,(k+m-2) - 36\,b$

Theorem 3 *Assume* $(a-1)^2 \geq 8\,b$

i) *Entropy production:* *If* $L_- \leq k+m \leq L_+$

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] \leq 0 \quad \forall\, t>0$$

ii) *Entropy production:* *If* $k+m+1 \neq 0$ *and* $L_- < k+m < L_+$,

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} \left|(u^{(k+m+1)/2})_{xx}\right|^2 dx \leq 0 \quad \forall\, t>0$$

If $k+m+1=0$ *and* $a+b+2-\mu \leq 0$ *for some* $0<\mu<1$, *then*

$$\frac{d}{dt}\Sigma_k[u(\cdot,t)] + \mu \int_{S^1} \left|(\log u)_{xx}\right|^2 dx \leq 0 \quad \forall\, t>0$$

Decay rates

Theorem 4 *Let* $k, m \in \mathbb{R}$ *be such that* $L_- \leq k + m \leq L_+$

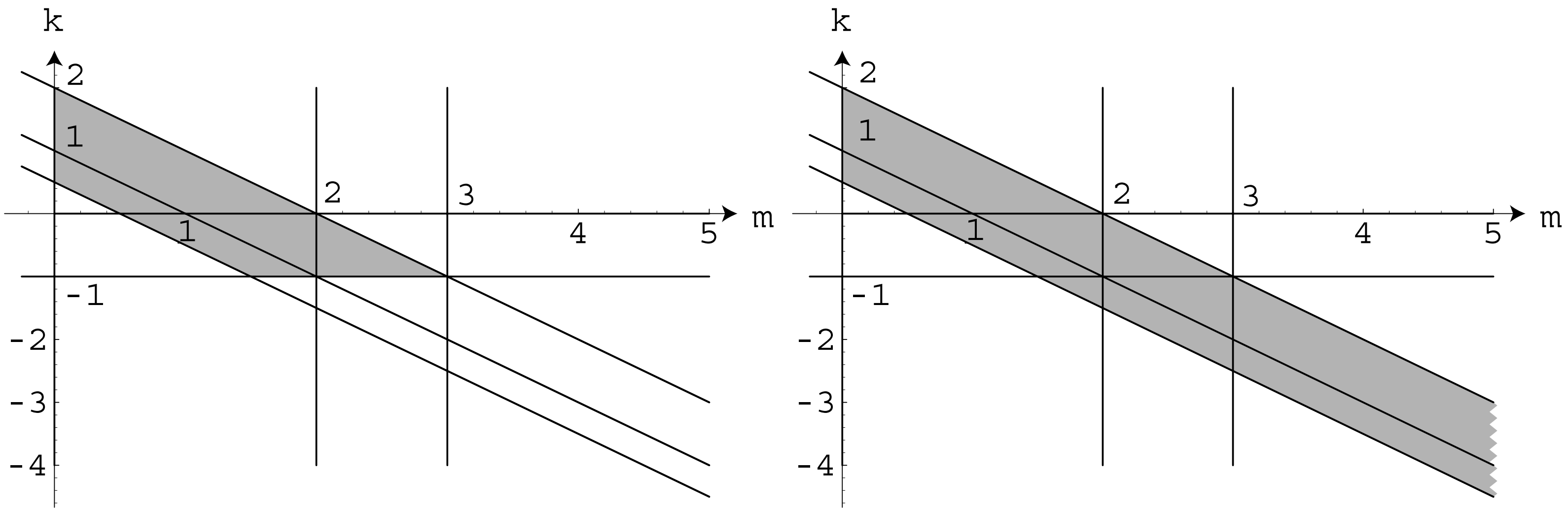
i) Short-time Algebraic Decay: If $k > -1$ *and* $m > 0$*, then*

$$\Sigma_k[u(\cdot, t)] \leq \left[\Sigma_k[u_0]^{-(2-q)/q} + 4\pi^2\, \mu\, \kappa_{p,q} \left(\frac{2}{q} - 1\right)\, t\right]^{-q/(2-q)}$$

ii) Asymptotically Exponential Decay: If $m + k + 1 > 0$*, then there exists* $C > 0$ *and* $t_1 > 0$ *such that*

$$\Sigma_k[u(\cdot, t)] \leq \Sigma_k[u(\cdot, t_1)]\, \exp\left(-\frac{32\, p^2\, \pi^4\, \mu\, \bar{u}^{p(2-q)}\, (t - t_1)}{1 + C\sqrt{\Sigma_k[u(\cdot, t_1)]}}\right)$$

Thin film equation: range of the parameters



Left: algebraic decay

Right: asymptotic exponential decay

. . . further references end directions of research

- [J.D., Nazaret, Savaré], preliminary (formal): what has been done in terms of gradient flows for the linear case (Fokker-Planck equation) seems generalizable to the porous medium case
- Forth higher order equations: not much is understood from the entropy (PDE) point of view, [Jüngel, Matthes], [Laugesen], or from the gradient flow point of view. Gradient flow of the Fisher information: [Gianazza-Savaré-Toscani]

L^q Poincaré inequalities for general measures and consequences for the porous medium equation

J.D., Ivan Gentil, Arnaud Guillin and Feng-Yu Wang

Goal

L^q-Poincaré inequalities, $q \in (1/2, 1]$

$$\left[\mathbf{Var}_\mu(f^q)\right]^{1/q} := \left[\int f^{2q}\,d\mu - \left(\int f^q\,d\mu\right)^2\right]^{1/q} \leqslant \mathrm{C_P} \int |\nabla f|^2\,d\nu$$

Application to the weighted porous media equation, $m \geq 1$

$$\frac{\partial u}{\partial t} = \Delta u^m - \nabla\psi \cdot \nabla u^m\,, \quad t \geqslant 0\,, \quad x \in \mathbb{R}^d$$

(Ornstein-Uhlenbeck form). With $d\mu = d\nu = d\mu_\psi = e^{-\psi}\,dx / \int e^{-\psi}\,dx$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = -\frac{8}{(m+1)^2}\int |\,\nabla u^{\frac{m+1}{2}}\,|^2\,d\mu_\psi$$

Outline

Equivalence between the following properties:

- L^q-Poincaré inequality
- Capacity-measure criterion
- Weak Poincaré inequality
- BCR (Barthe-Cattiaux-Roberto) criterion

In dimension $d = 1$, there are necessary and sufficient conditions to satisfy the BCR criterion

Motivation: large time asymptotics in connection with functional inequalities

L^q-Poincaré inequality

M Riemanian manifold
Let μ a probability measure, ν a positive measure on M

We shall say that (μ, ν) satisfies a L^q-Poincaré inequality with constant $\mathrm{C_P}$ if for all non-negative functions $f \in \mathcal{C}^1(M)$ one has

$$\Big[\mathbf{Var}_\mu(f^q)\Big]^{1/q} \leqslant \mathrm{C_P} \int |\nabla f|^2 \, d\nu$$

$q \in (0, 1]$ (false for $q > 1$ unless μ is a Dirac measure)
$\mathbf{Var}_\mu(g^2) = \int g^2 \, d\mu - \left(\int g \, d\mu\right)^2 = \mu(g^2) - \mu(g)^2$
$q \mapsto \Big[\mathbf{Var}_\mu(f^q)\Big]^{1/q}$ increasing wrt $q \in (0, 1]$: L^q-Poincaré inequalities form a hierarchy

Capacity-measure criterion

Capacity $\mathrm{Cap}_\nu(A,\Omega)$ of two measurable sets A and Ω such that $A \subset \Omega \subset M$

$$\mathrm{Cap}_\nu(A,\Omega) := \inf\left\{ \int |\nabla f|^2 \, d\nu \; : \; f \in \mathcal{C}^1(M) \, , \, \mathbb{I}_A \leqslant f \leqslant \mathbb{I}_\Omega \right\}$$

$$\beta_{\mathrm{P}} := \sup\left\{ \sum_{k\in\mathbb{Z}} \frac{\left[\mu(\Omega_k)\right]^{1/(1-q)}}{\left[\mathrm{Cap}_\nu(\Omega_k,\Omega_{k+1})\right]^{q/(1-q)}} \right\}^{(1-q)/q}$$

over all $\Omega \subset M$ with $\mu(\Omega) \leq 1/2$ and all sequences $(\Omega_k)_{k\in\mathbb{Z}}$ such that for all $k \in \mathbb{Z}$, $\Omega_k \subset \Omega_{k+1} \subset \Omega$

Theorem 1 (i) *If* $q \in [1/2, 1)$, *then* $\beta_{\mathrm{P}} \leqslant 2^{1/q}\, \mathrm{C}_{\mathrm{P}}$

(ii) *If* $q \in (0,1)$ *and* $\beta_{\mathrm{P}} < +\infty$, *then* $\mathrm{C}_{\mathrm{P}} \leqslant \kappa_{\mathrm{P}}\, \beta_{\mathrm{P}}$

Weak Poincaré inequalities

Definition 2 [Röckner and Wang] (μ, ν) *satisfies a weak Poincaré inequality if there exists a non-negative non increasing function* $\beta_{\rm WP}(s)$ *on* $(0, 1/4)$ *such that, for any bounded function* $f \in \mathcal{C}^1(M)$*,*

$$\forall\ s > 0\,, \quad \mathbf{Var}_\mu(f) \leqslant \beta_{\rm WP}(s) \int |\nabla f|^2\, d\nu + s\left[\mathbf{Osc}_\mu(f)\right]^2$$

$\mathbf{Var}_\mu(f) \leqslant \mu((f-a)^2)\ \forall\ a \in \mathbb{R}$

For $a = (\mathrm{supess}_\mu f + \mathrm{infess}_\mu f)/2$, $\mathbf{Var}_\mu(f) \leqslant \left[\mathbf{Osc}_\mu(f)\right]^2/4$: $s \leqslant 1/4$.

Proposition 3 *Let* $q \in [1/2, 1)$*. If* (μ, ν) *satisfies the* L^q*-Poincaré inequality, then it also satisfies a weak Poincaré inequality with* $\beta_{\rm WP}(s) = (11 + 5\sqrt{5})\, \beta_{\rm P}\, s^{1-1/q}/2$, $K := (11 + 5\sqrt{5})/2$.

L^q-Poincaré $\Longrightarrow$ BCR criterion $\Longrightarrow$ weak Poincaré

Theorem 4 [Maz'ja] *Let $q \in [1/2, 1)$. For all bounded open set $\Omega \subset M$, if $(\Omega_k)_{k\in\mathbb{Z}}$ is a sequence of open sets such that $\Omega_k \subset \Omega_{k+1} \subset \Omega$, then*

$$\sum_{k\in\mathbb{Z}} \frac{\mu(\Omega_k)^{1/(1-q)}}{[\mathrm{Cap}_\nu(\Omega_k, \Omega_{k+1})]^{q/(1-q)}} \leqslant \frac{1}{1-q} \int_0^{\mu(\Omega)} \left(\frac{t}{\Phi(t)}\right)^{q/(1-q)} dt$$

where $\Phi(t) := \inf\{\mathrm{Cap}_\nu(A, \Omega) \,:\, A \subset \Omega\,,\ \mu(A) \geqslant t\}$

As a consequence: $\beta_{\mathrm{P}} \leqslant (1-q)^{-(1-q)/q} \, \|t/\Phi(t)\|_{L^{q/(1-q)}(0,\mu(\Omega))}$

Corollary 5 *Let $q \in [1/2, 1)$. If (μ, ν) satisfies a weak Poincaré inequality with function β_{WP}, then it satisfies a L^q-Poincaré inequality with*

$$\beta_{\mathrm{P}} \leqslant \frac{11 + 5\sqrt{5}}{2} \left(\frac{4}{1-q}\right)^{\frac{1-q}{q}} \|\beta_{\mathrm{WP}}(\cdot/4)\|_{L^{\frac{q}{1-q}}(0,1/2)}$$

L^q-Poincaré	$\Longrightarrow$	Weak Poincaré with $\beta_{\mathrm{WP}}(s) = C\, s^{\frac{q-1}{q}}$	$\Longrightarrow$	$L^{q'}$-Poincaré $\forall\, q' \in (0, q)$

BCR criterion (1/2)

A variant of two results of [Barthe, Cattiaux, Roberto, 2005] (no absolute continuity of the measure μ with respect to the volume measure)

Theorem 6 [BCR] *Let μ be a probability measure and ν a positive measure on M such that (μ, ν) satisfies a weak Poincaré inequality with function $\beta_{\mathrm{WP}}(s)$. Then for every measurable subsets A, B of M such that $A \subset B$ and $\mu(B) \leqslant 1/2$,*

$$\mathrm{Cap}_\nu(A, B) \geq \frac{\mu(A)}{\gamma(\mu(A))} \quad \textit{with} \quad \gamma(s) := 4\,\beta_{\mathrm{WP}}(s/4)$$

Proof $\triangleleft$ Take f such that $\mathbb{I}_A \leqslant f \leqslant \mathbb{I}_B$: $\mathbf{Osc}_\mu(f) \leqslant 1$

By Cauchy-Schwarz, $\left(\int f\,d\mu\right)^2 \leqslant \mu(B) \int f^2\,d\mu \leqslant \frac{1}{2} \int f^2\,d\mu$

$$\beta_{\mathrm{WP}}(s) \int |\nabla f|^2\,d\nu + s \geqslant \mathbf{Var}_\mu(f) \geq \frac{1}{2} \int f^2\,d\mu \geq \frac{\mu(A)}{2}$$

$\frac{a}{\gamma(a)} = \frac{a}{4\,\beta_{\mathrm{WP}}(a/4)} \leqslant \sup_{s \in (0,1/4)} \frac{a/2 - s}{\beta_{\mathrm{WP}}(s)}$ with $a/2 = \mu(A)/2 \leqslant 1/4$ $\triangleright$

BCR criterion (2/2)

Lemma 7 *Take μ and ν as before, $\theta \in (0,1)$, γ a positive non increasing function on $(0,\theta)$. If $\forall\ A, B \subset M$ such that $A \subset B$ are measurable and $\mu(B) \leqslant \theta$,*

$$\mathrm{Cap}_\nu(A,B) \geq \frac{\mu(A)}{\gamma(\mu(A))}$$

then for every function $f \in \mathcal{C}^1(M)$ such that $\mu(\Omega_+) \leqslant \theta$, $\Omega_+ := \{f > 0\}$

$$\int f_+^2 \leq \frac{11+5\sqrt{5}}{2}\,\gamma(s)\int_{\Omega_+} |\nabla f|^2\,d\nu + s\left[\text{supess}_\mu\, f\right]^2 \quad \forall\ s \in (0,1)$$

Theorem 8 *Same assumptions, $\theta = 1/2$. Then $\forall f \in \mathcal{C}^1(M)$*

$$\mathbf{Var}_\mu(f) \leq \frac{11+5\sqrt{5}}{2}\,\gamma(s)\int |\nabla f|^2\,d\nu + s\left[\mathbf{Osc}_\mu(f)\right] \quad \forall\ s \in (0,1/4)$$

$\theta = 1/2$: use the median $m_\mu(f)$, $\mu(f \geqslant m_\mu(f)) \geqslant 1/2$, $\mu(f \leqslant m_\mu(f)) \geqslant 1/2$

Using the BCR criterion: a "Hardy condition"

[Muckenhoupt, 1972] [Bobkov-Götze, 1999] [Barthe-Roberto, 2003] [Barthe-Cattiaux-Roberto, 2005]

$M = \mathbb{R}$, $d\mu = \rho_\nu \, dx$ with median m_μ, $d\nu = \rho_\nu \, dx$

$$R(x) := \mu([x, +\infty)) \;, \quad L(x) := \mu((-\infty, x])$$

$$r(x) := \int_{m_\mu}^{x} \frac{1}{\rho_\nu} \, dx \quad \text{and} \quad \ell(x) := \int_{x}^{m_\mu} \frac{1}{\rho_\nu} \, dx$$

Proposition 9 *Let $q \in [1/2, 1]$. (μ, ν) satisfies a L^q-Poincaré inequality if*

$$\int_{m_\mu}^{\infty} |\, r\, R\,|^{q/(1-q)} \, d\mu < \infty \quad \textit{and} \quad \int_{-\infty}^{m_\mu} |\, \ell\, L\,|^{q/(1-q)} \, d\mu < \infty$$

Proof

Proof $\triangleleft$ Method: $\mathbf{Var}_\mu(f) \leqslant \mu(|F_-|^2) + \mu(|F_+|^2))$ with $g = (f - f(m_\mu))_\pm$ and prove that

$$\mu(|g|^2) \leqslant \frac{11 + 5\sqrt{5}}{2}\, \gamma(s) \int |\nabla g|^2 \, d\nu + s \left[\text{supess}_\mu g\right]^2 \quad \forall\, s \in (0, 1/2)$$

Let $A \subset B \subset M = (m_\mu, \infty)$ such that $A \subset B$ and $\mu(B) \leqslant 1/2$

$$\mathrm{Cap}_\nu(A, B) \geqslant \mathrm{Cap}_\nu\big(A, (m_\mu, \infty)\big) = \mathrm{Cap}_\nu\big((a, \infty), (m_\mu, \infty)\big) = \frac{1}{r(a)}$$

where $a = \inf A$. Change variables: $t = R(a)$ and choose $\gamma(t) := t\,(r \circ R)^{-1}(t)$ for any $t \in (0, 1/2)$ $\triangleright$

Porous media equation

With $\psi \in \mathcal{C}^2(\mathbb{R}^d)$, $d\mu_\psi := \frac{e^{-\psi}\,dx}{Z_\psi}$, define $\mathcal{L}$ on $\mathcal{C}^2(\mathbb{R}^d)$ by

$$\forall\, f \in \mathcal{C}^2(\mathbb{R}^d) \quad \mathcal{L}f := \Delta f - \nabla\psi \cdot \nabla f$$

Such a generator $\mathcal{L}$ is symmetric in $L^2_{\mu_\psi}(\mathbb{R}^d)$,

$$\forall\, f,\, g \in \mathcal{C}^1(\mathbb{R}^d) \quad \int f\,\mathcal{L}g\,d\mu_\psi = -\int \nabla f \cdot \nabla g\,d\mu_\psi$$

Consider for $m > 1$ the weighted porous media equation

$$\begin{cases} \frac{\partial u}{\partial t} = \mathcal{L}\,u^m & \text{in} \quad \mathrm{Q} \\ u(\cdot, 0) = u_0 & \text{in} \quad \Omega \\ n \cdot \nabla u = 0 & \text{on} \quad \Sigma \end{cases}$$

$\Omega \subset \mathbb{R}^d$, $Q = \Omega \times [0, +\infty)$, $\Sigma = \partial\Omega \times [0, +\infty)$

$u \in \mathcal{C}^2$, L^1-contraction, existence and uniqueness

Asymptotic behavior

Theorem 10 *Let* $m \geqslant 1$ *and assume that* (μ_ψ, μ_ψ) *satisfies a* L^q*-Poincaré inequality,* $q = 2/(m+1)$

$$\mathbf{Var}_{\mu_\psi}(u(\cdot,t)) \leqslant \left(\left[\mathbf{Var}_{\mu_\psi}(u_0) \right]^{-(m-1)/2} + \frac{4\,m\,(m-1)}{(m+1)^2}\,\mathrm{C_P}\,t \right)^{-2/(m-1)}$$

Reciprocally, if the above inequality is satisfied for any u_0*, then* (μ_ψ, μ_ψ) *satisfies a* L^q*-Poincaré inequality with constant* $\mathrm{C_P}$

Proof $\triangleleft$

$$\frac{d}{dt}\mathbf{Var}_{\mu_\psi}(u) = 2\int u_t\,u\,d\mu_\psi = 2\int u\,\mathcal{L}u^m\,d\mu_\psi = -\frac{8m}{(m+1)^2}\int |\nabla u^{\frac{m+1}{2}}|^2\,d\mu_\psi$$

Apply the L^q-Poincaré inequality with $u = f^{2/(m+1)}$, $q = 2/(m+1)$

Reciprocally, a derivation at $t = 0$ gives the L^q-Poincaré inequality $\triangleright$

A conclusion on L^q-Poincaré inequalities

- The Hardy criterion makes the link with mass transport in dimension 1
- Observe that we have only algebraic rates
- Weak logarithmic Sobolev inequalities [Cattiaux-Gentil-Guillin, 2006], L^q-logarithmic Sobolev inequalities [D.-Gentil-Guillin-Wang, 2006]

$$\left(\int f^{2q} \frac{\log f^{2q}}{\int f^{2q}\,d\mu}\,d\mu\right) =: \mathbf{Ent}_\mu\left(f^{2q}\right)^{1/q} \leq \mathrm{C_{LS}} \int |\nabla f|^2\,d\mu$$

- Orlicz spaces, duality, connections with mass transport theory [Bobkov-Götze, 1999] [Cattiaux-Gentil-Guillin, 2006] [Wang, 2006] [Roberto-Zegarlinski, 2003] [Barthe-Cattiaux-Roberto, 2005]

Conclusion

- Entropy methods for higher order equations are not yet well understood, except from an algebraic point of view: [Jüngel, Matthes]
- Mass transport: ongoing work [J.D., Nazaret, Savaré]
- Diffusion limits: [J.D., Markowich, Ölz, Schmeiser]
- Applications to models in gravitation [McCann], [J.D., Fernández]
- Keller-Segel model: [Blanchet, J.D., Perthame], [J.D., Schmeiser], [Blanchet, Carrillo, Calvez]