# Generalized entropy methods and stability in Sobolev and related inequalities


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# Outline

1. Constructive stability results and entropy methods
   - Entropy methods
   - Stability for Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$
2. LSI and GNS inequalities: sphere, Gaussian measure
   - Stability results on the sphere
   - Interpolation and LSI inequalities: Gaussian measure
   - More results on logarithmic Sobolev inequalities
3. Stability for Sobolev and logarithmic Sobolev inequalities
   - Constructive results on $\mathbb{R}^d$
   - Strategy of the proof
   - Dimensional dependence and stability results for the log-Sobolev inequality

# Some references on “entropy methods”

- Model inequalities: [Gagliardo, 1958], [Nirenberg, 1958]
*Carré du champ*: [Bakry, Emery, 1985]
- Motivated by asymtotic rates of convergence in kinetic equations:
▷ linear diffusions: [Toscani, 1998], [Arnold, Markowich, Toscani, Unterreiter, 2001]
▷ nonlinear diffusion for the carré du champ [Carrillo, Toscani], [Carrillo, Vázquez], [Carrillo, Jüngel, Markowich, Toscani, Unterreiter]
▷ sharp global decay rates, nonlinear diffusions: [del Pino, JD, 2001] (variational methods), [Carrillo, Jüngel, Markowich, Toscani, Unterreiter] (carré du champ), [Jüngel], [Demange] (manifolds)
- Refinements and stability [Arnold, Dolbeault], [Blanchet, Bonforte, JD, Grillo, Vázquez], [JD, Toscani], [JD, Esteban, Loss], [Bonforte, JD, Nazaret, Simonov]
- Detailed stability results [JD, Brigati, Simonov]

▷ Side results: hypocoercivity; symmetry in CKN inequalities
▷ Angle of attack: *entropy methods and diffusion flows as a tool*

# Constructive stability results Gagliardo-Nirenberg-Sobolev inequalities – entropy methods

*A joint work with M. Bonforte, B. Nazaret and N. Simonov*
***Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method***
arXiv:2007.03674, to appear in *Memoirs of the AMS*

# Gagliardo-Nirenberg-Sobolev inequalities on $\mathbb{R}^d$

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}\geq\mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}\qquad\text{(GNS)}$$

***Strategy.*** Rewrite (GNS) in non-scale invariant form

$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{2}+\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{p+1}\geq\mathcal{K}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p\,\gamma}$$

Use the fast diffusion flow

$$\frac{\partial\rho}{\partial t}=\Delta\rho^m\quad(t,x)\in\mathbb{R}^+\times\mathbb{R}^d$$

with initial datum $\rho(t=0,\cdot)=|f|^{2p}$ and apply entropy methods

Range of exponents

$$1<p\leq\tfrac{d}{d-2}\quad\Longleftrightarrow\quad\tfrac{d-1}{d}=:m_1\leq m<1$$

- Sobolev inequality: $p=\frac{d}{d-2}$, $m=m_1$
- Logarithmic Sobolev inequality: $p=1$, $m=1$

# Entropy – entropy production inequality

*Fast diffusion equation (written in self-similar variables)*

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left( v \left( \nabla v^{m-1} - 2\,x \right) \right) = 0 \qquad (r\,\mathrm{FDE})$$

*Generalized entropy (free energy)* and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left( v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v - \mathcal{B}) \right) dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2 dx$$

satisfy an *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

[del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-4\,t}$$

The *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

is equivalent to the *Gagliardo-Nirenberg-Sobolev inequalities*

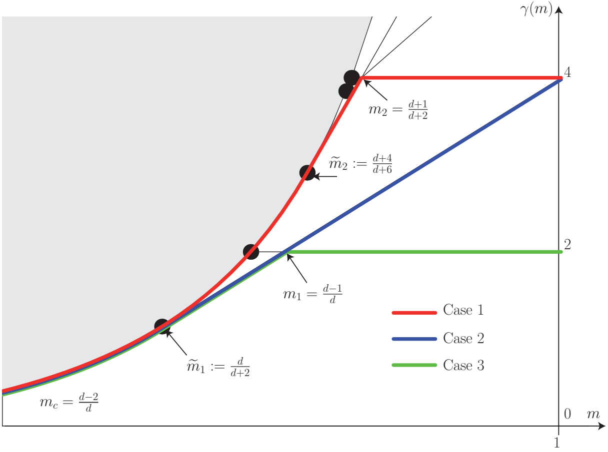
$$\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \qquad \text{(GNS)}$$

with equality if and only if $|f|^{2\,p}$ is the *Barenblatt profile* such that

$$|f(x)|^{2p} = \mathcal{B}(x) = \left(1+|x|^2\right)^{\frac{1}{m-1}}$$

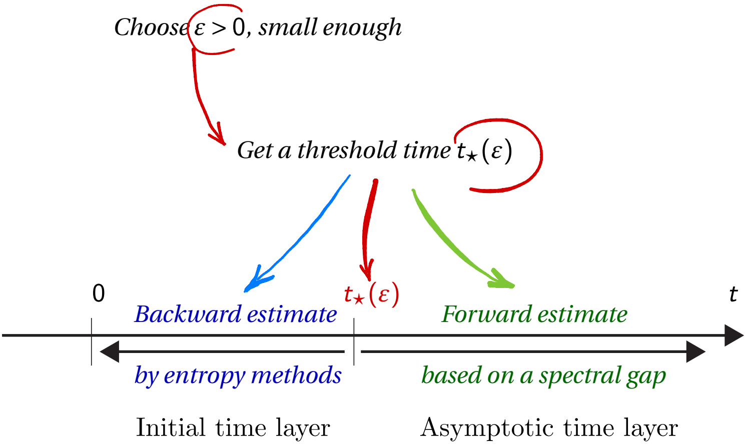
$v = f^{2\,p}$ so that $v^m = f^{p+1}$ and $v\left|\nabla v^{m-1}\right|^2 = (p-1)^2\,|\nabla f|^2$

# Spectral gap and Taylor expansion around $\mathcal{B}$




[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

# Strategy of the method



# A constructive stability result (subcritical case)

The ***stability in the entropy - entropy production estimate*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

$$\mathrm{A}[v] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} v\,dx < \infty$$

## Theorem

*Let $d \geq 1$ and $p \in (1,p^*)$. There is an explicit $\mathcal{C} = \mathcal{C}[f] > 0$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$ s.t. $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathrm{A}\big[f^{2p}\big] < \infty$*

$$(p-1)^2\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + 4\,\frac{d-p\,(d-2)}{p+1}\,\|f\|^{p+1}_{\mathrm{L}^{p+1}(\mathbb{R}^d)} - \mathcal{K}_{\mathrm{GNS}}\,\|f\|^{2p\gamma}_{\mathrm{L}^{2p}(\mathbb{R}^d)}$$
$$\geq \mathcal{C}[f] \inf_{\varphi\in\mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx$$

# The asymptotic time layer improvement

*Linearized free energy* and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2} \int_{\mathbb{R}^d} g^2\, \mathcal{B}^{2-m}\, dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m) \int_{\mathbb{R}^d} |\nabla g|^2\, \mathcal{B}\, dx$$

*Hardy-Poincaré inequality.* Let $d \geq 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m}\, dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}\, dx)$, $\int_{\mathbb{R}^d} g\, \mathcal{B}^{2-m}\, dx = 0$ and $\int_{\mathbb{R}^d} x\, g\, \mathcal{B}^{2-m}\, dx = 0$

$$\mathsf{I}[g] \geq 4\,\alpha\, \mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

## Proposition

*Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\, v\, dx = 0$ and*

$$(1-\varepsilon)\,\mathcal{B} \leq v \leq (1+\varepsilon)\,\mathcal{B}$$

*for some $\varepsilon \in (0, \chi\,\eta)$, then*

$$\mathcal{I}[v] \geq (4+\eta)\, \mathcal{F}[v]$$

# The initial time layer improvement: backward estimate

For some strictly convex function $\psi$ with $\psi(0)=0$, $\psi'(0)=1$, we have

$$\mathcal{I} - 4\,\mathcal{F} \ge 4\,(\psi(\mathcal{F}) - \mathcal{F}) \ge 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions), we expect an improvement

Rephrasing the *carré du champ* method, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

$$\frac{d\mathcal{Q}}{dt} \le \mathcal{Q}\,(\mathcal{Q} - 4)$$

**Lemma**

*Assume that $m > m_1$ and $v$ is a solution to $(r\,\mathrm{FDE})$ with nonnegative initial datum $v_0$. If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star,\cdot)] \ge 4 + \eta$, then*

$$\mathcal{Q}[v(t,\cdot)] \ge 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\, t \in [0, t_\star]$$

# Threshold time: uniform convergence in relative error

## Theorem

*[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $T \geq 0$ such that, if u is a solution of*

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

*with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying*

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \tag{$\mathrm{H}_A$}$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$ *and* $\mathcal{F}[u_0] \leq G$*, then*

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq T$$

# A constructive stability result (critical case)

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\ |x|\,f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

**Theorem**

*Let $d \geq 3$ and $A > 0$. For any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that*

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\,dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\,dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\,dx \leq A$$

*we have*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d^2\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\mathcal{C}_\star(A)}{4+\mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left|\nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}}\right|^2 dx$$

*$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\,\left(1 + A^{1/(2\,d)}\right)^{-1}$ and $\mathcal{C}_\star(0) > 0$ depends only on $d$*

# Comments, extensions

- *Gradient flows:* see for instance [Otto, 2001]
In the linear diffusion / Markov processes case, see [JD, Nazaret, Savaré] for various choices of the evolution equation / the entropy / the appropriate notion of distance to obtain a gradient flow

- Symmetry *without symmetrization:* rigidity results, as a byproduct of the entropy methods: [JD, Esteban, Loss, 2016] for an application to Caffarelli-Kohn-Nirenberg inequalities

- Some *rigidity results* in nonlinear elliptic PDEs can be reinterpreted using the *carré du champ* method:
[Gidas, Spruck, 1981], [Bidaut-Véron, Véron, 1991], [Demange, 2008] and also [Obata, 1971]

# Logarithmic Sobolev and Gagliardo-Nirenberg on the sphere

*A joint work with G. Brigati and N. Simonov*
***Logarithmic Sobolev and interpolation inequalities on the sphere: constructive stability results***
arXiv:2211.13180

▷ *Carré du champ methods combined with spectral information*

# (Improved) logarithmic Sobolev inequality

On the sphere $\mathbb{S}^d$ with $d \geq 1$

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d \geq \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left( \frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu_d \quad \forall \, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \tag{LSI}$$

$d\mu$: uniform probability measure; equality case: constant functions
Optimal constant: test functions $F_\varepsilon(x) = 1 + \varepsilon \, x \cdot \nu$, $\nu \in \mathbb{S}^d$, $\varepsilon \to 0$
▷ *improved inequality* under an appropriate *orthogonality condition*

**Theorem**

*Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that $\int_{\mathbb{S}^d} x \, F \, d\mu_d = 0$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log \left( \frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu_d \geq \frac{2}{d+2} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d$$

Improved ineq. $\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d \geq \left( \frac{d}{2} + 1 \right) \int_{\mathbb{S}^d} F^2 \log \left( F^2 / \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) d\mu_d$

# Logarithmic Sobolev inequality: stability (1)

What if $\int_{\mathbb{S}^d} x\,F\,d\mu_d \neq 0$ ? Take $F_\varepsilon(x)=1+\varepsilon\,x\cdot\nu$ and let $\varepsilon\to0$

$$\|\nabla F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\frac{d}{2}\int_{\mathbb{S}^d}F_\varepsilon^2\,\log\left(\frac{F_\varepsilon^2}{\|F_\varepsilon\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)d\mu_d=O(\varepsilon^4)=O\left(\|\nabla F_\varepsilon\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

Such a behaviour is in fact optimal: *carré du champ* method

## Proposition

*Let $d\ge1$, $\gamma=1/3$ if $d=1$ and $\gamma=(4\,d-1)\,(d-1)^2/(d+2)^2$ if $d\ge2$. Then, for any $F\in\mathrm{H}^1(\mathbb{S}^d,d\mu)$ with $\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}=1$ we have*

$$\int_{\mathbb{S}^d}|\nabla F|^2\,d\mu_d-\frac{d}{2}\int_{\mathbb{S}^d}F^2\,\log F^2\,d\mu_d\ge\frac12\,\frac{\gamma\,\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\gamma\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+d}$$

In other words, if $\|\nabla F\|_{\mathrm{L}^2(\mathbb{S}^d)}$ is small

$$\int_{\mathbb{S}^d}|\nabla F|^2\,d\mu_d-\tfrac{d}{2}\int_{\mathbb{S}^d}F^2\,\log F^2\,d\mu_d\ge\tfrac{\gamma}{2\,d}\,\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}+o\left(\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

# Logarithmic Sobolev inequality: stability (2)

Let $\Pi_1 F$ denote the orthogonal projection of a function $F \in \mathrm{L}^2(\mathbb{S}^d)$ on the spherical harmonics corresponding to the first eigenvalue on $\mathbb{S}^d$

$$\Pi_1 F(x) = \frac{x}{d+1} \cdot \int_{\mathbb{S}^d} y\, F(y)\, d\mu(y) \quad \forall\, x \in \mathbb{S}^d$$

▷ a global (and detailed) stability result

**Theorem**

*Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu_d - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu_d$$
$$\geq \mathscr{S}_d \left(\frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{2}\, \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1)\, F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

*for some explicit stability constant $\mathscr{S}_d > 0$*

# Improved Gagliardo-Nirenberg(-Sobolev) inequalities

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d \geq \frac{d}{p-2} \left( \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \quad \forall\, F \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \tag{GNS}$$

for any $p \in [1,2) \cup (2,2^*)$, with $d\mu$: uniform probability measure
$2^* := 2\,d/(d-2)$ if $d \geq 3$ and $2^* = +\infty$ otherwise
Optimal constant: test functions $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$, $\nu \in \mathbb{S}^d$, $\varepsilon \to 0$
logarithmic Sobolev inequality: obtained by taking the limit as $p \to 2$

**Theorem**

*Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that $\int_{\mathbb{S}^d} x\, F \, d\mu_d = 0$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d - \frac{d}{p-2} \left( \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \geq \mathscr{C}_{d,p} \int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d$$

*with $\mathscr{C}_{d,p} = \frac{2\,d - p\,(d-2)}{2\,(d+p)}$*

# Gagliardo-Nirenberg inequalities: stability (1)

With $F_\varepsilon(x) = 1 + \varepsilon\, x \cdot \nu$, the deficit is of order $\varepsilon^4$ as $\varepsilon \to 0$

**Proposition**

*Let $d \geq 1$ and $p \in (1,2) \cup (2,2^*)$. There is a convex function $\psi$ on $\mathbb{R}^+$ with $\psi(0) = \psi'(0) = 0$ such that, for any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d - \frac{d}{p-2} \left( \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \geq \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} \, \psi \left( \frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)}} \right)$$

This is also a consequence of the *carré du champ* method, with an explicit construction of $\psi$, and $\psi(s) = O(s^2)$ as $s \to 0_+$
There is no orthogonality constraint

# Gagliardo-Nirenberg inequalities: stability (2)

As in the case of the logarithmic Sobolev inequality, the improved inequality under orthogonality constraint and the stability inequality arising from the *carré du champ* method can be combined

**Theorem**

*Let $d \geq 1$ and $p \in (1,2) \cup (2,2^*)$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have*

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu_d - \frac{d}{p-2} \left( \|F\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right) \\ \geq \mathscr{S}_{d,p} \left( \frac{\|\nabla \Pi_1 F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla (\mathrm{Id} - \Pi_1) F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

*for some explicit stability constant $\mathscr{S}_{d,p} > 0$*

*On $\mathbb{R}^d$ with the Gaussian measure, Logaritmic Sobolev Inequalites are the limit case as $p \to 2$ and previous carré du champ methods fail*

# Gaussian interpolation inequalities

*Joint work with G. Brigati and N. Simonov*
***Gaussian interpolation inequalities***
arXiv:2302.03926

▷ *The large dimensional limit of the sphere*

# Large dimensional limit

*Gagliardo-Nirenberg-Sobolev inequalities* on $\mathbb{S}^d$, $p \in [1,2)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)}\right)$$

**Theorem**

*Let $v \in \mathrm{H}^1(\mathbb{R}^n, dx)$ with compact support, $d \ge n$ and*

$$u_d(\omega) = v\left(\omega_1/\sqrt{d}, \omega_2/\sqrt{d}, \ldots, \omega_n/\sqrt{d}\right)$$

*where $\omega \in \mathbb{S}^d \subset \mathbb{R}^{d+1}$. With $d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$,*

$$\lim_{d\to+\infty} d\left(\|\nabla u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{2-p}\left(\|u_d\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \|u_d\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)}\right)\right)$$
$$= \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right)$$

*Gaussian interpolation inequalities* on $\mathbb{R}^n$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \ge \frac{1}{2-p}\left(\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)}\right) \tag{1}$$

- $1 \le p < 2$ [Beckner, 1989], [Bakry, Emery, 1984]
- Poincaré inequality corresponding: $p = 1$
- *Gaussian logarithmic Sobolev inequality* $p \to 2$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} \ge \frac{1}{2}\int_{\mathbb{R}^n} |v|^2 \log\left(\frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right) d\gamma$$

$$d\gamma(y) := (2\,\pi)^{-n/2}\, e^{-\frac{1}{2}\,|y|^2}\, dy$$

# Carré du champ on $\mathbb{S}^d$

If $\Delta$ denotes the Laplace-Beltrami operator on $\mathbb{S}^d$ and

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left( \Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

then $\frac{d}{dt}\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} = 0$, $\frac{d}{dt}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 2\,(p-2)\int_{\mathbb{S}^d} u^{-\,p\,(1-m)}\,|\nabla u|^2\,d\mu_d$

$$m_\pm(d,p) := \frac{1}{(d+2)\,p} \left( d\,p + 2 \pm \sqrt{d\,(p-1)\,\big(2\,d-(d-2)\,p\big)} \right)$$

## Proposition

*If $m \in [m_-(d,p), m_+(d,p)]$, then for any $t>0$*

$$\frac{d}{dt} \left( \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} - \frac{d}{p-2} \left( \|u\|^2_{\mathrm{L}^p(\mathbb{S}^d,d\mu_d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d,d\mu_d)} \right) \right) \le 0$$

# Admissible parameters on $\mathbb{S}^d$

Monotonicity of the deficit along

$$\frac{\partial u}{\partial t} = u^{-p\,(1-m)} \left( \Delta u + (m\,p-1)\,\frac{|\nabla u|^2}{u} \right)$$

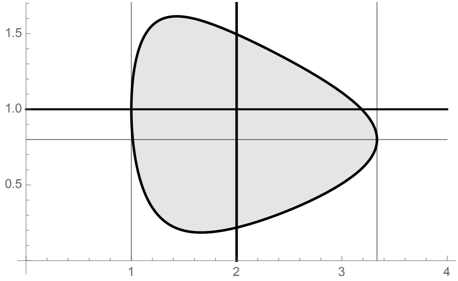



Figure: Case $d=5$: admissible parameters $1 \le p \le 2^* = 10/3$ and $m$ (horizontal axis: $p$, vertical axis: $m$)

# Gaussian carré du champ and nonlinear diffusion

$$\frac{\partial v}{\partial t} = v^{-p\,(1-m)}\left(\mathcal{L}v + (m\,p-1)\,\frac{|\nabla v|^2}{v}\right) \quad \text{on} \quad \mathbb{R}^n$$

*Ornstein-Uhlenbeck operator:* $\mathcal{L} = \Delta - x \cdot \nabla$

$$m_\pm(p) := \lim_{d\to+\infty} m_\pm(d,p) = 1 \pm \frac{1}{p}\sqrt{(p-1)\,(2-p)}$$

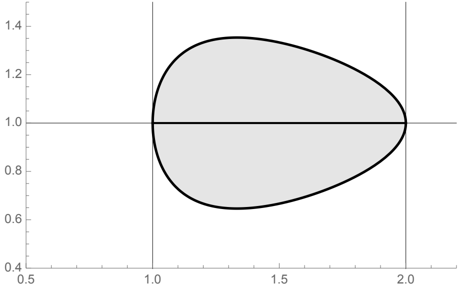



Figure: The admissible parameters $1 \leq p \leq 2$ and $m$ are independent of $n$

# A stability result for Gaussian interpolation inequalities

**Theorem**

*For all $n \geq 1$, and all $p \in (1,2)$, there is an explicit constant $c_{n,p} > 0$ such that, for all $v \in \mathrm{H}^1(d\gamma)$,*

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} - \frac{1}{p-2}\left(\|v\|^2_{\mathrm{L}^p(\mathbb{R}^n,d\gamma)} - \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}\right)$$
$$\geq c_{n,p}\left(\|\nabla(\mathrm{Id}-\Pi_1)v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \frac{\|\nabla\Pi_1 v\|^4_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)} + \|v\|^2_{\mathrm{L}^2(\mathbb{R}^n,d\gamma)}}\right)$$

# More results on logarithmic Sobolev inequalities

*Joint work with G. Brigati and N. Simonov*
***Stability for the logarithmic Sobolev inequality***
arXiv:2303.12926

▷ *Entropy methods, with constraints*

# Stability under a constraint on the second moment

$u_\varepsilon(x) = 1 + \varepsilon\, x$ in the limit as $\varepsilon \to 0$

$$d(u_\varepsilon, 1)^2 = \|u_\varepsilon'\|^2_{\mathrm{L}^2(\mathbb{R}, d\gamma)} = \varepsilon^2 \quad \text{and} \quad \inf_{w \in \mathscr{M}} \mathsf{d}(u_\varepsilon, w)^\alpha \le \frac{1}{2}\, \varepsilon^4 + O\big(\varepsilon^6\big)\,.$$

$\mathscr{M} := \big\{ w_{a,c} \,:\, (a, c) \in \mathbb{R}^d \times \mathbb{R} \big\}$ where $w_{a,c}(x) = c\, e^{-a \cdot x}$

## Proposition

*For all* $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ *such that* $\|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$ *and* $\|x\, u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le d$, *we have*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\sigma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\sigma \ge \frac{1}{2\, d} \left( \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\sigma \right)^2$$

*and, with* $\psi(s) := s - \frac{d}{4} \log\left(1 + \frac{4}{d}\, s\right)$,

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\sigma)} - \frac{1}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\, d\sigma \ge \psi\left( \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d, d\sigma)} \right)$$

# Stability under log-concavity

$$\mathscr{C}_\star = 1 + \frac{1}{1728} \approx 1.0005787$$

**Theorem**

*For all $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ such that $u^2\,\gamma$ is log-concave and such that*

$$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\sigma = (1,0) \quad \text{and} \quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\sigma \le d$$

*we have*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} - \frac{\mathscr{C}_\star}{2} \int_{\mathbb{R}^d} |u|^2 \log |u|^2\,d\sigma \ge 0$$

## Theorem

*Let $d \ge 1$. For any $\varepsilon > 0$, there is some explicit $\mathscr{C} > 1$ depending only on $\varepsilon$ such that, for any $u \in \mathrm{H}^1(\mathbb{R}^d, d\gamma)$ with*

$$\int_{\mathbb{R}^d} (1,x)\,|u|^2\,d\sigma = (1,0)\,,\quad \int_{\mathbb{R}^d} |x|^2\,|u|^2\,d\sigma \le d\,,\quad \int_{\mathbb{R}^d} |u|^2\,e^{\,\varepsilon\,|x|^2}\,d\sigma < \infty$$

*for some $\varepsilon > 0$, then we have*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\sigma)} \ge \frac{\mathscr{C}}{2}\int_{\mathbb{R}^d} |u|^2\,\log|u|^2\,d\sigma$$

*Additionally, if $u$ is compactly supported in a ball of radius $R > 0$, then*

$$\mathscr{C} = 1 + \frac{\mathscr{C}_\star - 1}{1 + \mathscr{C}_\star\,R^2}$$

# Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence

*A joint work with JD, M.J. Esteban, A. Figalli, R. Frank, M. Loss*
***Sharp stability for Sobolev and log-Sobolev inequalities, with optimal dimensional dependence***
arXiv: 2209.08651

# Stability results for the Sobolev inequality

Sobolev inequality on $\mathbb{R}^d$ with $d \geq 3$

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \geq \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$$

with equality on the manifold $\mathcal{M}$ of the Aubin–Talenti functions

$$g(x) = c\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}\,, \quad a \in (0,\infty)\,, \quad b \in \mathbb{R}^d\,, \quad c \in \mathbb{R}$$

**Theorem**

*There is a constant $\beta > 0$ with an explicit lower estimate which does not depend on $d$ such that for all $d \geq 3$ and all $f \in \mathrm{H}^1(\mathbb{R}^d) \setminus \mathcal{M}$ we have*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

[JD, Esteban, Figalli, Frank, Loss]

Some important features of this result:
- No compactness argument
- The (estimate of the) constant $\beta$ is explicit
- The decay rate $\beta/d$ is optimal as $d \to +\infty$

# Some history

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mathsf{S}_d \, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$$

▷ $2^* = 2\,d/(d-2)$ is the *critical Sobolev exponent*

▷ $\mathsf{S}_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$ is the sharp Sobolev constant [Rodemich, 1966], [Rosen, 1971], [Aubin, 1976] and [Talenti, 1976]

▷ [Brezis, Lieb, 1985]: is it possible to bound the *Sobolev deficit* $\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ on $\dot{\mathrm{H}}^1(\mathbb{R}^d)$ from below by a distance to $\mathcal{M}$ ?

▷ [Lions, 1985] if the deficit is small for some function $f$, then $f$ has to be close to $\mathcal{M}$

▷ [Bianchi, Egnell, 1991] for any $d \ge 3$ there is a constant $c_{\mathrm{BE}} > 0$ s.t.

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \ge c_{\mathrm{BE}} \quad \forall\, f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)\setminus\mathcal{M}$$

▷ [Figalli, Glaudo, 2020] $\mathrm{H}^1$ distance to... by $\|-\Delta u + u^{2^*-1}\|_{\mathrm{H}^{-1}}$

▷ [König, 2022] $c_{\mathrm{BE}}$ is achieved and $c_{\mathrm{BE}} < 4/(d+4)$

# Comments

- The power two of the distance to $\mathcal{M}$ is optimal
- The strategy of Bianchi-Egnell is based

▷ on a local analysis in a neighbourhood of $\mathcal{M}$ (spectral analysis)
▷ on a reduction of the global estimate to a local estimate by the concentration-compactness method based on Lions's analysis

Our strategy is to make both steps of the strategy of Bianchi-Egnell constructive and based on

- The "far away" regime and the "neighbourhood" of $\mathcal{M}$
- Competing symmetries and a notion of a continuous flow (based on Steiner's symmetrization)

# Stability for Sobolev: main steps of the proof

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq \frac{\beta}{d}\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

▷ Step 1: Detailed Taylor expansion close to $\mathcal{M}$, with explicit remainder term

▷ Step 2: Far from $\mathcal{M}$: *competing symmetries* and continuous symmetrization (for *nonnegative* functions f)

▷ Step 3: *sign changing* functions by convexity arguments

▷ Step 4: Asymptotic dimensional dependence: refined local analysis

# A preliminary result (without optimal dependence in $d$)

$$\mathcal{E}[f] := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,,\quad \nu(\delta) := \sqrt{\frac{\delta}{1-\delta}}$$

## Theorem

[JD, Esteban, Figalli, Frank, Loss] Let $d \ge 3$, $q = 2\,d/(d-2)$. If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a *non-negative* function, then

$$\mathcal{E}[f] \ge \sup_{0<\delta<1} \delta\,\mu(\delta)$$

where $\mu(\delta) \ge \mathsf{m}\big(\nu(\delta)\big)$ and

$$\mathsf{m}(\nu) := \tfrac{4}{d+4} - \tfrac{2}{q}\,\nu^{q-2} \qquad \text{if} \quad d \ge 6$$
$$\mathsf{m}(\nu) := \tfrac{4}{d+4} - \tfrac{1}{3}\,(q-1)\,(q-2)\,\nu - \tfrac{2}{q}\,\nu^{q-2} \qquad \text{if} \quad d = 4,5$$
$$\mathsf{m}(\nu) := \tfrac{4}{7} - \tfrac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \tfrac{1}{3}\,\nu^4 \qquad \text{if} \quad d = 3$$

# Strategy: two regions

- *Taylor expansion, spectral estimates:* in the region $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, prove that

$$\mathcal{E}[f] \ge \mu(\delta)$$

- *Continuous flow argument:* [Christ, 2017] if $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \delta \, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, build a *flow* $(f_\tau)_{0\le\tau<\infty}$ s.t.

$$f_0 = f\,, \quad \|f_\tau\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,, \quad \tau \mapsto \|\nabla f_\tau\|_{\mathrm{L}^2(\mathbb{R}^d)} \text{ is } \searrow$$

$$\lim_{\tau\to\infty} \inf_{g\in\mathcal{M}} \|\nabla(f_\tau - g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = 0$$

$$\mathcal{E}[f] \ge \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} = 1 - S_d\,\frac{\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \ge \frac{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f_{\tau_0}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}$$

for some $\tau_0$ (it exists ?) s.t. $\inf_{g\in\mathcal{M}} \|\nabla(f_{\tau_0} - g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = \delta\,\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

... then $\mathcal{E}[f] \ge \mathcal{E}(f_{\tau_0}) \ge \delta\,\mu(\delta)$

# Step 1: Taylor expansion in the neighbourhood of $\mathcal{M}$

## Proposition

*Let* $\nu > 0$, $r \in \mathrm{H}^1(\mathbb{S}^d)$ *such that* $1 + r \geq 0$, $\|r\|_{\mathrm{L}^q(\mathbb{S}^d)} \leq \nu$ *and*

$$\int_{\mathbb{S}^d} r\, d\mu_d = 0 = \int_{\mathbb{S}^d} \omega_j\, r\, d\mu_d\,, \quad j = 1, \ldots, d+1$$

$$\int_{\mathbb{S}^d} \left(|\nabla r|^2 + A\,(1+r)^2\right) d\mu_d - A \left(\int_{\mathbb{S}^d} (1+r)^q\, d\mu_d\right)^{2/q} \geq \mathrm{m}(\nu) \int_{\mathbb{S}^d} \left(|\nabla r|^2 + A\, r^2\right) d\mu_d$$

$$\mathrm{m}(\nu) := \frac{4}{d+4} - \frac{2}{q}\,\nu^{q-2} \qquad \textit{if} \quad d \geq 6$$

$$\mathrm{m}(\nu) := \frac{4}{d+4} - \frac{1}{3}\,(q-1)\,(q-2)\,\nu - \frac{2}{q}\,\nu^{q-2} \qquad \textit{if} \quad d = 4\,,\,5$$

$$\mathrm{m}(\nu) := \frac{4}{7} - \frac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \frac{1}{3}\,\nu^4 \qquad \textit{if} \quad d = 3$$

# Analysis close to the manifold of optimizers

**Proposition**

*Let $X$ be a measure space and $u, r \in \mathrm{L}^q(X)$ for some $q \geq 2$ with $u \geq 0$ and $u + r \geq 0$. Assume also that $\int_X u^{q-1}\, r\, dx = 0$. If $2 \leq q \leq 3$, then*

$$\|u + r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left( (q-1) \int_X u^{q-2}\, r^2\, dx + \frac{2}{q} \int_X r_+^q\, dx \right)$$

$2 \leq q = \frac{2\,d}{d-2} \leq 3$ means $d \geq 6$ and is the most difficult case for Taylor

**Corollary**

*Let $q = 2^*$, $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ which realizes*

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$$

*Set $r := f - u$ and $\sigma := \|r\|_q / \|u\|_q$. If $d \geq 6$, we have*

$$\|\nabla f\|_2^2 - S_d\, \|f\|_q^2 \geq \int_{\mathbb{R}^d} \left( |\nabla r|^2 - S_d\, (q-1)\, \|u\|_q^{2-q}\, u^{q-2}\, r^2 \right) dx - \frac{2}{q}\, \|\nabla r\|_2^2\, \sigma^{q-2}$$

# Spectral gap estimate

*Cf.* [Rey, 1990] and [Bianchi, Egnell, 1991]

**Lemma**

*Let $d \geq 3$, $q = 2^*$, $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that $\|\nabla f - \nabla u\| = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|$. Then $r := f - u$ satisfies*

$$\int_{\mathbb{R}^d} \left( |\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,|u|^{q-2}\,r^2 \right) dx \geq \frac{4}{d+4} \int_{\mathbb{R}^d} |\nabla r|^2\,dx$$

**Corollary**

*Let $q = 2^*$ and $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$. Set $\mathcal{D}[f] := \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$ and $\tau := \mathcal{D}[f]/(\|\nabla f\|_2^2 - \mathcal{D}[f]^2)^{1/2}$. If $d \geq 6$, we have*

$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \geq \left( \tfrac{4}{d+4} - \tfrac{2}{q}\,\tau^{q-2} \right) \mathcal{D}[f]^2$$

# Step 2: The "far away" regime for nonnegative solutions

▷ We prove the inequality for *nonnegative* functions far from $\mathcal{M}$ using the method of *competing symmetries* and a continuous symmetrization

# Competing symmetries

[Carlen, Loss, 1990]

- On $\mathbb{S}^d$ (inverse stereographic projection) use the *conformal rotation*

$$(UF)(s) = F(s_1, s_2, \ldots, s_{d+1}, -s_d)$$

On $\mathbb{R}^d$, the function that corresponds to $UF$ on $\mathbb{R}^d$ is given by

$$(Uf)(x) = \left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x-e_d|^2}, \ldots, \frac{x_{d-1}}{|x-e_d|^2}, \frac{|x|^2-1}{|x-e_d|^2}\right)$$

where $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$ and $\mathcal{E}(Uf) = \mathcal{E}[f]$

- *Symmetric decreasing rearrangement*: if $f \geq 0$, let

$$\mathcal{R}f(x) = f^*(x)$$

$f$ and $f^*$ are equimeasurable and $\|\nabla f^*\|_2 \leq \|\nabla f\|_2$

*... continuous Steiner symmetrization*

On $\mathbb{R}^d$, let

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

## Theorem

[Carlen, Loss] Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions

$$f_n = (\mathcal{R}U)^n f$$

Then $h_f = \|f\|_{2^*}\, g_* \in \mathcal{M}$ and

$$\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$$

If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_2)_{n\in\mathbb{N}}$ is a non-increasing sequence

Define $\mathcal{M}_1$ to be the set of the elements in $\mathcal{M}$ with $2^*$-norm equal to 1

**Lemma**

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(f, g^{2^*-1}\right)^2$$

**Lemma**

*For the sequence $(f_n)_{n\in\mathbb{N}}$ of the Theorem of [Carlen, Loss] we have that*

$$n \mapsto \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_{2^*}^2 \text{ is strictly decreasing}$$

$$\lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d \|f\|_{2^*}^2$$

# Two alternatives

**Lemma**

Let $0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$ s.t. $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 \ge \delta\,\|\nabla f\|_2^2$
One of the following alternatives holds:

(a) for all $n = 0, 1, 2 \ldots$ $\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 \ge \delta\,\|\nabla f_n\|_2^2$

(b) $\exists n_0 \in \mathbb{N}$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|_2^2 \ge \delta\,\|\nabla f_{n_0}\|_2^2 \quad \text{and} \quad \inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|_2^2 < \delta\,\|\nabla f_{n_0+1}\|_2^2$$

In case (a) we have

$$\mathcal{E}[f] = \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \ge \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f\|_2^2} \ge \frac{\|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \ge \delta$$

because by the Theorem of [Carlen, Loss]

$$\lim_{n\to\infty} \|\nabla f_n\|_2^2 \le \frac{1}{\delta} \lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta}\left(\lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2\right)$$

# Continuous rearrangement

Let $\mathsf{f}_0 = U f_{n_0}$ and denote by $(\mathsf{f}_\tau)_{0\le\tau\le\infty}$ the continuous rearrangement starting at $\mathsf{f}_0$ and ending at $\mathsf{f}_\infty = f_{n_0+1}$
We find $\tau_0 \in [0,\infty)$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 = \delta\,\|\nabla \mathsf{f}_{\tau_0}\|_2^2$$

and conclude using

$$\mathcal{E}(\mathsf{f}_0) \ge 1 - S_d\,\frac{\|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} \ge 1 - S_d\,\frac{\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_{\tau_0}\|_2^2} = \delta\,\frac{\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d\,\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2} \ge \delta\,\mu(\delta)$$

Existence of $\tau_0$ not granted: a discussion is needed !
(use a semi-continuity property)

# Step 3: removing the positivity assumption

The *Bianchi-Egnell stability estimate*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq c_{\mathrm{BE}}\,\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

Nonnegative functions: $c_{\mathrm{BE}}^{\mathrm{pos}} \geq \delta\,\mu(\delta)$ and $c_{\mathrm{BE}} \leq c_{\mathrm{BE}}^{\mathrm{pos}} \leq \frac{4}{d+4}$

*Sign-changing solutions.* Take $m := \|u_-\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and assume that $1-m = \|u_+\|^{2^*}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$. We argue that $2\,h(1/2)\,m \leq h(m)$ if

$$h(m) := m^{1-\frac{2}{d}} + (1-m)^{1-\frac{2}{d}} - 1$$

With $\mathrm{D}(v) := \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|v\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}$ and (...), we obtain

$$\mathrm{D}(u) \geq c_{\mathrm{BE}}^{\mathrm{pos}}\,\|\nabla u_+ - \nabla g_+\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \frac{2\,h(1/2)}{2\,h(1/2)+\xi_d}\,\|\nabla u_-\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$\boxed{c_{\mathrm{BE}} \geq \tfrac{1}{2}\,\delta\,\mu(\delta)}$$

# Dimensional dependence and stability results for the log-Sobolev inequality

# An equivalent form of the stability inequality

Bianchi-Egnell stability estimate

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \ge \frac{\beta(d)}{d}\,\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

*We know that* $\beta_\star = \liminf_{d\to+\infty}\beta(d) > 0$

With the Aubin-Talenti function $g_\star(x) := \left(1+|x|^2\right)^{1-\frac{d}{2}}$ and $u = f/g_\star$,

$$\int_{\mathbb{R}^d}|\nabla u|^2\,g_\star^2\,dx + d\,(d-2)\int_{\mathbb{R}^d}|u|^2\,g_\star^{2^*}\,dx$$
$$-\,d\,(d-2)\,\|g_\star\|^{2^*-2}_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\left(\int_{\mathbb{R}^d}|u|^{2^*}\,g_\star^{2^*}\,dx\right)^{2/2^*}$$
$$\ge \frac{\beta(d)}{d}\left(\int_{\mathbb{R}^d}|\nabla u|^2\,g_\star^2\,dx + d\,(d-2)\int_{\mathbb{R}^d}|u-1|^2\,F_\star^{2^*}\,dx\right)$$

# A rescaling

$$u(x)=v\,(r_d\,x)\quad\forall\,x\in\mathbb{R}^d\,,\quad r_d=\sqrt{\frac{d}{2\,\pi}}$$

$$\begin{aligned}\int_{\mathbb{R}^d}|\nabla v|^2\left(1+\tfrac{1}{r_d^2}\,|x|^2\right)^2d\mu_d\\ \ge\pi\,(d-2)\left[\left(\int_{\mathbb{R}^d}|v|^{2^*}\,d\mu_d\right)^{2/2^*}-\int_{\mathbb{R}^d}|v|^2\,d\mu_d\right]\\ +\frac{\beta(d)}{d}\left(\int_{\mathbb{R}^d}|\nabla v|^2\,d\mu_d+(d-2)\int_{\mathbb{R}^d}|v-1|^2\,d\mu_d\right)\end{aligned}$$

where $d\mu_d=Z_d^{-1}\,g_\star^{2^*}\,dx$ is the probability measure given by

$$d\mu_d(x):=Z_d^{-1}\left(1+\tfrac{1}{r_d^2}\,|x|^2\right)^{-d}dx\quad\text{with}\quad Z_d=\frac{2^{1-d}\,\sqrt{\pi}}{\Gamma\left(\frac{d+1}{2}\right)}\left(\frac{d}{2}\right)^{\frac{d}{2}}$$

# The large dimensions limit in the Sobolev inequality

Let us consider a function $v(x)$ which actually depends only on $y \in \mathbb{R}^N$, where we write that $x = (y, z) \in \mathbb{R}^N \times \mathbb{R}^{d-N} \approx \mathbb{R}^N$, for some integer $N$ such that $1 \leq N < d$. With $|x|^2 = |y|^2 + |z|^2$ and

$$1 + \frac{1}{r_d^2}\,|x|^2 = 1 + \frac{1}{r_d^2}\left(|y|^2 + |z|^2\right) = \left(1 + \frac{1}{r_d^2}\,|y|^2\right)\left(1 + \frac{|z|^2}{r_d^2 + |y|^2}\right)$$

we can integrate over the $z$ variable and notice that

$$\lim_{d\to+\infty}\left(1 + \frac{1}{r_d^2}\,|y|^2\right)^{-\frac{N+d}{2}} = e^{-\pi\,|y|^2}$$

$$\lim_{d\to+\infty}\int_{\mathbb{R}^d}|v(y)|^2\,d\mu_d = \int_{\mathbb{R}^N}|v|^2\,d\gamma$$

$$\lim_{d\to+\infty}\int_{\mathbb{R}^d}|\nabla v|^2\left(1 + \frac{1}{r_d^2}\,|x|^2\right)^2 d\mu_d = \int_{\mathbb{R}^N}|\nabla v|^2\,d\gamma$$

where $d\gamma(y) := e^{-\pi\,|y|^2}\,dy$ is a standard Gaussian probability measure
*Gaussian logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^N}|\nabla v|^2\,d\gamma \geq \frac{1}{2}\int_{\mathbb{R}^N}|v|^2\,\log\left(\frac{|v|^2}{\int_{\mathbb{R}^N}|v|^2\,d\gamma}\right)d\gamma$$

*Gaussian logarithmic Sobolev inequality*

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, d\gamma \geq \pi \int_{\mathbb{R}^N} |v|^2 \ln \left( \frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\gamma)}} \right) d\gamma$$

With $d\gamma = e^{-\pi\,|x|^2}$, the constant $\pi$ is optimal
[Carlen, 1991] equality holds if and only if

$$v(x) = c\, e^{a\cdot x}$$

for some $a \in \mathbb{R}^N$ and $c \in \mathbb{R}$

**Theorem**

*There is an explicit constant $\kappa > 0$ such that $\forall\, N \in \mathbb{N}$ and $\forall\, v \in \mathrm{H}^1(\gamma)$*

$$\int_{\mathbb{R}^N} |\nabla v|^2 \, d\gamma - \pi \int_{\mathbb{R}^N} v^2 \ln \left( \frac{|v|^2}{\|v\|^2_{\mathrm{L}^2(\gamma)}} \right) d\gamma \geq \kappa \inf_{a\in\mathbb{R}^N,\, c\in\mathbb{R}} \int_{\mathbb{R}^N} (v - c\, e^{a\cdot x})^2 \, d\gamma$$

# Step 4 for Sobolev stability: optimal dependence

Refinement of Step 1: cutting $r$ into pieces

$$(1+r)^q - 1 - q\,r$$

for real numbers $r$ in terms of three numbers

$$r_1 := \min\{r,\gamma\}\,, \quad r_2 := \min\{(r-\gamma)_+, M-\gamma\} \quad \text{and} \quad r_3 := (r-M)_+$$

where $\gamma$ and $M$ are parameters such that $0 < \gamma < M$

$$\theta := q-2 = 2^*-2 = \tfrac{4}{d-2} \to 0 \quad \text{as} \quad d \to +\infty$$

## Lemma

*Given $q \in [2,3]$, $r \in [-1,\infty)$ and $\overline{M} \in [\sqrt{e},+\infty)$, we have*

$$\begin{aligned}(1+r)^q - 1 - q\,r &\le \tfrac12\,q\,(q-1)\,(r_1+r_2)^2 + 2\,(r_1+r_2)\,r_3 + \Big(1 + C_M\,\theta\,\overline{M}^{-1}\ln\overline{M}\Big)\,r_3^q \\ &\quad + \Big(\tfrac32\,\gamma\,\theta\,r_1^2 + C_{M,\overline{M}}\,\theta\,r_2^2\Big)\,\mathbb{1}_{\{r\le M\}} + C_{M,\overline{M}}\,\theta\,M^2\,\mathbb{1}_{\{r>M\}}\end{aligned}$$

$$\int_{\mathbb{S}^d} \left(|\nabla r|^2 + A\,(1+r)^2\right) d\mu_d - A \left(\int_{\mathbb{S}^d} (1+r)^q \, d\mu_d\right)^{2/q} \ge \theta\,\epsilon_0 \int_{\mathbb{S}^d} \left(|\nabla r|^2 + A\, r^2\right) d\mu_d + \sum_{k=1}^{3} I_k$$

$$I_1 := (1-\theta\,\epsilon_0) \int_{\mathbb{S}^d} \left(|\nabla r_1|^2 + A\, r_1^2\right) d\mu_d - A\,(q-1+\epsilon_1\,\theta) \int_{\mathbb{S}^d} r_1^2\, d\mu_d + A\,\sigma_0\,\theta \int_{\mathbb{S}^d} (r_2^2 + r$$

$$I_2 := (1-\theta\,\epsilon_0) \int_{\mathbb{S}^d} \left(|\nabla r_2|^2 + A\, r_2^2\right) d\mu_d - A\left(q-1+(\sigma_0 + C_{\epsilon_1,\epsilon_2})\,\theta\right) \int_{\mathbb{S}^d} r_2^2\, d\mu_d$$

$$I_3 := (1-\theta\,\epsilon_0) \int_{\mathbb{S}^d} \left(|\nabla r_3|^2 + A\, r_3^2\right) d\mu_d - \frac{2}{q}\, A\,(1+\epsilon_2\,\theta) \int_{\mathbb{S}^d} r_3^q\, d\mu_d - A\,\sigma_0\,\theta \int_{\mathbb{S}^d} r_3^2\, d\mu_d$$

for some parameter $\sigma_0 > 0$
$I_1$: spectral gap estimates
$I_3$: use the Sobolev inequality. The extra coefficient $\frac{2}{q} < 1$ gives us enough room to accomodate all error terms $I_2$: an improved spectral gap inequality using that $\mu\big(\{r_2 > 0\}\big)$ is small

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Thank you for your attention !