Entropy methods for parabolic and elliptic equations

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

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Young PDE's @ Roma

Outline

▷ **Without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows**

- The Bakry-Emery method on the sphere
- Rényi entropy powers
- Self-similar variables and relative entropies
- The role of the spectral gap

▷ **Symmetry breaking and linearization**

- The critical Caffarelli-Kohn-Nirenberg inequality
- Linearization and spectrum
- A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities

▷ **With weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**

- Large time asymptotics and spectral gaps
- A discussion of optimality cases

Inequalities without weights and fast diffusion equations

- ▷ The Bakry-Emery method on the sphere: a parabolic method
- ▷ Euclidean space: Rényi entropy powers
- ▷ Euclidean space: self-similar variables and relative entropies
- ▷ The role of the spectral gap

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right)d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla\rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial\rho}{\partial t} = \Delta\rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

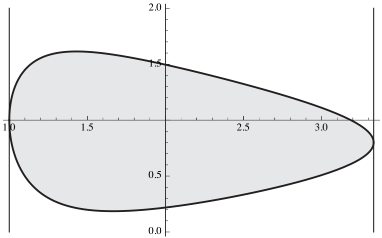
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

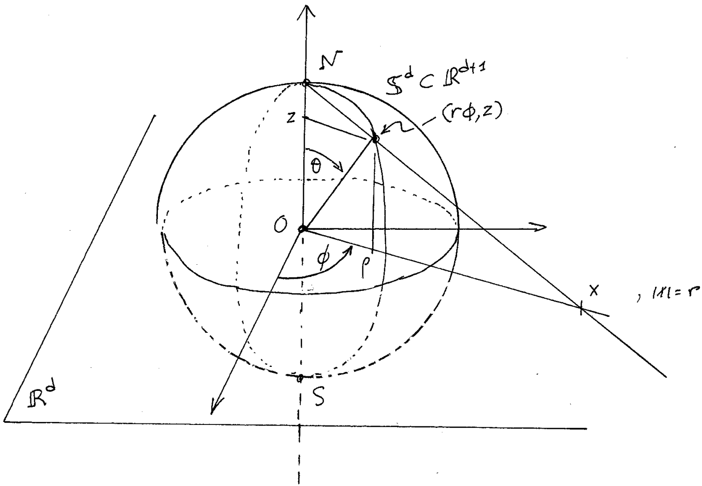
(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z = \cos\theta$, $v(\theta) = f(z)$, $d\nu_d := \nu^{\frac{d}{2}-1}\,dz/Z_d$, $\nu(z) := 1 - z^2$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\,f := (1 - z^2)\,f'' - d\,z\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1, \mathcal{L}\,f_2\rangle = -\int_{-1}^{1} f_1'\,f_2'\;\nu\;d\nu_d$

Proposition

Let $p \in [1,2) \cup (2,2^*]$, $d \geq 1$. For any $f \in \mathrm{H}^1([-1,1], d\nu_d)$,

$$-\langle f, \mathcal{L}\,f\rangle = \int_{-1}^{1} |f'|^2\;\nu\;d\nu_d \geq d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathcal{L}\, g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathcal{L}\, f\rangle = \langle \mathcal{L}\, f, \mathcal{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu, \mathcal{L}\, f\right\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

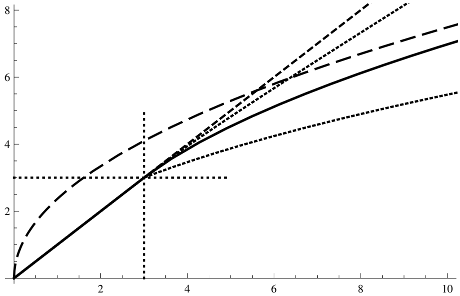
is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^{\#} < \frac{2\,d}{d-2} = 2^*$$

Bifurcation point of view



The interpolation inequality from the point of view of bifurcations

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) \le \lambda \quad \text{if (and only if)} \quad \lambda > \frac{d}{p-2}$$

▷ Improved inequalities under appropriate orthogonality constraints ?

Integral constraints

With the heat flow...

Proposition

For any $p \in (2, 2^\#)$, the inequality

$$\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{\lambda}{p-2} \, \|f\|_2^2 \geq \frac{\lambda}{p-2} \, \|f\|_p^2$$

$$\forall \, f \in \mathrm{H}^1((-1,1), d\nu_d) \text{ s.t. } \int_{-1}^{1} z \, |f|^p \, d\nu_d = 0$$

holds with

$$\lambda \geq d + \frac{(d-1)^2}{d\,(d+2)} \, (2^\# - p) \, (\lambda^\star - d)$$

... and with a nonlinear diffusion flow ?

Antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

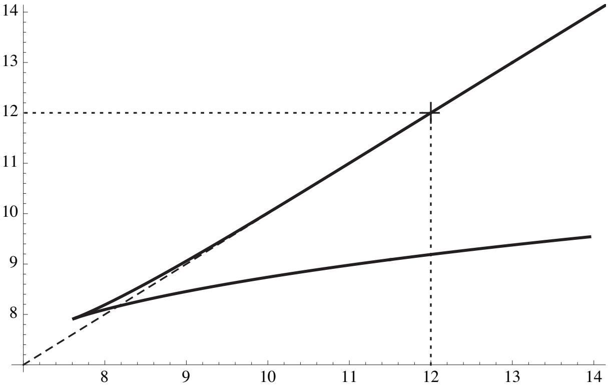
If $p \in (1,2) \cup (2,2^*)$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\mu \ge \frac{d}{p-2}\left[1+\frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1}\right]\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

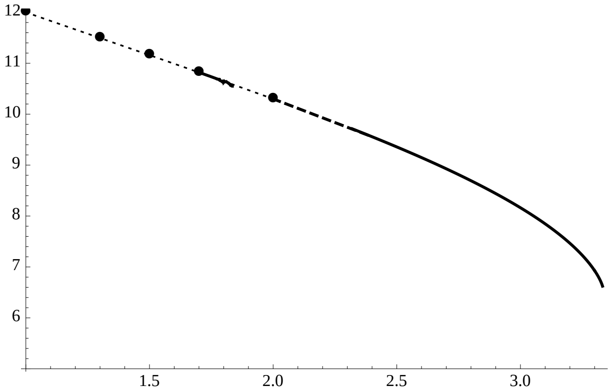
$$\int_{\mathbb{S}^d} |\nabla u|^2\,d\mu \ge \frac{d}{2}\,\frac{(d+3)^2}{(d+1)^2}\int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu$$

The larger picture: branches of antipodal solutions



Case $d = 5$, $p = 3$*: the branch* $\mu(\lambda)$ *as a function of* λ

The optimal constant in the antipodal framework



Numerical computation of the optimal constant when $d=5$ *and* $1\le p\le 10/3\approx 3.33$*. The limiting value of the constant is numerically found to be equal to* $\lambda_\star=2^{1-2/p}\,d\approx 6.59754$ *with* $d=5$ *and* $p=10/3$

Work in progress and open questions (1)

▷ **Adding weights**

- fractional dimensions: in the ultraspherical setting, if one adds the weight $(1-z^2)^{\frac{n-d}{2}}$ the *ultraspherical* operator becomes

$$\mathcal{L}\,f := (1-z^2)\,f'' - n\,z\,f' = \nu\,f'' + \frac{d}{2}\,\nu'\,f'$$

The Bakry-Emery method applies only for $p \le 2^\#$
The interpolation inequality is true up to $p = 2^*$ [Pearson]
Parabolic flows: [JD, Zhang] using $(\varepsilon^2 + 1 - z^2)^{\frac{n-d}{2}}$, in progress

- Open: general singular potentials beyond the range covered by linear flows ?

▷ **Constrained problems**

- The Lin-Ni problem in convex domains [JD, Kowalczyk]
- Branches with turning points corresponding to higher Morse index ?

The elliptic / rigidity point of view (nonlinear flow)

$u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$... Which computation do we have to do ?

$$-\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with

$$\int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d = 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

Rényi entropy powers and fast diffusion

- The Euclidean space without weights

▷ Rényi entropy powers, the entropy approach without rescaling: (Savaré, Toscani): scalings, nonlinearity and a concavity property inspired by information theory

▷ Faster rates of convergence: (Carrillo, Toscani), (JD, Toscani)

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t, x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The variation of the Fisher information

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $1_{\frac{1}{d}} \le m < 1$, then

$$\mathrm{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v\,|\nabla \mathrm{p}|^2\,dx = -\,2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathrm{p}\|^2 + (m-1)\,(\Delta \mathrm{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathrm{p}\|^2 - \frac{1}{d}\,(\Delta \mathrm{p})^2 = \left\| \mathrm{D}^2 \mathrm{p} - \frac{1}{d}\,\Delta \mathrm{p}\,\mathrm{Id} \right\|^2$$

.... there are no boundary terms in the integrations by parts ?

The concavity property

Theorem

[Toscani-Savaré] *Assume that $m \geq 1 - \frac{1}{d}$ if $d > 1$ and $m > 0$ if $d = 1$. Then $F(t)$ is increasing, $(1-m)\,\mathsf{F}''(t) \leq 0$ and*

$$\lim_{t\to+\infty} \frac{1}{t}\,\mathsf{F}(t) = (1-m)\,\sigma \lim_{t\to+\infty} \mathsf{E}^{\sigma-1}\,\mathsf{I} = (1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I} \geq \mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1 - \frac{1}{d} \leq m < 1$. Hint: $v^{m-1/2} = \frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}}$, $q = \frac{1}{2\,m-1}$

Euclidean space: self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type *Barenblatt solutions*

$$v_\star(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}}\, \mathcal{B}_\star\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mu := 2 + d\,(m-1)$$

where $\mathcal{B}_\star$ is the Barenblatt profile (with appropriate mass)

$$\mathcal{B}_\star(x) := \left(1 + |x|^2\right)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t,x) = \frac{1}{\kappa^d\, R^d}\, u\left(\tau, \frac{x}{\kappa\, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2}\,\log\left(\frac{R(t)}{R_0}\right)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla\cdot\Big[u\left(\nabla u^{m-1} - 2\,x\right)\Big] = 0$$

Free energy and Fisher information

- The function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[u \left(\nabla u^{m-1} - 2\,x \right) \right] = 0$$

- (Ralston, Newman, 1984) Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left(-\frac{u^m}{m} + |x|^2 u \right) dx - \mathcal{E}_0$$

- Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{E}[u] = -\,\mathcal{I}[u]\,, \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2\,x \right|^2 dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|u_\infty\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} = M > 0$

$$u_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

- *Entropy – entropy production inequality* (del Pino, JD)

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[u] \geq 4\,\mathcal{E}[u]$$

$p = \frac{1}{2m-1}$, $u = w^{2p}$: (GN) $\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \geq \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$

Corollary

(del Pino, JD) *A solution u with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies* $\mathcal{E}[u(t,\cdot)] \leq \mathcal{E}[u_0]\,e^{-4\,t}$

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - 2\,x \right) \Big] = 0 \quad \tau > 0\,, \quad x \in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R > 0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1} - 2\,x \right) \cdot \frac{x}{|x|} = 0 \quad \tau > 0\,, \quad x \in \partial B_R\,.$$

With $z(\tau,x) := \nabla \mathrm{Q}(\tau,x) := \nabla u^{m-1} - 2\,x$, the *relative Fisher information* is such that

$$\frac{d}{d\tau} \int_{B_R} u\,|z|^2\,dx + 4 \int_{B_R} u\,|z|^2\,dx$$
$$+\, 2\,\tfrac{1-m}{m} \int_{B_R} u^m \left(\left\| \mathrm{D}^2 \mathrm{Q} \right\|^2 - (1-m)\,(\Delta \mathrm{Q})^2 \right) dx$$
$$= \int_{\partial B_R} u^m \left(\omega \cdot \nabla |z|^2 \right) d\sigma \le 0 \quad \text{(by Grisvard's lemma)}$$

Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

(Blanchet, Bonforte, JD, Grillo, Vázquez) *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_* := \frac{d-4}{d-2}$, *the entropy decays according to*

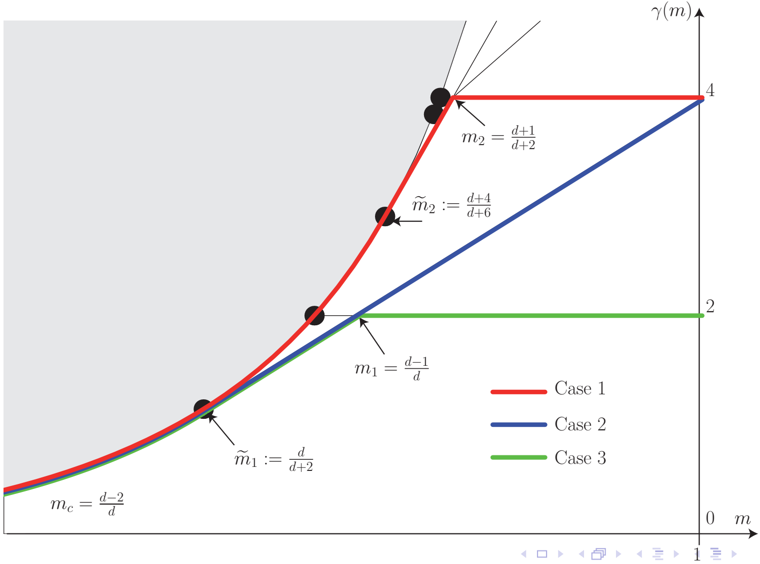
$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\ t \ge 0$$

where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\ f \in H^1(d\mu_\alpha)\,,\ \int_{\mathbb{R}^d} f d\mu_{\alpha-1} = 0$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gap and best constants



Symmetry and symmetry breaking results

▷ The critical Caffarelli-Kohn-Nirenberg inequality

▷ Linearization and spectrum

▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

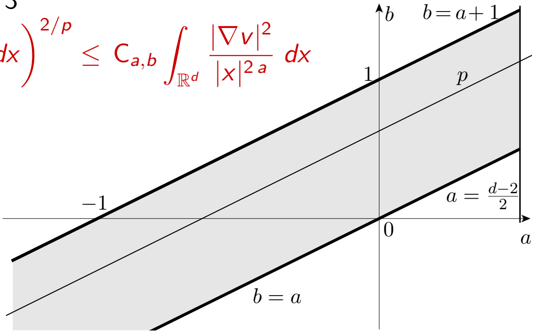
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d=2$, $a+1/2 < b \le a+1$ if $d=1$

and $a < a_c := (d-2)/2$

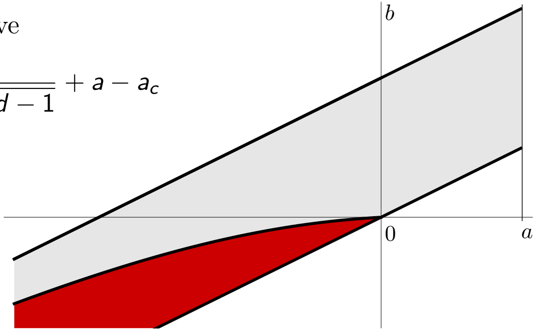
$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

(Glaser, Martin, Grosse, Thirring (1976))
(Caffarelli, Kohn, Nirenberg (1984))
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$



[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

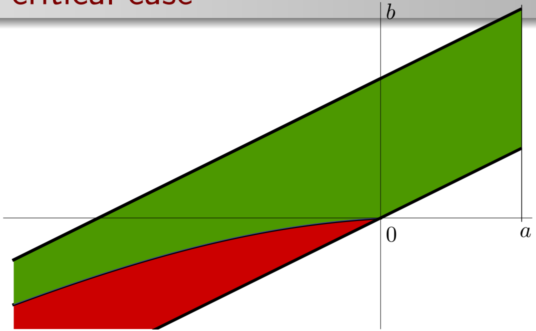
The functional

$$\mathsf{C}^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad\omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}\quad\forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda:=(a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}}\quad\text{with}\quad a=a_c\pm\sqrt{\Lambda}\quad\text{and}\quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s,\omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$\mathrm{H}^1(\mathcal{C}) \ni \varphi \mapsto \|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda \, \|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}$$

under a constraint on $\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}$

$\varphi_\star$ *is not* optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_\star^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*, i.e., for $\Lambda > \Lambda_1$ (explicit)

The variational problem on the cylinder

$$\Lambda \mapsto \mu(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathcal{C})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\, \|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

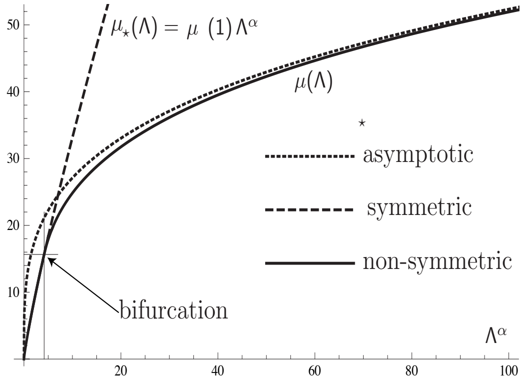
is a concave increasing function

Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathbb{R})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \Lambda\, \|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\varphi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}} = \mu_\star(1)\, \Lambda^\alpha$$

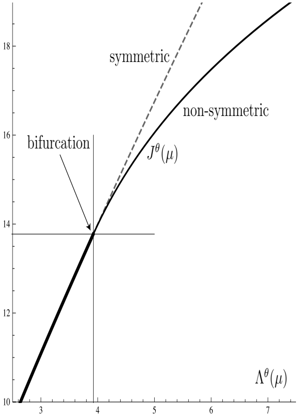
Symmetry means $\mu(\Lambda) = \mu_\star(\Lambda)$
Symmetry breaking means $\mu(\Lambda) < \mu_\star(\Lambda)$

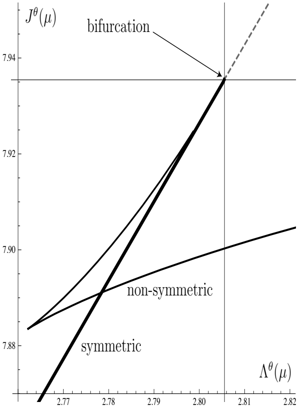
Numerical results

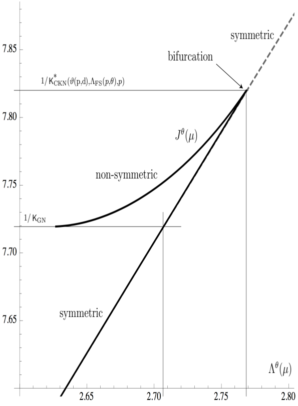


Parametric plot of the branch of optimal functions for $p = 2.8$*,* $d = 5$*. Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

what we have to to prove / discard...







When the local criterion (linear stability) differs from global results in a larger family of inequalities (center, right)...

The elliptic problem: rigidity

The symmetry issue can be reformulated as a uniqueness (rigidity) issue. An optimal function for the inequality

$$\left(\int_{\mathbb{R}^d}\frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}\le \mathsf{C}_{a,b}\int_{\mathbb{R}^d}\frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$

solves the (elliptic) Euler-Lagrange equation

$$-\nabla\cdot\left(|x|^{-2a}\,\nabla v\right)=|x|^{-bp}\,v^{p-1}$$

(up to a scaling and a multiplication by a constant). Is any nonnegative solution of such an equation equal to

$$v_\star(x)=\left(1+|x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}}$$

(up to invariances) ? On the cylinder

$$-\,\partial_s^2\,\varphi-\partial_\omega\varphi+\Lambda\,\varphi=\varphi^{p-1}$$

Up to a normalization and a scaling

$$\varphi_\star(s,\omega)=(\cosh s)^{-\frac{1}{p-2}}$$

Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^q(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \tag{CKN}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,,\quad \gamma-2 < \beta < \frac{d-2}{d}\,\gamma\,,\quad \gamma \in (-\infty, d)\,,\quad p \in (1, p_\star] \quad \text{with } p_\star := \frac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

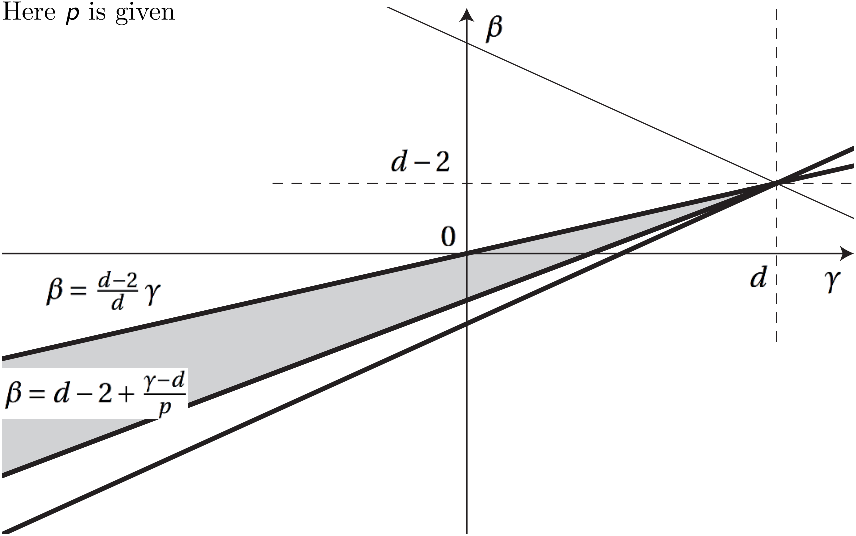
- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\,x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions ?

Range of the parameters

Here p is given



Symmetry and symmetry breaking

(M. Bonforte, JD, M. Muratori and B. Nazaret, 2016) Let us define $\beta_{\rm FS}(\gamma) := d - 2 - \sqrt{(d-\gamma)^2 - 4\,(d-1)}$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma < 0 \quad \text{and} \quad \beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$$

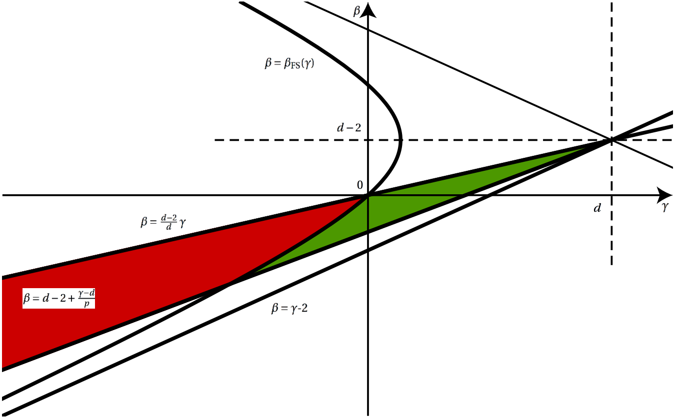
In the range $\beta_{\rm FS}(\gamma) < \beta < \frac{d-2}{d}\,\gamma$, $w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal

(JD, Esteban, Loss, Muratori, 2016)

Theorem

Symmetry holds in (CKN) if

$$\gamma \ge 0\,, \quad \text{or} \quad \gamma \le 0 \quad \text{and} \quad \gamma - 2 \le \beta \le \beta_{\rm FS}(\gamma)$$



The green area is the region of symmetry, while the red area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

▷ Entropy and Caffarelli-Kohn-Nirenberg inequalities

▷ Large time asymptotics and spectral gaps

▷ Optimality cases

CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{E}[v]\le\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}(x):=\left(1+|x|^{2+\beta-\gamma}\right)^{\frac{1}{m-1}}$
Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{E}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0\qquad\text{(WFDE-FP)}$$

then $\dfrac{d}{dt}\mathcal{E}[v(t,\cdot)]=-\dfrac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$

Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,,\quad \alpha = 1 + \frac{\beta-\gamma}{2}\quad\text{and}\quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|^{\vartheta}_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\,\|v\|^{1-\vartheta}_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathsf{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,,\quad \alpha > 0\,,\quad n > d\quad\text{and}\quad p\in(1,p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1+|x|^2\right)^{-1/(p-1)}\quad\forall\,x\in\mathbb{R}^d$$

The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathsf{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d}\,\frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d}\,dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

Looking for an optimal function in (CKN) is equivalent to minimize $\mathcal{G}$ under a mass constraint

With $\mathsf{L}_\alpha = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathsf{L}_\alpha\,u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma}\\
&\ge (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\left|\mathsf{L}_\alpha\mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}' - \frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right) u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\rm FS}^2 - \alpha^2\right)|\nabla_\omega\mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \le \alpha_{\rm FS}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result. *A quantifier elimination problem (Tarski, 1951) ?*

(3/3: elliptic regularity, boundary terms)

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathsf{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathsf{L}_\alpha u^m\,d\mu\ge\mathcal{H}[u]\ge 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \, \nabla \cdot \Big[|x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$$\zeta := 1 - \left(1 - \tfrac{2-m}{(1-m)\,q}\right)\left(1 - \tfrac{2-m}{1-m}\,\theta\right)$$

$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma}$ is in the range $0 < \theta < \frac{1-m}{2-m} < 1$

Theorem

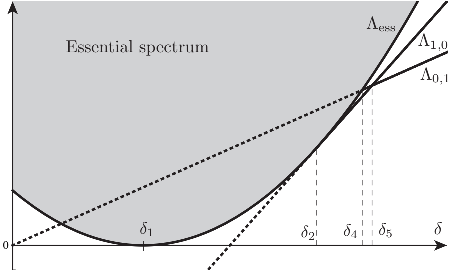
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

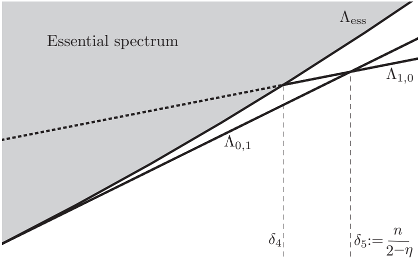
$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0,d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$





The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

Global vs. asymptotic estimates

- *Estimates on the global rates.* When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{E}[v]\le\frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{E}[v(t)]\le\mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t}\quad\forall\,t\ge0\quad\text{with}\quad\Lambda_\star:=\frac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

- *Optimal global rates.* Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{E}[v]\le\mathcal{I}[v]\quad\forall\,v\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)\quad\text{such that}\quad\|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}=M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M):=\frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{E}[v(t)]\le\mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t}\quad\forall\,t\ge0$$

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon\, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathsf{D}_\alpha^* \left(|x|^{-\beta}\, \mathcal{B}_\star\, \mathsf{D}_\alpha f\right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f_1 \cdot \mathsf{D}_\alpha f_2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

we compute

$$\frac{1}{2}\,\frac{d}{dt}\, \langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = \int_{\mathbb{R}^d} f\, (\mathcal{L}\, f)\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx = -\int_{\mathbb{R}^d} |\mathsf{D}_\alpha f|^2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

for any f smooth enough: with $\langle f, \mathcal{L}\, f \rangle = -\,\langle\!\langle f, f \rangle\!\rangle$

$$\frac{1}{2}\,\frac{d}{dt}\, \langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f \cdot \mathsf{D}_\alpha (\mathcal{L}\, f)\, \mathcal{B}_\star\, |x|^{-\beta}\, dx = -\,\langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$-\mathcal{L}\,f_1=\lambda_1\,f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g,g\rangle\!\rangle:=-\,\langle g,\mathcal{L}\,g\rangle\geq\lambda_1\,\|g-\bar g\|^2\,,\quad \bar g:=\langle g,1\rangle\,/\,\langle 1,1\rangle \tag{P1}$$

$$-\,\langle\!\langle g,\mathcal{L}\,g\rangle\!\rangle\geq\lambda_1\,\langle\!\langle g,g\rangle\!\rangle \tag{P2}$$

Proof by expansion of the square
$-\,\langle\!\langle(g-\bar g),\mathcal{L}\,(g-\bar g)\rangle\!\rangle=\langle\mathcal{L}\,(g-\bar g),\mathcal{L}\,(g-\bar g)\rangle=\|\mathcal{L}\,(g-\bar g)\|^2$

- (P1) is associated with the symmetry breaking issue
- (P2) is associated with the *carré du champ* method

The optimal constants / eigenvalues are the same

- Key observation: $\lambda_1\geq 4\quad\Longleftrightarrow\quad \alpha\leq\alpha_{\rm FS}:=\sqrt{\frac{d-1}{n-1}}$

A loss of compactness...

Work in progress with N. Simonov] Case $\lambda_1 > 4$, *i.e.*, $\alpha < \alpha_{\rm FS}$

- *free energy*, or *generalized relative entropy*

$$\mathcal{E}[v] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left(v^m - \mathfrak{B}^m - m\,\mathfrak{B}^{m-1} \left(v - \mathfrak{B} \right) \right) \frac{dx}{|x|^\gamma}$$

- *relative Fisher information*

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} - \nabla \mathfrak{B}^{m-1} \right|^2 \frac{dx}{|x|^\beta}$$

Proposition

[JD, Simonov] In the symmetry range, for any $M > 0$,

$$\inf \left\{ \frac{\mathcal{I}[v]}{\mathcal{E}[v]} \,:\, v \in \mathcal{D}_+(\mathbb{R}^d)\,,\ \int_{\mathbb{R}^d} v\,|x|^{-\gamma}\,dx = M \right\} = \frac{1-m}{m} \left(2 + \beta - \gamma \right)^2$$

Conjecture: in the symmetry breaking range, $\inf \frac{\mathcal{I}[v]}{\mathcal{E}[v]}$ is determined by the spectral gap

Work in progress and open questions (2)

- [with N. Simonov] Entropy – entropy production inequalities in the symmetry breaking range of CKN
- [with A. Zhang] Towards proofs in the weighted parabolic case (sphere: BGL Sobolev inequality and CKN): regularization of the weight ?
- [with N. Simonov and A. Zhang] Doubly nonlinear parabolic case (Euclidean space)
- [with M. García-Huidobro and R. Manásevich] Doubly nonlinear parabolic case (sphere)
- Full (analytic) parabolic proof based on the *carré du champ* method based on the analysis of the regularity of the flow in the neighborhood of degenerate points / singularities of the potentials [collaborators are welcome !]
- Hypo-coercive methods and (sharp) decay rates in coupled kinetic equations...

- Lecture notes on *Symmetry and nonlinear diffusion flows...* a course on entropy methods (see webpage)
- [JD, Maria J. Esteban, and Michael Loss] *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs* ... the elliptic point of view: arXiv: 1711.11291

These slides can be found at

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Thank you for your attention !