Stability estimates in some classical functional inequalities

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Differential Equation seminar

Roma 2 Tor Vergata

Outline

1. An introduction to entropy methods
   - The carré du champ method: φ-entropies
   - φ-entropies and diffusions
   - Interpolation inequalities on the sphere
2. Stability, fast diffusion equation and entropy methods
   - Rényi entropy powers, fast diffusion and Gagliardo-Nirenberg-Sobolev inequalities
   - The threshold time and the improved entropy – entropy production inequality (subcritical case)
   - Stability results (subcritical and critical case)
3. Stability in Caffarelli-Kohn-Nirenberg inequalities ?

An introduction to entropy methods

- Entropies and diffusions on $\mathbb{R}^d$ (linear case)
  - ▷ φ-entropies and entropy-entropy production inequalities
  - ▷ The Bakry-Emery or *carré du champ* method
  - ▷ Improvements and stability

- Interpolation inequalities on the sphere
  - ▷ From linear to nonlinear diffusion flows
  - ▷ Improved entropy-entropy production inequalities
  - ▷ Stability results

The Fokker-Planck equation (domain in $\mathbb{R}^d$)

The linear Fokker-Planck (FP) equation

$$\frac{\partial u}{\partial t} = \Delta u + \nabla \cdot (u \nabla \psi)$$

on a domain $\Omega \subset \mathbb{R}^d$, with no-flux boundary conditions

$$(\nabla u + u \nabla \psi) \cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

is equivalent to the Ornstein-Uhlenbeck (OU) equation

$$\frac{\partial v}{\partial t} = \Delta v - \nabla \psi \cdot \nabla v =: \mathscr{L} v$$

[Bakry, Emery, 1985], [Arnold, Markowich, Toscani, Unterreiter, 2001]
With mass normalized to 1, the unique stationary solution of (FP) is

$$u_s = e^{-\psi} \iff v_s = 1$$

Definition of the φ-entropies

If $d\gamma = e^{-\psi}\,dx$ is the invariant probability measure, let

$$\mathscr{E}[v] := \int_{\mathbb{R}^d} \varphi(v)\,d\gamma$$

φ is a nonnegative convex continuous function on $\mathbb{R}^+$ such that $\varphi(1) = 0$ and $1/\varphi''$ is concave on $(0,+\infty)$:

$$\varphi'' \ge 0\,, \quad \varphi \ge \varphi(1) = 0 \quad \text{and} \quad (1/\varphi'')'' \le 0$$

Classical examples

$$\varphi_p(v) := \tfrac{1}{p-1}\left(v^p - 1 - p\,(v-1)\right) \quad p \in (1,2]$$

$$\varphi_1(v) := v\log v - (v-1)\,, \qquad \varphi_2(v) := |v-1|^2$$

The invariant measure

$$d\gamma = e^{-\psi}\,dx$$

where ψ is a *potential* such that $e^{-\psi}$ is in $\mathrm{L}^1(\mathbb{R}^d, dx)$
$d\gamma$ is a probability measure

Entropy – entropy production inequalities, linear flows

Case of a smooth convex bounded domain Ω

$$\frac{\partial v}{\partial t} = \mathscr{L}\, v := \Delta v - \nabla\psi\cdot\nabla v\,,\quad \nabla v\cdot \nu = 0 \quad \text{on} \quad \partial\Omega$$

$$\frac{d}{dt}\int_\Omega \frac{v^q-1}{q-1}\,d\gamma = -\frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma \quad \text{and} \quad w = v^{q/2}$$

$$\frac{d}{dt}\int_\Omega |\nabla w|^2\,d\gamma \le -2\,\Lambda(q)\int_\Omega |\nabla w|^2\,d\gamma$$

where $\Lambda(q) > 0$ is the best constant in the inequality

$$\frac{2}{q}\,(q-1)\int_\Omega |\nabla X|^2\,d\gamma + \int_\Omega \mathrm{Hess}\,\psi : X\otimes X\,d\gamma \ge \Lambda(q)\int_\Omega |X|^2\,d\gamma$$

Proposition

$$\int_\Omega \frac{v^q-1}{q-1}\,d\gamma \le \frac{4}{q\,\Lambda(q)}\int_\Omega \left|\nabla v^{q/2}\right|^2 d\gamma \quad \textit{for any } v \textit{ s.t.} \quad \int_\Omega v\,d\gamma = 1$$

[Bakry, Emery, 1984] [JD, Nazaret, Savaré, 2008]

The Bakry-Emery method (domain in $\mathbb{R}^d$)

With $d\gamma = u_s\,dx$ and v such that $\int_\Omega v\,d\gamma = 1$, $q \in (1,2]$

- *q-entropy*

$$\mathcal{E}_q[v] := \frac{1}{q-1}\int_\Omega \left(v^q - 1 - q\,(v-1)\right) d\gamma$$

- *q-Fisher information* with $w = v^{q/2}$

$$\mathcal{I}_q[v] := \frac{4}{q}\int_\Omega |\nabla w|^2\,d\gamma$$

▷ The strategy

$$\frac{d}{dt}\mathcal{E}_q[v(t,\cdot)] = -\mathcal{I}_q[v(t,\cdot)] \quad \text{and} \quad \frac{d}{dt}\Big(\mathcal{I}_q[v] - 2\,\lambda\,\mathcal{E}_q[v]\Big) \le 0$$

▷ The decay rates

$$\mathcal{I}_q[v(t,\cdot)] \le \mathcal{I}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t} \quad \text{and} \quad \mathcal{E}_q[v(t,\cdot)] \le \mathcal{E}_q[v(0,\cdot)]\,e^{-2\,\lambda\,t}$$

▷ The entropy-entropy production inequality

$$\mathcal{I}_q[v] \ge 2\,\lambda\,\mathcal{E}_q[v] \quad \forall\, v \in \mathrm{H}^1(\Omega, d\gamma)$$

Properties of the φ-entropies

- Generalized Csiszár-Kullback-Pinsker inequality: [Pinsker], [Csiszár 1967], [Kullback 1967], [Cáceres, Carrillo, JD, 2002]

$$\mathcal{E}[v] \geq \mathscr{C}_q \, \|v-1\|^2_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}\,, \quad \mathscr{C}_q = \inf_{s\in(0,\infty)} \frac{s^{2-q}\,\varphi''(s)}{2^{2/q}} \min\left\{1, \|v\|^{q-2}_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}\right\}$$

- Tensorization and sub-additivity

$$\iint_{\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}} \varphi''(v)\,|\nabla v|^2\,d\gamma_1\,d\gamma_2 \geq \min\{\Lambda_1,\Lambda_2\}\,\mathcal{E}_{\gamma_1\otimes\gamma_2}[v]$$

- Holley-Stroock type perturbation results: if for some constants $a,\,b\in\mathbb{R}$, $e^{-b}\,d\gamma \leq d\mu \leq e^{-a}\,d\gamma$, then

$$e^{a-b}\,\Lambda \int_{\mathbb{R}^d} \left[\varphi(v)-\varphi(\tilde v)-\varphi'(\tilde v)(v-\tilde v)\right] d\mu \leq \int_{\mathbb{R}^d} \varphi''(v)\,|\nabla v|^2\,d\mu$$

Improved entropy – entropy production inequalities

In the special case $\psi(x)=|x|^2/2+\frac{d}{2}\log(2\pi)$, with $w=v^{q/2}$, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^d}|\nabla w|^2\,d\gamma+\int_{\mathbb{R}^d}|\nabla w|^2\,d\gamma\le-\frac{2}{q}\kappa_q\int_{\mathbb{R}^d}\frac{|\nabla w|^4}{w^2}\,d\gamma$$

with $\kappa_q=(q-1)(2-q)/q$

Cauchy-Schwarz: $\left(\int_{\mathbb{R}^d}|\nabla w|^2\,d\gamma\right)^2\le\int_{\mathbb{R}^d}\frac{|\nabla w|^4}{w^2}\,d\gamma\int_{\mathbb{R}^d}w^2\,d\gamma$

$$\frac{d}{dt}\mathscr{I}[v]+2\,\mathscr{I}[v]\le-\kappa_q\,\frac{\mathscr{I}[v]^2}{1+(q-1)\,\mathscr{E}[v]}$$

Proposition

Assume that $q\in(1,2)$ and $d\gamma=(2\pi)^{-d/2}\,e^{-|x|^2/2}\,dx$. There exists a strictly convex function Ψ such that $\Psi(0)=0$ and $\Psi'(0)=1$ and

$$\Psi\left(\|f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}-1\right)\le\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d,d\gamma)}\quad\text{if}\quad\|f\|_{\mathrm{L}^q(\mathbb{R}^d,d\gamma)}=1$$

Two references

- J.D. and X. Li. Phi-Entropies: convexity, coercivity and hypocoercivity for Fokker-Planck and kinetic Fokker-Planck equations. Mathematical Models and Methods in Applied Sciences, 28 (13): 2637-2666, 2018.

- D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators, volume 348 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Cham, 2014.

Improved inequalities and stability results

Entropy – entropy production inequality

$$\boxed{\mathcal{I}[u] \geq \Lambda\,\mathcal{E}[u]}$$

▷ ***Improved entropy – entropy production inequality*** (weaker form)

$$\mathcal{I} \geq \Lambda\,\Psi(\mathcal{E})$$

for some Ψ such that $\Psi(0)=0$, $\Psi'(0)=1$ and $\Psi''>0$

$$\mathcal{I} - \Lambda\,\mathcal{E} \geq \Lambda\,\big(\Psi(\mathcal{E}) - \mathcal{E}\big) \geq 0$$

▷ ***Improved constant*** means ***stability***
Under some restrictions on the functions, there is some $\Lambda_\star > \Lambda$ such that

$$\mathcal{I} - \Lambda\,\mathcal{E} \geq (\Lambda_\star - \Lambda)\,\mathcal{E} \geq 0 \quad \text{or} \quad \mathcal{I} - \Lambda\,\mathcal{E} \geq \left(1 - \frac{\Lambda}{\Lambda_\star}\right)\mathcal{I} \geq 0$$

Interpolation inequalities on the sphere

Interpolation inequalities on the sphere

- ▷ From linear to nonlinear diffusion flows
- ▷ Improved entropy-entropy production inequalities
- ▷ Stability results

A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

$p \in [1,2) \cup (2,2^*]$ if $d \ge 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1, 2$

Theorem

If $\lambda \le d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas, Spruck, 1981], [Bidaut-Véron, Véron, 1991]

Bifurcation point of view and symmetry breaking

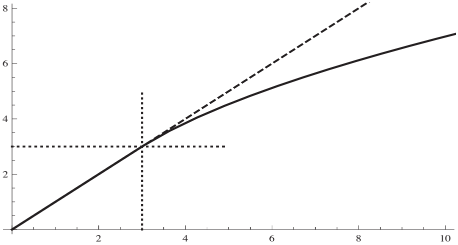


Figure: $(p-2)\,\lambda \mapsto (p-2)\,\mu(\lambda)$ *with* $d=3$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p-2}$$

▷ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry, Emery, 1985] [Beckner, 1993], [Bidaut-Véron, Véron, 1991, Corollary 6.1]

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

[Bakry, Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

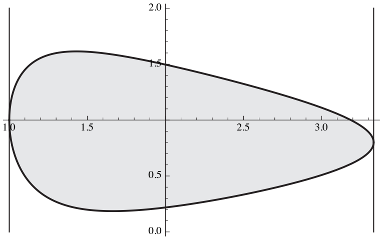
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^\#$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) admissible region, $d = 5$

Computation of the admissible region

With $\rho = |u|^{\beta p}$ and $m = 1 + \frac{2}{p}\left(\frac{1}{\beta} - 1\right)$, $\kappa = \beta\,(p-2)+1$, with the *trace free Hessian*

$$\mathrm{L}u := \mathrm{H}u - \frac{1}{d}\,(\Delta u)\,g_d$$

and the trace free tensor

$$\mathrm{M}u := \frac{\nabla u \otimes \nabla u}{u} - \frac{1}{d}\,\frac{|\nabla u|^2}{u}\,g_d$$

we have

$$\frac{d}{dt}\Big(\mathscr{I}_p[\rho] - d\,\mathscr{E}_p[\rho]\Big) = -\frac{d}{d-1}\left(\mathsf{a}\,\|\mathrm{L}u\|^2 - 2\,\mathsf{b}\,\mathrm{L}u : \mathrm{M}u + \mathsf{c}\,\|\mathrm{M}u\|^2\right)$$

$$\mathsf{a} = 1\,,\quad \mathsf{b} = (\kappa+\beta-1)\,\frac{d-1}{d+2}\,,\quad \mathsf{c} = (\kappa+\beta-1)\,\frac{d}{d+2} + \kappa\,(\beta-1)$$

so that the *admissible region* is defined by $\mathsf{b}^2 - \mathsf{a}\,\mathsf{c} \le 0$

The proof: two identities

Let us denote the *Hessian* by $\mathrm{H}v$ and define the *trace free Hessian* by

$$\mathrm{L}v := \mathrm{H}v - \frac{1}{d}\,(\Delta v)\,g_d$$

We also consider the following trace free tensor

$$\mathrm{M}v := \frac{\nabla v \otimes \nabla v}{v} - \frac{1}{d}\,\frac{|\nabla v|^2}{v}\,g_d$$

- first identity

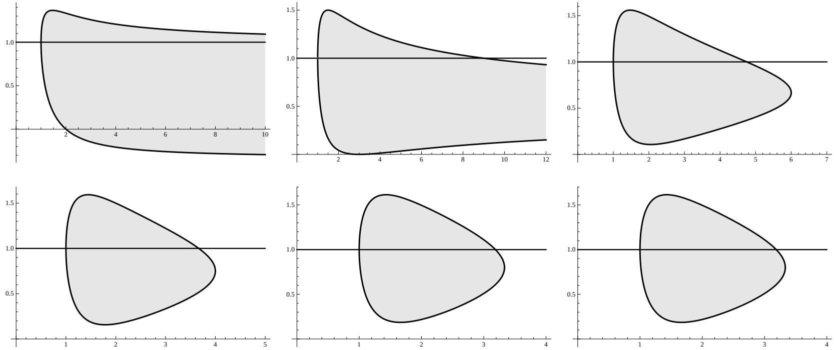
$$\int_{\mathbb{S}^d} \Delta v\,\frac{|\nabla v|^2}{v}\,d\mu = \frac{d}{d+2}\left(\frac{d}{d-1}\int_{\mathbb{S}^d} \|\mathrm{M}v\|^2\,d\mu - 2\int_{\mathbb{S}^d} \mathrm{L}v : \mathrm{M}v\,d\mu\right).$$

- second identity

$$\int_{\mathbb{S}^d} (\Delta v)^2\,d\mu = \frac{d}{d-1}\int_{\mathbb{S}^d} \|\mathrm{L}v\|^2\,d\mu + d\int_{\mathbb{S}^d} |\nabla v|^2\,d\mu$$

arises as a consequence of the Bochner-Lichnerowicz-Weitzenböck formula on $\mathbb{S}^d$

$$\frac{1}{2}\,\Delta\left(|\nabla v|^2\right) = \|\mathrm{H}v\|^2 + \nabla(\Delta v)\cdot\nabla v + (d-1)\,|\nabla v|^2$$



The admissible range for $d = 1, 2, 3$ *(first line), and* $d = 4, 5$ *and* 10 *(from left to right)*

Improved inequalities

▷ the monotonicity result

$$\frac{d}{dt}\Big(\mathscr{I}_p[\rho]-d\,\mathscr{E}_p[\rho]\Big)=-\frac{d}{d-1}\,\mathsf{a}\,\Big\|\mathrm{L}u-\frac{\mathsf{b}}{\mathsf{a}}\,\mathrm{M}\Big\|^2-\frac{d}{d-1}\left(\mathsf{c}-\frac{\mathsf{b}^2}{\mathsf{a}}\right)\|\mathrm{M}u\|^2$$

▷ improved inequalities [Arnold, JD, 2005], [JD, Nazaret, Savaré, 2008], [JD, Toscani, 2013], [JD, Esteban, Kowalczyk, Loss, 2014], [JD, Esteban, 2020]

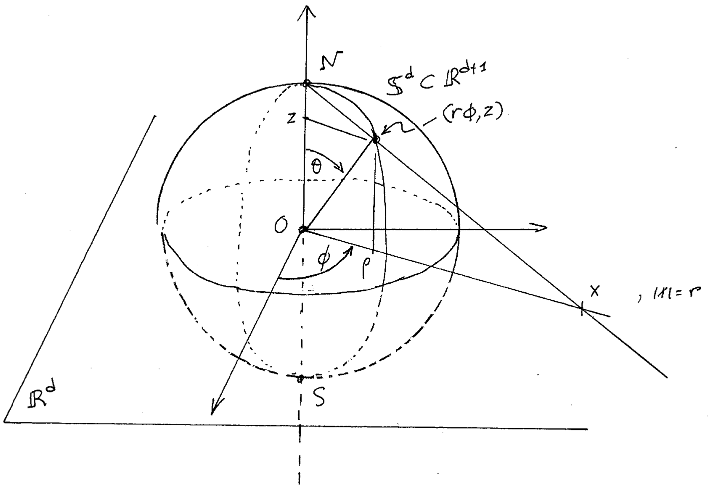
$$\mathscr{I}_p[\rho]\ge d\,\Psi\big(\mathscr{E}_p[\rho]\big)$$

for some convex Φ with $\Phi(0)=0$ and $\Phi'(0)=1$

▷ Application: with $d\ge2$, $2-p\neq\gamma:=\left(\frac{d-1}{d+2}\right)^2(p-1)\,(2^\#-p)>0$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge\frac{d}{2-p-\gamma}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\|u\|^{2-\frac{2\gamma}{2-p}}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{\frac{2\gamma}{2-p}}_{\mathrm{L}^2(\mathbb{S}^d)}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

Cylindrical coordinates, Schwarz symmetrization, stereographic projection...



... and the ultra-spherical operator

Change of variables $z=\cos\theta$, $v(\theta)=f(z)$, $d\nu_d:=\nu^{\frac{d}{2}-1}\,dz/Z_d$

$$\nu(z):=1-z^2$$

The self-adjoint *ultraspherical* operator is

$$\mathscr{L}\,f:=(1-z^2)\,f''-d\,z\,f'=\nu\,f''+\frac{d}{2}\,\nu'\,f'$$

which satisfies $\langle f_1,\mathscr{L}\,f_2\rangle=-\int_{-1}^{1}f_1'\,f_2'\,\nu\,d\nu_d$

Proposition

Let $p\in[1,2)\cup(2,2^*]$, $d\ge1$. *For any* $f\in\mathrm{H}^1([-1,1],d\nu_d)$,

$$-\langle f,\mathscr{L}\,f\rangle=\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d\ge d\,\frac{\|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2}$$

The heat equation $\frac{\partial g}{\partial t} = \mathscr{L}\, g$ for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathscr{L}\, f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f, \mathscr{L}\, f\rangle = \langle \mathscr{L}\, f, \mathscr{L}\, f\rangle + (p-1)\left\langle \frac{|f'|^2}{f}\,\nu, \mathscr{L}\, f\right\rangle$$

$$\frac{d}{dt}\mathscr{I}[g(t,\cdot)] + 2\,d\,\mathscr{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^{\#} < \frac{2\,d}{d-2} = 2^*$$

With integral constraints

With the heat flow...

Proposition

For any $p\in(2,2^\#)$, the inequality

$$\int_{-1}^1|f'|^2\,\nu\,d\nu_d+\frac{\lambda}{p-2}\,\|f\|_2^2\ge\frac{\lambda}{p-2}\,\|f\|_p^2$$

$$\forall\,f\in\mathrm H^1((-1,1),d\nu_d)\;s.t.\;\int_{-1}^1 z\,|f|^p\,d\nu_d=0$$

holds with

$$\lambda\ge d+\frac{(d-1)^2}{d\,(d+2)}\,(2^\#-p)\,(\lambda^\star-d)$$

... and with a nonlinear diffusion flow ?

With antipodal symmetry

With the additional restriction of *antipodal symmetry*, that is

$$u(-x) = u(x) \quad \forall\, x \in \mathbb{S}^d$$

Theorem

If $p \in (1,2) \cup (2,2^*)$, we have

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{p-2} \left[1 + \frac{(d^2-4)\,(2^*-p)}{d\,(d+2)+p-1} \right] \left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \right)$$

for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ with antipodal symmetry. The limit case $p=2$ corresponds to the improved logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\mu \ge \frac{d}{2} \frac{(d+3)^2}{(d+1)^2} \int_{\mathbb{S}^d} |u|^2 \log \left(\frac{|u|^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \right) d\mu$$

References

- JD, M. J. Esteban, M. Kowalczyk, and M. Loss. Improved interpolation inequalities on the sphere. Discrete and Continuous Dynamical Systems Series S, 7 (4): 695-724, 2014.
- JD, M.J. Esteban, and M. Loss. Interpolation inequalities on the sphere: linear *versus* nonlinear flows. Annales de la faculté des sciences de Toulouse Sér. 6, 26 (2): 351-379, 2017
- JD, M.J. Esteban. Improved interpolation inequalities and stability. Advanced Nonlinear Studies, 20 (2): 277-291, 2020.

Subcritical interpolation inequalities on the sphere: stability

[Frank, 2022] *Degenerate stability of some Sobolev inequalities*
Annales IHP C (2022), arXiv:2107.11608
If $d \geq 2$ and $2 < p < 2^*$, there is $c_{d,p} > 0$ such that, if $\int_{\mathbb{S}^d} u \, d\mu = 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d \, \frac{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2} \geq c_{d,p} \, \frac{\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2} \, \|u-1\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2}{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2} \, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

An optimal result: take $u(x) = 1 + \varepsilon \, z$

Theorem

If $d \geq 2$ and $2 < p < 2^*$, there is $\mathscr{C}_{d,p} > 0$ such that for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d \, \frac{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2} \geq \mathscr{C}_{d,p} \int_{\mathbb{S}^d} |\nabla u^\perp|^2 \, d\mu$$

with optimal constant $\mathscr{C}_{d,p} = \frac{2\,d - p\,(d-2)}{2\,d\,(d+p)}$ *[Brigati, JD, Simonov]*

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

A joint project with M. Bonforte, B. Nazaret and N. Simonov
Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method
arXiv:2007.03674, to appear in *Memoirs of the AMS*

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

- The Rényi entropy powers and the Gagliardo-Nirenberg inequalities
- Self-similar solutions and the entropy – entropy production method
- Large time asymptotics, spectral analysis (Hardy-Poincaré inequality)
- Initial time layer: improved entropy – entropy production estimates

Rényi entropy powers and Gagliardo-Nirenberg-Sobolev inequalities

[Toscani, Savaré, 2014]
[JD, Toscani, 2016]
[JD, Esteban, Loss, 2016]

Mass, moment, entropy and Fisher information

(i) *Mass conservation.* With $m \ge m_c := (d-2)/d$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u(t,x)\,dx = 0$$

(ii) *Second moment.* With $m > d/(d+2)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,dx)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} |x|^2\,u(t,x)\,dx = 2\,d\int_{\mathbb{R}^d} u^m(t,x)\,dx$$

(iii) *Entropy estimate.* With $m \ge m_1 := (d-1)/d$, $u_0^m \in \mathrm{L}^1(\mathbb{R}^d)$ and $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d,(1+|x|^2)\,dx)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d} u^m(t,x)\,dx = \frac{m^2}{1-m}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy functional and *Fisher information functional*

$$\mathsf{E}[u] := \int_{\mathbb{R}^d} u^m\,dx \quad \text{and} \quad \mathsf{I}[u] := \frac{m^2}{(1-m)^2}\int_{\mathbb{R}^d} u\,|\nabla u^{m-1}|^2\,dx$$

Entropy growth rate

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\ge\mathscr{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p}\qquad\text{(GNS)}$$

$$p=\frac{1}{2m-1}\quad\Longleftrightarrow\quad m=\frac{p+1}{2p}\in[m_1,1)$$

$u=f^{2p}$ so that $u^m=f^{p+1}$ and $u\,|\nabla u^{m-1}|^2=(p-1)^2\,|\nabla f|^2$

$$\mathscr{M}=\|f\|_{2p}^{2p}\,,\quad \mathsf{E}[u]=\|f\|_{p+1}^{p+1}\,,\quad \mathsf{I}[u]=(p+1)^2\,\|\nabla f\|_2^2$$

If u solves (FDE) $\frac{\partial u}{\partial t}=\Delta u^m$

$$\mathsf{E}'\ge\frac{p-1}{2p}\,(p+1)^2\,\Big(\mathscr{C}_{\mathrm{GNS}(p)}\Big)^{\frac{2}{\theta}}\,\|f\|_{2p}^{\frac{2}{\theta}}\,\|f\|_{p+1}^{-\frac{2(1-\theta)}{\theta}}=C_0\,\mathsf{E}^{1-\frac{m-m_c}{1-m}}$$

$$\int_{\mathbb{R}^d}u^m(t,x)\,dx\ge\left(\int_{\mathbb{R}^d}u_0^m\,dx+\frac{(1-m)\,C_0}{m-m_c}\,t\right)^{\frac{1-m}{m-m_c}}\quad\forall\,t\ge0$$

Equality case: $u(t,x)=\frac{c_1}{R(t)^d}\,\mathscr{B}\left(\frac{c_2\,x}{R(t)}\right)$, $\mathscr{B}(x):=\left(1+|x|^2\right)^{\frac{1}{m-1}}$

Pressure variable and decay of the Fisher information

The t-derivative of the *Rényi entropy power* $\mathsf{E}^{\frac{2}{d}\,\frac{1}{1-m}-1}$ is proportional to

$$\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}$$

The nonlinear *carré du champ method* can be used to prove (GNS) :

▷ *Pressure variable*

$$\mathsf{P} := \frac{m}{1-m}\,u^{m-1}$$

▷ *Fisher information*

$$\mathsf{I}[u] = \int_{\mathbb{R}^d} u\,|\nabla \mathsf{P}|^2\,dx$$

If u solves (FDE), then

$$\mathsf{I}' = \int_{\mathbb{R}^d} \Delta(u^m)\,|\nabla \mathsf{P}|^2\,dx + 2\int_{\mathbb{R}^d} u\,\nabla \mathsf{P}\cdot\nabla\Big((m-1)\,\mathsf{P}\,\Delta \mathsf{P} + |\nabla \mathsf{P}|^2\Big)\,dx$$

$$= -2\int_{\mathbb{R}^d} u^m\left(\|\mathrm{D}^2\mathsf{P}\|^2 - (1-m)\,(\Delta \mathsf{P})^2\right)dx$$

Rényi entropy powers and interpolation inequalities

▷ Integrations by parts and completion of squares: with $m_1 = \frac{d-1}{d}$

$$-\frac{\mathsf{I}}{2\theta}\,\frac{\mathrm{d}}{\mathrm{d}t}\log\left(\mathsf{I}^{\theta}\,\mathsf{E}^{2\,\frac{1-\theta}{p+1}}\right)$$

$$=\int_{\mathbb{R}^d} u^m\left\|\mathrm{D}^2\mathrm{P}-\frac{1}{d}\,\Delta\mathrm{P}\,\mathrm{Id}\right\|^2 dx+(m-m_1)\int_{\mathbb{R}^d} u^m\left|\Delta\mathrm{P}+\frac{\mathsf{I}}{\mathsf{E}}\right|^2 dx$$

▷ Analysis of the asymptotic regime as $t\to+\infty$

$$\lim_{t\to+\infty}\frac{\mathsf{I}[u(t,\cdot)]^{\theta}\,\mathsf{E}[u(t,\cdot)]^{2\,\frac{1-\theta}{p+1}}}{\mathscr{M}^{\frac{2\theta}{p}}}=\frac{\mathsf{I}[\mathscr{B}]^{\theta}\,\mathsf{E}[\mathscr{B}]^{2\,\frac{1-\theta}{p+1}}}{\|\mathscr{B}\|_1^{\frac{2\theta}{p}}}=(p+1)^{2\theta}\,(\mathscr{C}_{\mathrm{GNS}}(p))^{2\theta}$$

We recover the (GNS) Gagliardo-Nirenberg-Sobolev inequalities

$$\mathsf{I}[u]^{\theta}\,\mathsf{E}[u]^{2\,\frac{1-\theta}{p+1}}\ge(p+1)^{2\theta}\,(\mathscr{C}_{\mathrm{GNS}}(p))^{2\theta}\,\mathscr{M}^{\frac{2\theta}{p}}$$

The fast diffusion equation in self-similar variables

- ▷ Rescaling and self-similar variables
- ▷ Relative entropy and the entropy – entropy production inequality
- ▷ Large time asymptotics and spectral gaps

Entropy – entropy production inequality

With a time-dependent rescaling based on *self-similar variables*

$$u(t,x)=\frac{1}{\kappa^d\,R^d}\,v\Big(\tau,\frac{x}{\kappa\,R}\Big)\quad\text{where}\quad\frac{dR}{dt}=R^{1-\mu},\quad\tau(t):=\tfrac12\log R(t)$$

$\frac{\partial u}{\partial t}=\Delta u^m$ is changed into *a Fokker-Planck type equation*

$$\frac{\partial v}{\partial\tau}+\nabla\cdot\Big[v\Big(\nabla v^{m-1}-2\,x\Big)\Big]=0\qquad\qquad(r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathscr{F}[v]:=-\frac1m\int_{\mathbb{R}^d}\Big(v^m-\mathscr{B}^m-m\,\mathscr{B}^{m-1}\,(v-\mathscr{B})\Big)\,dx$$

$$\mathscr{I}[v]:=\int_{\mathbb{R}^d}v\,\Big|\nabla v^{m-1}+2\,x\Big|^2\,dx$$

are such that $\mathscr{I}[v]\ge4\,\mathscr{F}[v]$ by (GNS) [del Pino, JD, 2002] so that

$$\mathscr{F}[v(t,\cdot)]\le\mathscr{F}[v_0]\,e^{-4\,t}$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

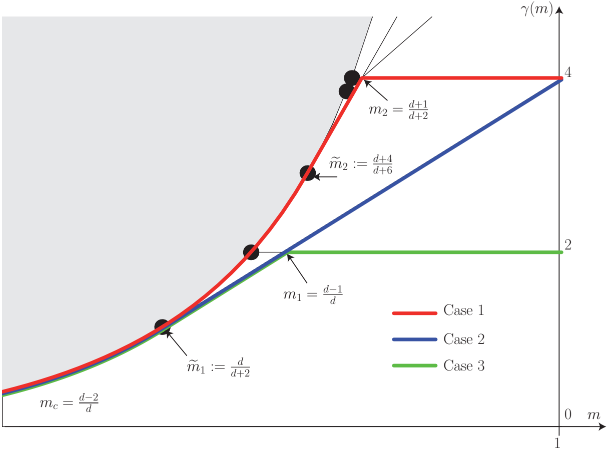
Lemma

Under assumption (H),

$$\mathscr{F}[v(t,\cdot)] \le C\,e^{-2\gamma(m)\,t} \quad \forall\, t\ge 0\,, \quad \gamma(m) := (1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m) := 2$ *if* $\frac{d-1}{d} = m_1 \le m < 1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2}\int_{\mathbb{R}^d} g^2\,\mathscr{B}^{2-m}\,dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m)\int_{\mathbb{R}^d} |\nabla g|^2\,\mathscr{B}\,dx$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}^{2-m}\,dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathscr{B}\,dx)$, $\int_{\mathbb{R}^d} g\,\mathscr{B}^{2-m}\,dx = 0$ and $\int_{\mathbb{R}^d} x\,g\,\mathscr{B}^{2-m}\,dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\,\mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\,dx = \mathscr{M}$, $\int_{\mathbb{R}^d} x\,v\,dx = 0$ and

$$(1-\varepsilon)\,\mathscr{B} \le v \le (1+\varepsilon)\,\mathscr{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathscr{I}[v] \ge (4+\eta)\,\mathscr{F}[v]$$

The initial time layer improvement: backward estimate

Hint: for some strictly convex function ψ with $\psi(0)=0$, $\psi'(0)=1$, we have

$$\mathscr{I}-4\,\mathscr{F}\ge 4\left(\psi(\mathscr{F})-\mathscr{F}\right)\ge 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement

Rephrasing the *carré du champ* method, $\mathscr{Q}[v]:=\frac{\mathscr{I}[v]}{\mathscr{F}[v]}$ is such that

$$\frac{d\mathscr{Q}}{dt}\le\mathscr{Q}\,(\mathscr{Q}-4)$$

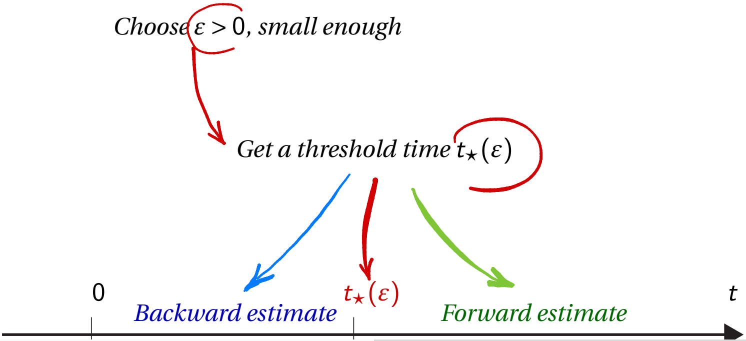
Lemma

Assume that $m>m_1$ and v is a solution to $(r\,\mathrm{FDE})$ with nonnegative initial datum v_0. If for some $\eta>0$ and $t_\star>0$, we have $\mathscr{Q}[v(t_\star,\cdot)]\ge 4+\eta$, then

$$\mathscr{Q}[v(t,\cdot)]\ge 4+\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}\quad\forall\,t\in[0,t_\star]$$

Stability in Gagliardo-Nirenberg-Sobolev inequalities

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (r FDE)

(and in particular for the initial datum)

If v is a solves $(r\,\mathsf{FDE})$ for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} v_0\,dx \le A < \infty \qquad (\mathsf{H}_A)$$

then

$$(1-\varepsilon)\,\mathscr{B} \le v(t,\cdot) \le (1+\varepsilon)\,\mathscr{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and A

Global Harnack Principle

The *Global Harnack Principle* holds if for some $t>0$ large enough

$$\mathscr{B}_{M_1}(t-\tau_1,x)\le u(t,x)\le \mathscr{B}_{M_2}(t+\tau_2,x) \qquad \text{(GHP)}$$

[Vázquez, 2003], [Bonforte, Vázquez, 2006]: (GHP) holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$
[Vázquez, 2003], [Bonforte, Simonov, 2020]: (GHP) holds if

$$\mathsf{A}[u_0] := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d\setminus B_R(0)} |u_0|\,dx < \infty$$

Theorem

[Bonforte, Simonov, 2020] If $M+\mathsf{A}[u_0]<\infty$, then

$$\lim_{t\to\infty} \left\| \frac{u(t)-B(t)}{B(t)} \right\|_\infty = 0$$

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $T \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$ and $\mathcal{F}[u_0] \leq G$, then

$$\sup_{x \in \mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq T$$

The threshold time

Proposition

Let $m\in(m_1,1)$ *if* $d\ge2$, $m\in(1/3,1)$ *if* $d=1$, $\varepsilon\in(0,\varepsilon_{m,d})$, $A>0$ *and* $G>0$

$$T=\mathsf{c}_\star\,\frac{1+A^{1-m}+G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a}=\frac{\alpha}{\vartheta}\,\frac{2-m}{1-m}$, $\alpha=d\,(m-m_c)$ *and* $\vartheta=\nu/(d+\nu)$

$$\mathsf{c}_\star=\mathsf{c}_\star(m,d)=\sup_{\varepsilon\in(0,\varepsilon_{m,d})}\max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m):=\max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m}-1},\frac{2^{3-m}\,\kappa_\star}{1-(1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m):=\frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\,\frac{\alpha}{\vartheta}}}\quad\text{and}\quad\kappa_3(\varepsilon,m):=\frac{8\,\alpha^{-1}}{1-(1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. Then there is a positive number ζ such that

$$\mathscr{I}[v]\ge(4+\zeta)\,\mathscr{F}[v]$$

for any nonnegative function $v\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v]=G$, $\int_{\mathbb{R}^d}v\,dx=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v\,dx=0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon\in(0,2\varepsilon_\star)\,,\quad \varepsilon_\star:=\frac12\min\{\varepsilon_{m,d},\chi\,\eta\}\quad\text{with}\quad t_\star=t_\star(\varepsilon)=\frac12\log R(T)$$

$$(1-\varepsilon)\,\mathscr{B}\le v(t,\cdot)\le(1+\varepsilon)\,\mathscr{B}\quad\forall\,t\ge t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathscr{I}[v(t,.)]\ge(4+\zeta)\,\mathscr{F}[v(t,.)]\quad\forall\,t\in[0,t_\star]\quad\text{where}\quad \zeta=\frac{4\,\eta\,e^{-4\,t_\star}}{4+\eta-\eta\,e^{-4\,t_\star}}$$

Two consequences

$$\zeta=\mathsf{Z}\big(A,\mathscr{F}[u_0]\big)\,,\quad \mathsf{Z}(A,G):=\frac{\zeta_\star}{1+A^{(1-m)\frac{2}{\alpha}}+G}\,,\quad \zeta_\star:=\frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^a}{2\,\alpha\,c_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. If v is a solution of (r FDE) with nonnegative initial datum $v_0\in\mathrm{L}^1(\mathbb{R}^d)$ such that $\mathscr{F}[v_0]=G$, $\int_{\mathbb{R}^d}v_0\,dx=\mathscr{M}$, $\int_{\mathbb{R}^d}x\,v_0\,dx=0$ and v_0 satisfies (H_A), then

$$\mathscr{F}[v(t,.)]\le\mathscr{F}[v_0]\,e^{-\,(4+\zeta)\,t}\quad\forall\,t\ge0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathscr{I}[v]-4\,\mathscr{F}[v]\ge\zeta\,\mathscr{F}[v]$ also holds in a stronger sense

$$\mathscr{I}[v]-4\,\mathscr{F}[v]\ge\frac{\zeta}{4+\zeta}\,\mathscr{I}[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f] := \left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2p}{d-p(d-4)}}\,,\quad \kappa[f] := \frac{\mathscr{M}^{\frac{1}{2p}}}{\|f\|_{2p}}$$

$$\mathsf{A}[f] := \frac{\mathscr{M}}{\lambda[f]^{\frac{d-p(d-4)}{p-1}}\,\|f\|_{2p}^{2p}}\,\sup_{r>0} r^{\frac{d-p(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2p}\,dx$$

$$\mathsf{E}[f] := \frac{2p}{1-p}\int_{\mathbb{R}^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2p}}}\,f^{p+1} - \mathsf{g}^{p+1} - \frac{1+p}{2p}\,\mathsf{g}^{1-p}\left(\frac{\kappa[f]^{2p}}{\lambda[f]^2}\,f^{2p}-\mathsf{g}^{2p}\right)\right)dx$$

$$\mathfrak{S}[f] := \frac{\mathscr{M}^{\frac{p-1}{2p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf{Z}(\mathsf{A}[f],\mathsf{E}[f])$$

Theorem

Let $d\ge1$, $p\in(1,p^*)$

If $f\in\mathscr{W}_p(\mathbb{R}^d) := \left\{f\in\mathrm{L}^{2p}(\mathbb{R}^d)\,:\,\nabla f\in\mathrm{L}^2(\mathbb{R}^d),\ |x|\,f^p\in\mathrm{L}^2(\mathbb{R}^d)\right\}$,

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2p\gamma} - \left(\mathscr{C}_{\mathrm{GN}}\,\|f\|_{2p}\right)^{2p\gamma} \ge \mathfrak{S}[f]\,\|f\|_{2p}^{2p\gamma}\,\mathsf{E}[f]$$

With $\mathscr{K}_{\rm GNS} = C(p,d)\,\mathscr{C}_{\rm GNS}^{2p\gamma}$, $\gamma = \frac{d+2-p(d-2)}{d-p(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathscr{K}_{\rm GNS}\,\|f\|_{2p}^{2p\gamma}$$

Theorem

Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathscr{C} = \mathscr{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}(\mathbb{R}^d,(1+|x|^2)\,dx)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}[f^{2p}] < \infty$,

$$\delta[f] \ge \mathscr{C}[f] \inf_{\varphi \in \mathfrak{M}} \int_{\mathbb{R}^d} \left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx$$

▷ The dependence of $\mathscr{C}[f]$ on $\mathsf{A}[f^{2p}]$ and $\mathscr{F}[f^{2p}]$ is explicit and does not degenerate if $f \in \mathfrak{M}$

▷ Can we remove the condition $\mathsf{A}[f^{2p}] < \infty$?

Stability in Sobolev's inequality (critical case)

▷ A constructive stability result

▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \ge 3$ and

$$\mathscr{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) : \nabla f \in \mathrm{L}^2(\mathbb{R}^d),\ |x|\, f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Theorem

Let $d \ge 3$ and $A > 0$. Then for any nonnegative $f \in \mathscr{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \quad \text{and} \quad \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \le A$$

we have

$$\delta[f] := \|\nabla f\|_2^2 - \mathsf{S}_d^2\, \|f\|_{2^*}^2 \ge \frac{\mathscr{C}_\star(A)}{4 + \mathscr{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathscr{C}_\star(A) = \mathfrak{C}_\star \left(1 + A^{1/(2d)}\right)^{-1}$ *and $\mathfrak{C}_\star > 0$ depends only on d*

Peculiarities of the critical case

▷ We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\right)$$

the *Bianchi-Egnell type result*

$$\delta[f] \geq \frac{\mathfrak{C}_\star\,Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathscr{J}[f|g]$$

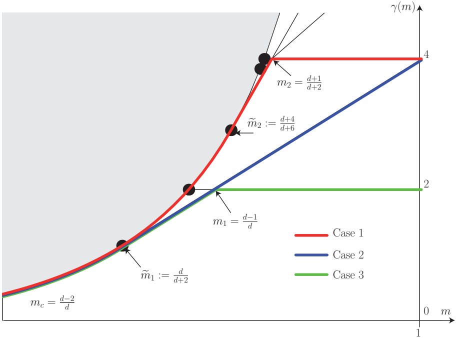
with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

▷ Notion of time delay [JD, Toscani, 2014, 2015]

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathscr{B}_\lambda(x) = \lambda^{-d/2}\,\mathscr{B}\big(x/\sqrt{\lambda}\big)$ in order to match $\int_{\mathbb{R}^d} |x|^2\, v\,dx$ where the function v solves (r FDE) or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\, w\left(t+\tau(t), \frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d} |x|^2\, v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,, \quad \tau(0)=0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

Stability in Caffarelli-Kohn-Nirenberg inequalities ?

Caffarelli-Kohn-Nirenberg inequalities

Let $\mathscr{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) : |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathscr{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

▷ *An optimal function among radial functions:*

$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Theorem

Let $d \ge 2$ and $p < 2^*$. $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ (symmetry) if and only if either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \ge b_{\mathrm{FS}}(a)$

[JD, Esteban, Loss, 2016]

More Caffarelli-Kohn-Nirenberg inequalities

On $\mathbb{R}^d$ with $d \ge 1$, let us consider the *Caffarelli-Kohn-Nirenberg interpolation inequalities*

$$\|f\|_{2p,\gamma} \le \mathscr{C}_{\beta,\gamma,p} \, \|\nabla f\|_{2,\beta}^{\theta} \, \|f\|_{p+1,\gamma}^{1-\theta}$$

$$\gamma - 2 < \beta < \frac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with} \quad p_\star := \frac{d-\gamma}{d-\beta-2}\,,$$

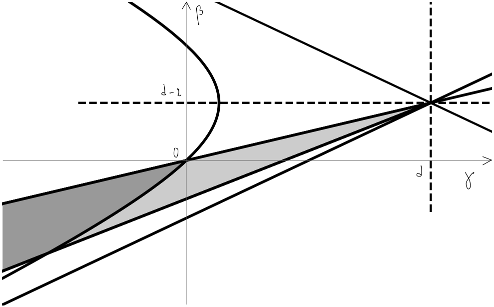
with $\theta = \dfrac{(d-\gamma)(p-1)}{p\big(d+\beta+2-2\gamma-p(d-\beta-2)\big)}$ and $\|f\|_{q,\gamma} := \left(\int_{\mathbb{R}^d} |f|^q \, |x|^{-\gamma} \, dx\right)^{1/q}$

Symmetry means that equality is achieved by the *Aubin-Talenti type functions*

$$\mathsf{g}(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-\frac{1}{p-1}}$$

[JD, Esteban, Loss, Muratori, 2017] Symmetry holds if and only if

$$\gamma < d\,, \quad \text{and} \quad \gamma - 2 < \beta < \frac{d-2}{d}\,\gamma \quad \text{and} \quad \beta \le \beta_{\rm FS}(\gamma)$$



$d = 4$ and $p = 6/5$: (γ, β) admissible region

An improved decay rate along the flow

In self-similar variables, with $m=(p+1)/(2p)$

$$|x|^{-\gamma}\,\frac{\partial v}{\partial t}+\nabla\cdot\Big(|x|^{-\beta}\,v\,\nabla v^{m-1}\Big)=\sigma\,\nabla\cdot\big(x\,|x|^{-\gamma}\,v\big)$$

$$\mathscr{F}[v]=\frac{2\,p}{1-p}\int_{\mathbb{R}^d}\left(v^{\frac{p+1}{2\,p}}-\mathrm{g}^{p+1}-\frac{p+1}{2\,p}\,\mathrm{g}^{1-p}\left(v-\mathrm{g}^{2p}\right)\right)|x|^{-\gamma}\,dx$$

Theorem

In the symmetry region, if $v\ge0$ is a solution with a initial datum v_0 s.t.

$$A[v_0]:=\sup_{R>0}R^{\frac{2+\beta-\gamma}{1-m}-(d-\gamma)}\int_{|x|>R}v_0(x)\,|x|^{-\gamma}\,dx<\infty$$

then there are some $\zeta>0$ and some $T>0$such that

$$\mathscr{F}[v(t,.)]\le\mathscr{F}[v_0]\,e^{-\,(4\,\alpha^2+\zeta)\,t}\quad\forall\,t\ge2\,T$$

[Bonforte, JD, Nazaret, Simonov, 2022]

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Thank you for your attention !