Stability estimates in critical functional inequalities

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Outline

1. Sobolev and HLS inequalities
   - Duality
   - Yamabe flow
   - Entropy methods, improvements
2. A constructive Bianchi-Egnell stability result
   - Constructive stability for Sobolev: a statement
   - A flow based on competing symmetries
   - Analysis close to the manifold of optimizers
3. Stability, fast diffusion equation and entropy methods
   - GNS inequality and the fast diffusion equation
   - The threshold time and consequences (subcritical case)
   - Stability results (subcritical and critical case)

Sobolev and Hardy-Littlewood-Sobolev inequalities

▷ Stability in a weaker norm, with explicit constants

▷ From duality to improved estimates based on Yamabe's flow

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \qquad \text{(S)}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\, dx \quad \forall\, v \in \mathcal{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \qquad \text{(HLS)}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$

Improved Sobolev inequality by duality

Theorem

[JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq 1$ such that

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \leq \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

Proof: the completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1} v|^2 \, dx = \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla(-\Delta)^{-1} v \, dx = \int_{\mathbb{R}^d} u \, v \, dx = \int_{\mathbb{R}^d} u^{2^*} \, dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \mathsf{S}_d \, \|u\|^{\frac{4}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \nabla u - \nabla(-\Delta)^{-1} v \right|^2 dx$$

shows that

$$0 \le \mathsf{S}_d \, \|u\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right] - \left[\mathsf{S}_d \, \|u^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} u^q \, (-\Delta)^{-1} u^q \, dx \right]$$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{FDE}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\,dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\,dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\,dx$$

where $v = v(t,\cdot)$ is a solution of (FDE). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2}$

A simple observation

Proposition

[JD] Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If v is a solution of (FDE) with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, then

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (S) if v is optimal for (HLS)

Improved Sobolev inequality

By integrating along the flow defined by (FDE), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (S), but only when $d \ge 5$ for integrability reasons

Theorem

[JD] Assume that $d \ge 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \le \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \le \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

Proof: use the convexity properties of $t \mapsto \mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\, dx$ to get an estimate of the *extinction time* and combine with a differential inequality for $t \mapsto \mathsf{H}(t)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[del Pino, Saez], [Vázquez, Esteban, Rodriguez] For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot)-1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Another improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

Theorem

[JD, Jankowiak] Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right]\right)$$

$$\forall\, u \in \mathcal{D}\,,\quad v = u^{\frac{d+2}{d-2}}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ for any $x \ge 0$

Proof: $\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t))\ \forall\, t \in [0,T)$, $\kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}$ and consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

$\mathcal{C} = 1$ is not optimal

$\mathcal{C} = 1$ is the constant in the expansion of the square method

Theorem

[JD, Jankowiak] In the inequality

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \le \mathcal{C}_d \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

we have

$$\frac{d}{d+4} \le \mathcal{C}_d < 1$$

based on a (painful) linearization

Extensions:

- Moser-Trudinger-Onofri inequality
- fractional Laplacian operator [Jankowiak, Nguyen]

Towards a constructive Bianchi-Egnell stability result

- ▷ A constructive estimate for the Bianchi-Egnell stability result
- ▷ Competing symmetries and the construction of a flow
- ▷ Explicit estimates close to the manifold of optimizers

Stability for Sobolev

With $d \geq 3$, $2^* = 2\,d/(d-2)$, we consider the *stability* inequality

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq c_{\mathrm{BE}} \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

for functions in $\dot{\mathrm{H}}^1(\mathbb{R}^d) = \left\{f \in \mathrm{L}^q(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\right\}$
$S_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$ is the optimal constant in Sobolev's inequality
$\mathcal{M}$ is the manifold of the optimal *Aubin-Talenti* functions

$$f(x) = c\,\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}$$

$c_{\mathrm{BE}} = 0$: [Rodemich, 1966], [Aubin, 1976], [Talenti, 1976]
$c_{\mathrm{BE}} > 0$? [Brezis, Lieb, 1985]
$c_{\mathrm{BE}} > 0$: [Bianchi, Egnell, 1991]
$c_{\mathrm{BE}} >?$

Results in collaboration with
M.J. Esteban, A. Figalli, R.L. Frank, M. Loss

Main result

$$\mathcal{E}(f) := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,, \quad \nu(\delta) := \sqrt{\frac{\delta}{1-\delta}}$$

Theorem

Let $d \geq 3$, $q = 2\,d/(d-2)$. If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a non-negative function, then

$$\mathcal{E}(f) \geq \kappa := \sup_{0<\delta<1} \delta\,\mu(\delta)$$

where $\mu(\delta) \geq \mathsf{m}\big(\nu(\delta)\big)$ and

$$\begin{aligned}
\mathsf{m}(\nu) &:= \tfrac{4}{d+4} - \tfrac{2}{q}\,\nu^{q-2} && \text{if} \quad d \geq 6\\
\mathsf{m}(\nu) &:= \tfrac{4}{d+4} - \tfrac{1}{3}\,(q-1)\,(q-2)\,\nu - \tfrac{2}{q}\,\nu^{q-2} && \text{if} \quad d = 4,5\\
\mathsf{m}(\nu) &:= \tfrac{4}{7} - \tfrac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \tfrac{1}{3}\,\nu^4 && \text{if} \quad d = 3
\end{aligned}$$

Comments

Upper bound: $C_{\rm BE} \le \inf_{0 \le f \in \dot{\rm H}^1(\mathbb{R}^d)\setminus\mathcal{M}} \mathcal{E}(f) \le \frac{4}{d+4}$

We also have a *lower bound* on $C_{\rm BE} := \inf_{f \in \dot{\rm H}^1(\mathbb{R}^d)\setminus\mathcal{M}} \mathcal{E}(f)$

[work in progress]

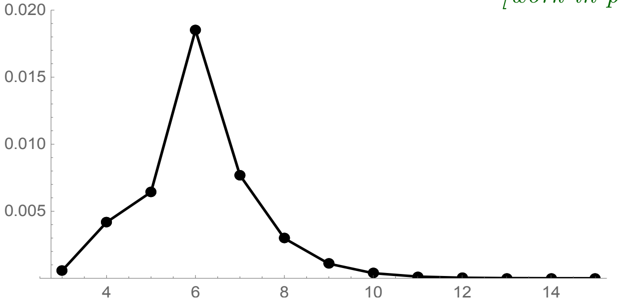


Figure: Plot of $d \mapsto \kappa(d)$ for $d = 3, 4, \ldots, 15$

Large dimensions: $\kappa(d) \sim 2^{d+1}\, d^{-1}\,(d+4)^{-\frac{d}{2}}$ as $d \to +\infty$

Strategy: two regions

- In the region $\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, prove that

$$\mathcal{E}(f)\ge\mu(\delta)$$

- If $\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \delta\,\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, build a *flow* $(f_\tau)_{0\le\tau<\infty}$ s.t.

$$f_0=f\,,\quad \|f_\tau\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}=\|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,,\quad \tau\mapsto\|\nabla f_\tau\|_{\mathrm{L}^2(\mathbb{R}^d)}\ \text{is}\ \searrow$$

$$\lim_{\tau\to\infty}\inf_{g\in\mathcal{M}}\|\nabla(f_\tau-g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}=0$$

$$\mathcal{E}(f)\ge\frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}=1-S_d\,\frac{\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\ge\frac{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}-S_d\,\|f_{\tau_0}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}$$

for some τ_0 (it exists ?) s.t. $\inf_{g\in\mathcal{M}}\|\nabla(f_{\tau_0}-g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)}=\delta\,\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

... then $\mathcal{E}(f)\ge\mathcal{E}(f_{\tau_0})\ge\delta\,\mu(\delta)$

Inverse stereographic projection

Denote by $s=(s_1,s_2,\ldots,s_{d+1})$ the coordinates in $\mathbb{R}^{d+1}$: $\mathbb{S}^d\subset\mathbb{R}^{d+1}$ can be parametrized in terms of stereographic coordinates by

$$s_j=\frac{2\,x_j}{1+|x|^2}\,,\quad j=1,\ldots,d\,,\quad s_{d+1}=\frac{1-|x|^2}{1+|x|^2}$$

We set

$$F(s)=\left(\tfrac{1+|x|^2}{2}\right)^{\frac{d-2}{2}}f(x)$$

$$\mathcal{E}(f)=\frac{\|\nabla f\|_2^2-S_d\,\|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2}=\frac{\|\nabla F\|_{\mathrm{L}^2(\mathbb{S}^d)}^2+\frac14\,d\,(d-2)\,\|F\|_{\mathrm{L}^2(\mathbb{S}^d)}^2-S_d\,\|F\|_{\mathrm{L}^{2^*}(\mathbb{S}^d)}^2}{\inf_{G\in\mathcal{M}}\left\{\|\nabla F-\nabla G\|_{\mathrm{L}^2(\mathbb{S}^d)}^2+\frac14\,d\,(d-2)\,\|F-G\|_{\mathrm{L}^2(\mathbb{S}^d)}^2\right\}}$$

where $G(s)=c\,(a+b\cdot s)^{-\frac{d-2}{2}}$, $a>0$, $b\in\mathbb{R}^d$ and $c\in\mathbb{C}$ are constants

Competing symmetries

[Carlen, Loss, 1990]

- *Conformal rotation*

$$(UF)(s) = F(s_1, s_2, \ldots, s_{d+1}, -s_d)$$

On $\mathbb{R}^d$, the function that corresponds to UF on $\mathbb{R}^d$ is given by

$$(Uf)(x) = \left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x-e_d|^2}, \ldots, \frac{x_{d-1}}{|x-e_d|^2}, \frac{|x|^2-1}{|x-e_d|^2}\right)$$

where $e_d = (0, \ldots, 0, 1) \in \mathbb{R}^d$ and $\mathcal{E}(Uf) = \mathcal{E}(f)$

- *Symmetric decreasing rearrangement*: if $f \geq 0$, let

$$\mathcal{R}f(x) = f^*(x)$$

f and f^* are equimeasurable and $\|\nabla f^*\|_2 \leq \|\nabla f\|_2$

On $\mathbb{R}^d$, let

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

Theorem

[Carlen, Loss] Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions

$$f_n = (\mathcal{R}U)^n f$$

Then $h_f = \|f\|_{2^*}\, g_* \in \mathcal{M}$ and

$$\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$$

If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_2)_{n\in\mathbb{N}}$ is a non-increasing sequence

Define $\mathcal{M}_1$ to be the set of the elements in $\mathcal{M}$ with 2^*-norm equal to 1

$$\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 = \|\nabla f\|_2^2 - S_d \sup_{g\in\mathcal{M}_1} \left(f, g^{2^*-1}\right)^2$$

Lemma

For the sequence $(f_n)_{n\in\mathbb{N}}$ of the Theorem of [Carlen, Loss] we have that

$n \mapsto \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_{2^*}^2$ *is strictly decreasing*

$$\lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d \|f\|_{2^*}^2$$

Two alternatives

Lemma

Let $0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$ s.t. $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 \ge \delta\,\|\nabla f\|_2^2$
One of the following alternatives holds:

(a) for all $n = 0, 1, 2\ldots$ $\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 \ge \delta\,\|\nabla f_n\|_2^2$

(b) $\exists n_0 \in \mathbb{N}$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|_2^2 \ge \delta\,\|\nabla f_{n_0}\|_2^2 \quad \text{and} \quad \inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|_2^2 < \delta\,\|\nabla f_{n_0+1}\|_2^2$$

In case (a) we have

$$\mathcal{E}(f) = \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \ge \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f\|_2^2} \ge \frac{\|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \ge \delta$$

because by the Theorem of [Carlen, Loss]

$$\lim_{n\to\infty} \|\nabla f_n\|_2^2 \le \frac{1}{\delta} \lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta}\left(\lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2\right)$$

Continuous rearrangement

Let $\mathsf{f}_0 = U f_{n_0}$ and denote by $(\mathsf{f}_\tau)_{0\le\tau\le\infty}$ the continuous rearrangement starting at f_0 and ending at $\mathsf{f}_\infty = f_{n_0+1}$
We find $\tau_0 \in [0,\infty)$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 = \delta\,\|\nabla \mathsf{f}_{\tau_0}\|_2^2$$

and conclude using

$$\mathcal{E}(\mathsf{f}_0) \ge 1 - S_d\,\frac{\|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} \ge 1 - S_d\,\frac{\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_{\tau_0}\|_2^2} = \delta\,\frac{\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d\,\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2} \ge \delta\,\mu(\delta)$$

Existence of τ_0 not granted: a discussion is needed !

Remark. We can build a ***flow*** by gluing continuous symmetrization at each step of the sequence $(f_n)_{n\in\mathbb{N}}$

Analysis close to the manifold of optimizers

Proposition

Let X be a measure space and $u, r \in \mathrm{L}^q(X)$ for some $q \geq 2$ with $u \geq 0$ and $u + r \geq 0$. Assume also that $\int_X u^{q-1}\, r\, dx = 0$. If $2 \leq q \leq 3$, then

$$\|u + r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q} \left((q-1) \int_X u^{q-2}\, r^2\, dx + \frac{2}{q} \int_X r_+^q\, dx \right)$$

$2 \leq q = \frac{2\,d}{d-2} \leq 3$ means $d \geq 6$ and is the most difficult case for Taylor

Corollary

Let $q = 2^*$, $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ whish realizes$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$
Set $r := f - u$ and $\sigma := \|r\|_q / \|u\|_q$. If $d \geq 6$, we have

$$\|\nabla f\|_2^2 - S_d\, \|f\|_q^2 \geq \int_{\mathbb{R}^d} \left(|\nabla r|^2 - S_d\, (q-1)\, \|u\|_q^{2-q}\, u^{q-2}\, r^2 \right) dx - \frac{2}{q}\, \|\nabla r\|_2^2\, \sigma^{q-2}$$

Spectral gap estimate

Cf. [Rey, 1990] and [Bianchi, Egnell, 1991]

Lemma

Let $d \geq 3$, $q = 2^*$, $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that $\|\nabla f - \nabla u\| = \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|$. Then $r := f - u$ satisfies

$$\int_{\mathbb{R}^d} \left(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,|u|^{q-2}\,r^2 \right) dx \geq \frac{4}{d+4} \int_{\mathbb{R}^d} |\nabla r|^2\,dx$$

Corollary

Let $q = 2^*$ and $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$. Set $\mathcal{D}(f) := \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$ and $\tau := \mathcal{D}(f)/(\|\nabla f\|_2^2 - \mathcal{D}(f)^2)^{1/2}$. If $d \geq 6$, we have

$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \geq \left(\tfrac{4}{d+4} - \tfrac{2}{q}\,\tau^{q-2} \right) \mathcal{D}(f)^2$$

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

A joint project with M. Bonforte, B. Nazaret and N. Simonov

Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method

arXiv:2007.03674, to appear in *Memoirs of the AMS*

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \ge \mathcal{C}_{\mathrm{GNS}}(p) \, \|f\|_{2p} \qquad \text{(GNS)}$$

Range of exponents:

$$1 < p \le \frac{d}{d-2} \quad \Longleftrightarrow \quad \frac{d-1}{d} =: m_1 \le m < 1$$

- Sobolev inequality: $p = \frac{d}{d-2}$, $m = m_1$
- Logarithmic Sobolev inequality

Entropy – entropy production inequality

Fast diffusion equation (written in self-similar variables)

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \Big[v \left(\nabla v^{m-1} - 2\,x \right) \Big] = 0 \qquad (r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v - \mathcal{B}) \right)\,dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v\,\left| \nabla v^{m-1} + 2\,x \right|^2\,dx$$

satisfy an *entropy – entropy production inequality*

$$\mathcal{I}[v] \ge 4\,\mathcal{F}[v]$$

[del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \le \mathcal{F}[v_0]\,e^{-\,4\,t}$$

The *entropy – entropy production inequality* $\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$ is equivalent to the *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p) \, \|f\|_{2p} \qquad \text{(GNS)}$$

with equality if and only if $|f(x)|^{2p} = \mathcal{B}(x) = \left(1 + |x|^2\right)^{\frac{1}{m-1}}$.

$$p = \frac{1}{2\,m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2\,p} \in [m_1, 1) \quad \text{with} \quad m_1 = \frac{d-1}{d}$$

$u = f^{2\,p}$ so that $u^m = f^{\,p+1}$ and $u\left|\nabla u^{m-1}\right|^2 = (p-1)^2\,|\nabla f|^2$

$$\mathcal{M} = \|f\|_{2\,p}^{2\,p}\,, \quad \mathsf{E}[u] = \|f\|_{p+1}^{p+1}\,, \quad \mathsf{I}[u] = (p+1)^2\,\|\nabla f\|_2^2$$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the Hardy–Poincaré inequality

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,,\quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

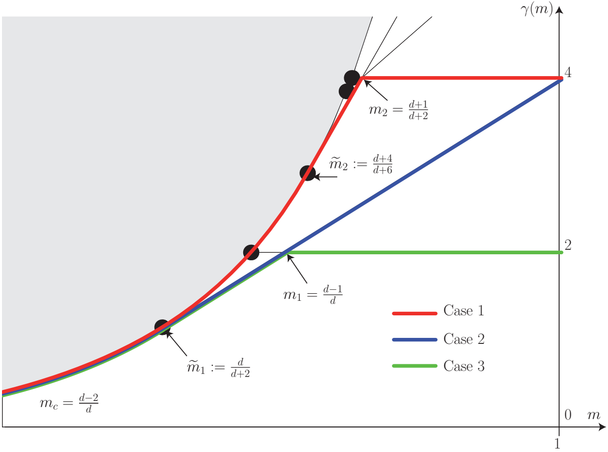
Lemma

Under assumption (H),

$$\mathcal{F}[v(t,\cdot)] \le C\,e^{-2\,\gamma(m)\,t} \quad \forall\, t\ge 0\,,\quad \gamma(m):=(1-m)\,\Lambda_{1/(m-1),d}$$

Moreover $\gamma(m):=2$ *if* $\frac{d-1}{d}=m_1\le m<1$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2} \int_{\mathbb{R}^d} g^2\, \mathcal{B}^{2-m}\, dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m) \int_{\mathbb{R}^d} |\nabla g|^2\, \mathcal{B}\, dx$$

Hardy-Poincaré inequality. Let $d \ge 1$, $m \in (m_1, 1)$ and $g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}^{2-m}\, dx)$ such that $\nabla g \in \mathrm{L}^2(\mathbb{R}^d, \mathcal{B}\, dx)$, $\int_{\mathbb{R}^d} g\, \mathcal{B}^{2-m}\, dx = 0$ and $\int_{\mathbb{R}^d} x\, g\, \mathcal{B}^{2-m}\, dx = 0$

$$\mathsf{I}[g] \ge 4\,\alpha\, \mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

Let $m \in (m_1, 1)$ if $d \ge 2$, $m \in (1/3, 1)$ if $d = 1$, $\eta = 2\,(d\,m - d + 1)$ and $\chi = m/(266 + 56\,m)$. If $\int_{\mathbb{R}^d} v\, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\, v\, dx = 0$ and

$$(1-\varepsilon)\,\mathcal{B} \le v \le (1+\varepsilon)\,\mathcal{B}$$

for some $\varepsilon \in (0, \chi\,\eta)$, then

$$\mathcal{I}[v] \ge (4+\eta)\,\mathcal{F}[v]$$

The initial time layer improvement: backward estimate

Hint: for some strictly convex function ψ with $\psi(0) = 0$, $\psi'(0) = 1$, we have

$$\mathcal{I} - 4\,\mathcal{F} \geq 4\,(\psi(\mathcal{F}) - \mathcal{F}) \geq 0$$

Far from the equality case (*i.e.*, close to an initial datum away from the Barenblatt solutions) for (FDE), we expect some improvement
Rephrasing the *carré du champ* method, $\mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$ is such that

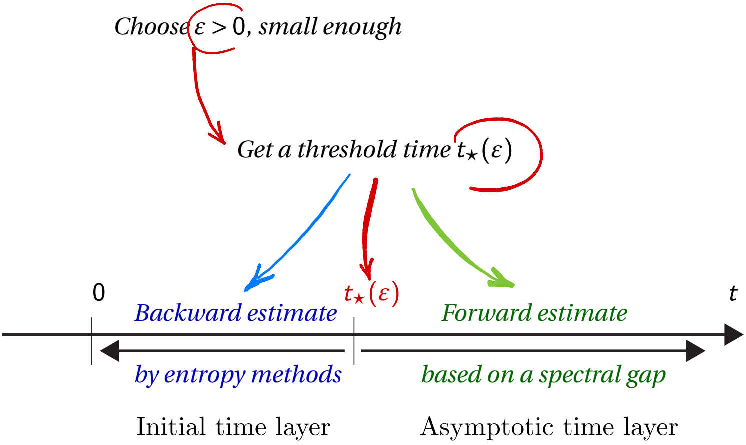
$$\frac{d\mathcal{Q}}{dt} \leq \mathcal{Q}\,(\mathcal{Q} - 4)$$

Lemma

Assume that $m > m_1$ *and* v *is a solution to* $(r\,\mathrm{FDE})$ *with nonnegative initial datum* v_0. *If for some* $\eta > 0$ *and* $t_\star > 0$, *we have* $\mathcal{Q}[v(t_\star,\cdot)] \geq 4 + \eta$, *then*

$$\mathcal{Q}[v(t,\cdot)] \geq 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\,t \in [0, t_\star]$$

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (*r* FDE)

(and in particular for the initial datum)

If v is a solves $(r\,\mathsf{FDE})$ for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} v_0\,dx \le A < \infty \qquad (\mathsf{H}_A)$$

then

$$(1-\varepsilon)\,\mathcal{B} \le v(t,\cdot) \le (1+\varepsilon)\,\mathcal{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and A

Global Harnack Principle

The *Global Harnack Principle* holds if for some $t > 0$ large enough

$$\mathcal{B}_{M_1}(t-\tau_1,x) \le u(t,x) \le \mathcal{B}_{M_2}(t+\tau_2,x) \qquad \text{(GHP)}$$

[Vázquez, 2003], [Bonforte, Vázquez, 2006]: (GHP) holds if $u_0 \lesssim |x|^{-\frac{2}{1-m}}$
[Vázquez, 2003], [Bonforte, Simonov, 2020]: (GHP) holds if

$$\mathsf{A}[u_0] := \sup_{R>0} R^{\frac{2}{1-m}-d} \int_{\mathbb{R}^d\setminus B_R(0)} |u_0|\,dx < \infty$$

Theorem

[Bonforte, Simonov, 2020] If $M+\mathsf{A}[u_0]<\infty$, then

$$\lim_{t\to\infty} \left\| \frac{u(t)-B(t)}{B(t)} \right\|_\infty = 0$$

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $T \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$ and $\mathcal{F}[u_0] \leq G$, then

$$\sup_{x\in\mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq T$$

The threshold time

Proposition

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$, $\varepsilon \in (0, \varepsilon_{m,d})$, $A > 0$ and $G > 0$

$$T = \mathsf{c}_\star \frac{1 + A^{1-m} + G^{\frac{\alpha}{2}}}{\varepsilon^{\mathsf{a}}}$$

where $\mathsf{a} = \frac{\alpha}{\vartheta} \frac{2-m}{1-m}$, $\alpha = d\,(m - m_c)$ *and* $\vartheta = \nu/(d+\nu)$

$$\mathsf{c}_\star = \mathsf{c}_\star(m,d) = \sup_{\varepsilon \in (0, \varepsilon_{m,d})} \max\left\{\varepsilon\,\kappa_1(\varepsilon,m),\ \varepsilon^{\mathsf{a}}\kappa_2(\varepsilon,m),\ \varepsilon\,\kappa_3(\varepsilon,m)\right\}$$

$$\kappa_1(\varepsilon,m) := \max\left\{\frac{8\,c}{(1+\varepsilon)^{1-m} - 1}, \frac{2^{3-m}\,\kappa_\star}{1 - (1-\varepsilon)^{1-m}}\right\}$$

$$\kappa_2(\varepsilon,m) := \frac{(4\,\alpha)^{\alpha-1}\,\mathsf{K}^{\frac{\alpha}{\vartheta}}}{\varepsilon^{\frac{2-m}{1-m}\frac{\alpha}{\vartheta}}} \quad \text{and} \quad \kappa_3(\varepsilon,m) := \frac{8\,\alpha^{-1}}{1 - (1-\varepsilon)^{1-m}}$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathcal{I}[v] \geq (4 + \zeta)\, \mathcal{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v] = G$, $\int_{\mathbb{R}^d} v\, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\, v\, dx = 0$ and v satisfies (H_A)

We have the *asymptotic time layer estimate*

$$\varepsilon \in (0, 2\,\varepsilon_\star)\,, \quad \varepsilon_\star := \frac{1}{2} \min\left\{\varepsilon_{m,d},\, \chi\,\eta\right\} \quad \text{with} \quad t_\star = t_\star(\varepsilon) = \frac{1}{2} \log R(T)$$

$$(1 - \varepsilon)\, \mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon)\, \mathcal{B} \quad \forall\, t \geq t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathcal{I}[v(t, .)] \geq (4 + \zeta)\, \mathcal{F}[v(t, .)] \quad \forall\, t \in [0, t_\star] \quad \text{where} \quad \zeta = \frac{4\,\eta\, e^{-4\,t_\star}}{4 + \eta - \eta\, e^{-4\,t_\star}}$$

Two consequences

$$\zeta=\mathsf{Z}\big(A,\mathcal F[u_0]\big)\,,\quad \mathsf{Z}(A,G):=\frac{\zeta_\star}{1+A^{(1-m)\,\frac{2}{\alpha}}+G}\,,\quad \zeta_\star:=\frac{4\,\eta\,c_\alpha}{4+\eta}\left(\frac{\varepsilon_\star^{\mathsf a}}{2\,\alpha\,\mathsf c_\star}\right)^{\frac{2}{\alpha}}$$

▷ Improved decay rate for the fast diffusion equation in rescaled variables

Corollary

Let $m\in(m_1,1)$ if $d\ge2$, $m\in(1/2,1)$ if $d=1$, $A>0$ and $G>0$. If v is a solution of $(r\,\mathrm{FDE})$ with nonnegative initial datum $v_0\in\mathrm L^1(\mathbb R^d)$ such that $\mathcal F[v_0]=G$, $\int_{\mathbb R^d}v_0\,dx=\mathcal M$, $\int_{\mathbb R^d}x\,v_0\,dx=0$ and v_0 satisfies (H_A), then

$$\mathcal F[v(t,.)]\le\mathcal F[v_0]\,e^{-\,(4+\zeta)\,t}\quad\forall\,t\ge0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathcal I[v]-4\,\mathcal F[v]\ge\zeta\,\mathcal F[v]$ also holds in a stronger sense

$$\mathcal I[v]-4\,\mathcal F[v]\ge\frac{\zeta}{4+\zeta}\,\mathcal I[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f]:=\left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2\,p}{d-p\,(d-4)}}\quad,\quad\kappa[f]:=\frac{\mathcal M^{\frac{1}{2\,p}}}{\|f\|_{2\,p}}$$

$$\mathsf A[f]:=\frac{\mathcal M}{\lambda[f]^{\frac{d-p\,(d-4)}{p-1}}\,\|f\|_{2\,p}^{2\,p}}\,\sup_{r>0}r^{\frac{d-p\,(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2\,p}\,dx$$

$$\mathsf E[f]:=\frac{2\,p}{1-p}\int_{\mathbb R^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2\,p}}}\,f^{p+1}-\mathrm g^{p+1}-\frac{1+p}{2\,p}\,\mathrm g^{1-p}\left(\frac{\kappa[f]^{2\,p}}{\lambda[f]^2}\,f^{2\,p}-\mathrm g^{2\,p}\right)\right)dx$$

$$\mathfrak S[f]:=\frac{\mathcal M^{\frac{p-1}{2\,p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf Z\,(\mathsf A[f],\,\mathsf E[f])$$

Theorem

Let $d\ge1$, $p\in(1,p^*)$

If $f\in\mathcal W_p(\mathbb R^d):=\left\{f\in\mathrm L^{2p}(\mathbb R^d)\,:\,\nabla f\in\mathrm L^2(\mathbb R^d)\,,\ |x|\,f^{\,p}\in\mathrm L^2(\mathbb R^d)\right\}$,

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2\,p\,\gamma}-\left(\mathcal C_{\rm GN}\,\|f\|_{2\,p}\right)^{2\,p\,\gamma}\ge\mathfrak S[f]\,\|f\|_{2\,p}^{2\,p\,\gamma}\,\mathsf E[f]$$

With $\mathcal{K}_{\rm GNS}=C(p,d)\,\mathcal{C}_{\rm GNS}^{2\,p\,\gamma}$, $\gamma=\frac{d+2-p\,(d-2)}{d-p\,(d-4)}$, consider the *deficit functional*

$$\delta[f]:=(p-1)^2\;\|\nabla f\|_2^2+4\,\frac{d-p\,(d-2)}{p+1}\;\|f\|_{p+1}^{p+1}-\mathcal{K}_{\rm GNS}\;\|f\|_{2\,p}^{2\,p\,\gamma}$$

Theorem

Let $d\ge1$ and $p\in(1,p^*)$. There is an explicit $\mathcal{C}=\mathcal{C}[f]$ such that, for any $f\in\mathrm{L}^{2p}\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$ such that $\nabla f\in\mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}\big[f^{2p}\big]<\infty$,

$$\delta[f]\ge\mathcal{C}[f]\,\inf_{\varphi\in\mathfrak{M}}\int_{\mathbb{R}^d}\big|(p-1)\,\nabla f+f^p\,\nabla\varphi^{1-p}\big|^2\,dx$$

▷ The dependence of $\mathcal{C}[f]$ on $\mathsf{A}\big[f^{2p}\big]$ and $\mathcal{F}\big[f^{2p}\big]$ is explicit and does not degenerate if $f\in\mathfrak{M}$

▷ Can we remove the condition $\mathsf{A}\big[f^{2p}\big]<\infty$?

Stability in Sobolev's inequality (critical case)

▷ A constructive stability result

▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star=2d/(d-2)=2^*$, $d\ge 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d)=\left\{f\in \mathrm{L}^{p^\star+1}(\mathbb{R}^d)\,:\,\nabla f\in \mathrm{L}^2(\mathbb{R}^d)\,,\;\;|x|\,f^{\,p^\star}\in \mathrm{L}^2(\mathbb{R}^d)\right\}$$

Theorem

Let $d\ge 3$ and $A>0$. Then for any nonnegative $f\in\mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}(1,x,|x|^2)\,f^{2^*}\,dx=\int_{\mathbb{R}^d}(1,x,|x|^2)\,\mathrm{g}\,dx\;\text{and}\;\sup_{r>0}r^d\int_{|x|>r}f^{2^*}\,dx\le A$$

we have

$$\delta[f]:=\|\nabla f\|_2^2-\mathsf{S}_d^2\,\|f\|_{2^*}^2\ge\frac{\mathcal{C}_\star(A)}{4+\mathcal{C}_\star(A)}\int_{\mathbb{R}^d}\left|\nabla f+\tfrac{d-2}{2}\,f^{\frac{d}{d-2}}\,\nabla \mathrm{g}^{-\frac{2}{d-2}}\right|^2dx$$

$\mathcal{C}_\star(A)=\mathfrak{C}_\star\left(1+A^{1/(2\,d)}\right)^{-1}$ *and* $\mathfrak{C}_\star>0$ *depends only on* d

Peculiarities of the critical case

▷ We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$\mathsf{A}[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad \mathsf{Z}[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,\mathsf{A}[f]\right)$$

the *Bianchi-Egnell type result*

$$\delta[f] \geq \frac{\mathfrak{C}_\star\,\mathsf{Z}[f]}{4+\mathsf{Z}[f]} \inf_{g\in\mathfrak{M}} \mathcal{J}[f|g]$$

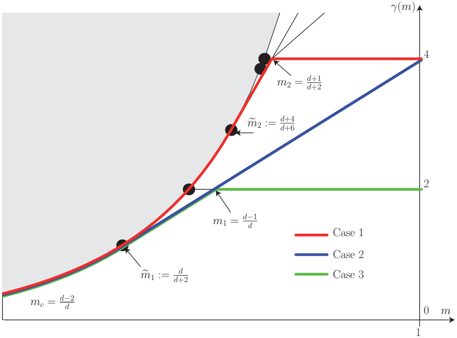
with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

▷ Notion of time delay [JD, Toscani, 2014, 2015]

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathcal{B}_\lambda(x) = \lambda^{-d/2}\,\mathcal{B}\left(x/\sqrt{\lambda}\right)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves $(r\,\mathrm{FDE})$ or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0)=0\quad\text{and}\quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

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Thank you for your attention !