Phase transitions and symmetry in PDEs

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Outline

- *Phase transition* and *symmetry breaking*
- Preliminaries: some observations
  - ▷ Ground state in Schrödinger eqations, a mechanism
  - ▷ Moving planes and eigenvalues
  - ▷ Phase transition and asymptotic behaviour in a *flocking* model
- *Symmetry and symmetry breaking in interpolation inequalities*
  - ▷ Gagliardo-Nirenberg-Sobolev inequalities on the sphere
  - ▷ [Keller-Lieb-Thirring inequalities on the sphere]
  - ▷ Caffarelli-Kohn-Nirenberg inequalities
- *Ground states with magnetic fields*
  - ▷ *Magnetic rings*, a one-dimensional magnetic interpolation inequality
  - ▷ Interpolation inequalities in dimensions 2 and 3, spectral estimates
- *Aharonov-Bohm magnetic fields in* $\mathbb{R}^2$
  - ▷ Aharonov-Bohm effect
  - ▷ Interpolation [and Keller-Lieb-Thirring] inequalities in $\mathbb{R}^2$
  - ▷ Aharonov-Symmetry and symmetry breaking

Phase transition and symmetry breaking

- The notion of ***phase transition*** in physics

▷ Ehrenfest's classification and more recent definitions

- ***Symmetry breaking***

▷ The principles of Pierre Curie
▷ A mathematical point of view: the symmetry of the ground state

- Bifurcations, interpolation inequalities and evolution equations

▷ Subcritical interpolation inequalities depending on a single parameter
▷ The non-linear problem *versus* the linearized spectral problem
▷ Nonlinear flows as a tool: generalized Bakry-Emery method
▷ Energy and relaxation

Preliminaries: some examples

- The ground state of a nonlinear Schrödinger equation: when the potential competes with the nonlinearity
- Moving planes and eigenvalues

with P. Felmer

- Phase transition and asymptotic behaviour in a *flocking* model with a mean field term:

the *homogeneous Cucker-Smale / McKean-Vlasov model*

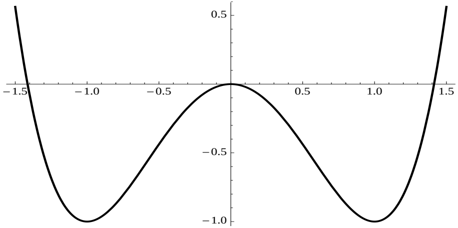
PhD thesis of Xingyu Li, https://arxiv.org/abs/1906.07517

Ground state of NLS

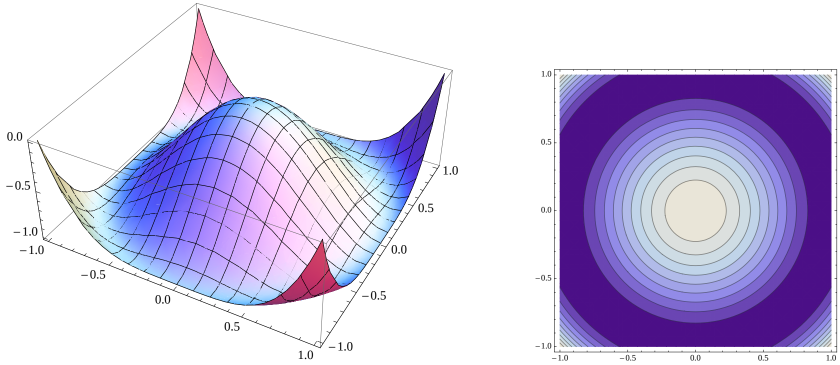
▷ *The typical issue is the competition between a potential or a weight and a nonlinearity*

Let us consider a nonlinear Schrödinger equation in presence of a radial external potential with a minimum which is not at the origin

$$-\Delta u + V(x)\, u - f(u) = 0$$

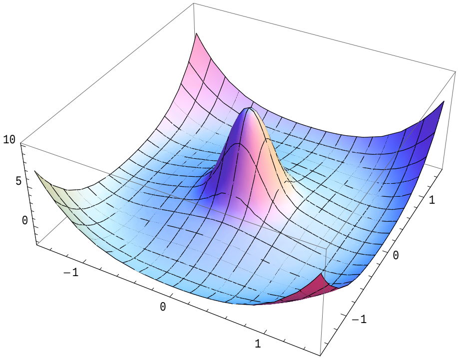


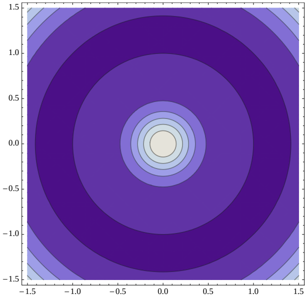
A one-dimensional potential $V(x)$



A two-dimensional potential $V(x)$ *with mexican hat shape*

Radial solutions to $-\Delta u + V(x)\, u - F'(u) = 0$

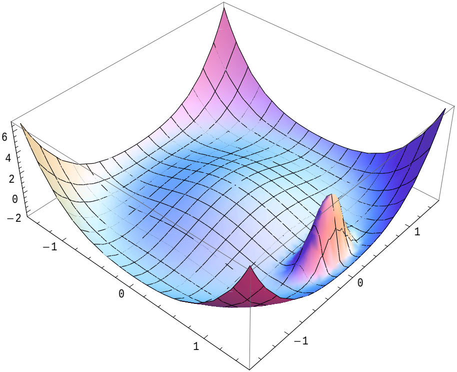


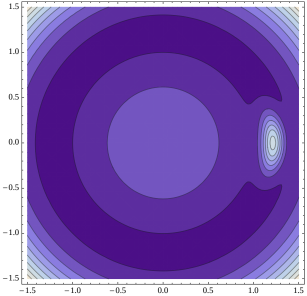


... *give rise to a radial density of energy* $x \mapsto V\,|u|^2 + F(u)$

symmetry breaking

... but in some cases minimal energy solutions





... *give rise to a non-radial density of energy* $x \mapsto V\,|u|^2 + F(u)$

The theorem of Gidas, Ni and Nirenberg

Theorem

[Gidas, Ni and Nirenberg, 1979 and 1980] Let $u \in C^2(B)$, $B = B(0,1) \subset \mathbb{R}^d$, be a solution of

$$\Delta u + f(u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that f is Lipschitz. If u is positive, then it is radially symmetric and decreasing along any radius: $u'(r) < 0$ for any $r \in (0,1]$

Extension: $\Delta u + f(r,u) = 0$, $r = |x|$ if $\frac{\partial f}{\partial r} \le 0$... a "cooperative" case

An extension of the theorem of Gidas, Ni and Nirenberg

Theorem (JD, P. Felmer)

$$\Delta u + \lambda f(r,u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ *(no assumption on the sign of* $\frac{\partial f}{\partial r}$*)*

There exists λ_1, λ_2 *with* $0 < \lambda_1 \le \lambda_2$ *such that*

(i) Monotonicity: if $\lambda \in (0,\lambda_1)$*, then* $\frac{d}{dr}(u - \lambda\, u_0) < 0$ *where* u_0 *is the solution of* $\Delta u_0 + \lambda f(r,0) = 0$

(ii) Symmetry: if $\lambda \in (0,\lambda_2)$*, then* u *is radially symmetric*

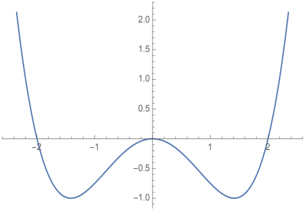
A simple version of the Cucker-Smale model

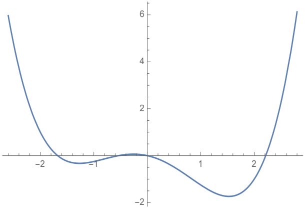
A model for bird flocking (simplified version)

$$\frac{\partial f}{\partial t} = D\,\Delta_v f + \nabla_v \cdot \left(\nabla_v \phi(v)\, f - \mathbf{u}_f\, f\right)$$

where $\mathbf{u}_f = \int v\, f\, dv$ is the average velocity, D is a measure of the noise, f is a probability measure

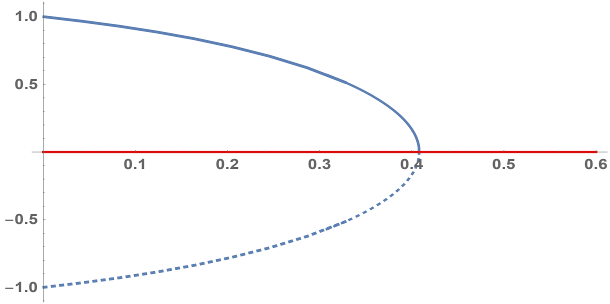
left: $\phi(v) = \frac{1}{4}\,|v|^4 - \frac{1}{2}\,|v|^2$ right: $\phi(v) - \mathbf{u}_f \cdot v$





[J. Tugaut, 2014]
[A. Barbaro, J. Cañizo, J.A. Carrillo, and P. Degond, 2016]

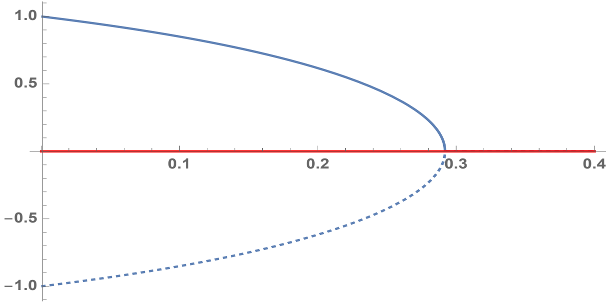
Stationary solutions: phase transition in dimension $d = 1$



- $d = 1$: there exists a bifurcation point $D = D_*$ such that the only stationary solution corresponds to $\mathbf{u}_f = 0$ if $D > D_*$ and there are three solutions corresponding to $\mathbf{u}_f = 0, \pm u(D)$ if $D < D_*$
- $\mathbf{u}_f = 0$ is linearly unstable if $D < D_*$

Notation: $f_\star^{(0)}$, $f_\star^{(+)}$, $f_\star^{(-)}$

Phase transition in dimension $d = 2$



Dynamics and free energy

The *free energy*

$$\mathcal{F}[f] := D \int_{\mathbb{R}^d} f \, \log f \, dv + \int_{\mathbb{R}^d} f \, \phi \, dv - \frac{1}{2} \, |\mathbf{u}_f|^2$$

decays according to

$$\frac{d}{dt} \mathcal{F}[f(t,\cdot)] = - \int_{\mathbb{R}^d} \left| D \, \frac{\nabla_v f}{f} + \nabla_v \phi - \mathbf{u}_f \right|^2 f \, dv$$

- $d = 1$: if $\mathcal{F}[f(t=0,\cdot)] < \mathcal{F}[f_\star^{(0)}]$ and $D < D_*$, then

$$\mathcal{F}[f(t,\cdot)] - \mathcal{F}\left[f_\star^{(\pm)}\right] \le C \, e^{-\lambda \, t}$$

- $d = 1$: λ is the eigenvalue of the linearized problem at $f_\star^{(\pm)}$ in the weighted space $\mathrm{L}^2\left((f_\star^{(\pm)})^{-1}\right)$ with scalar product

$$\langle f, g \rangle_\pm := D \int_{\mathbb{R}} f \, g \left(f_\star^{(\pm)}\right)^{-1} dv - \mathbf{u}_f \, \mathbf{u}_g$$

Symmetry and symmetry breaking in interpolation inequalities

- Gagliardo-Nirenberg-Sobolev inequalities on the sphere
- [Keller-Lieb-Thirring inequalities on the sphere]
- Caffarelli-Kohn-Nirenberg inequalities

Joint work with M.J. Esteban, M. Loss, M. Kowalczyk,...

A result of uniqueness on a classical example

On the sphere $\mathbb{S}^d$, let us consider the positive solutions of

$$-\Delta u + \lambda\, u = u^{p-1}$$

$p \in [1,2) \cup (2,2^*]$ if $d \geq 3$, $2^* = \frac{2\,d}{d-2}$

$p \in [1,2) \cup (2,+\infty)$ if $d = 1, 2$

Theorem

If $\lambda \leq d$, $u \equiv \lambda^{1/(p-2)}$ is the unique solution

[Gidas & Spruck, 1981], [Bidaut-Véron & Véron, 1991]

Bifurcation point of view and symmetry breaking

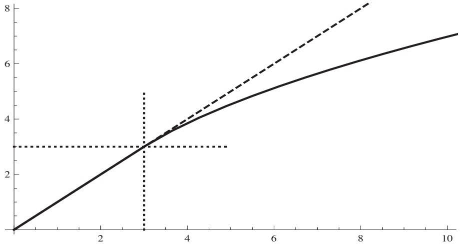


Figure: $(p-2)\,\lambda \mapsto (p-2)\,\mu(\lambda)$ *with* $d=3$

$$\boxed{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \lambda\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \mu(\lambda)\,\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}}$$

Taylor expansion of $u = 1 + \varepsilon\,\varphi_1$ as $\varepsilon \to 0$ with $-\Delta\varphi_1 = d\,\varphi_1$

$$\mu(\lambda) < \lambda \quad \text{if and only if} \quad \lambda > \frac{d}{p-2}$$

$\rhd$ The inequality holds with $\mu(\lambda) = \lambda = \frac{d}{p-2}$ [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

[Bakry & Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

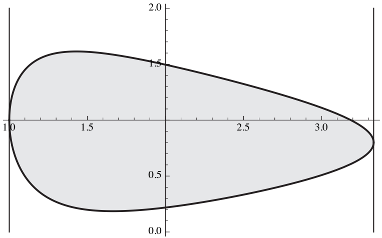
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

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- JD, M.J. Esteban, and M. Loss. Interpolation inequalities on the sphere: linear *versus* nonlinear flows. Annales de la faculté des sciences de Toulouse Sér. 6, 26 (2): 351-379, 2017
- JD, M.J. Esteban. Improved interpolation inequalities and stability. Preprint, 2019 arXiv:1908.08235

Optimal inequalities

With $\mu(\lambda)=\lambda=\frac{d}{p-2}$: [Bakry & Emery, 1985] [Beckner, 1993], [Bidaut-Véron & Véron, 1991, Corollary 6.1]

$$\boxed{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)}-\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u\in \mathrm{H}^1(\mathbb{S}^d)}$$

- $d\ge 3$, $p\in[1,2)$ or $p\in(2,\frac{2\,d}{d-2})$
- $d=1$ or $d=2$, $p\in[1,2)$ or $p\in(2,\infty)$
- $p=-2=2\,d/(d-2)=-2$ with $d=1$ [Exner, Harrell, Loss, 1998]

$$\boxed{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}+\frac{1}{4}\left(\int_{\mathbb{S}^1}\frac{1}{u^2}\,d\mu\right)^{-1} \ge \frac{1}{4}\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \quad \forall\, u\in \mathrm{H}^1_+(\mathbb{S}^1)}$$

The elliptic point of view (nonlinear flow)

$\frac{\partial u}{\partial t} = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$, $\kappa = \beta\,(p-2)+1$

$$-\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with

$$\int_{-1}^{1}\left|u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u}\right|^2\nu^2\;d\nu_d = 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

- Extension to compact Riemannian manifolds of dimension 2
+ another nonlinearity

We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2$$

We define

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{\displaystyle\int_{\mathfrak{M}} \left[\,\|\,\mathrm{L}_g u - \tfrac{1}{2}\,\mathrm{M}_g u\,\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right] e^{-u/2}\,d\,v_g}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2\,e^{-u/2}\,d\,v_g}$$

Theorem

Assume that $d=2$ and $\lambda_\star>0$. If u is a smooth solution to

$$-\frac{1}{2}\,\Delta_g u+\lambda=e^u$$

then u is a constant function if $\lambda\in(0,\lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathfrak{M}$

$$\frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})}+\lambda\int_{\mathfrak{M}} u\,d\,v_g\geq\lambda\,\log\left(\int_{\mathfrak{M}} e^u\,d\,v_g\right)\quad\forall\,u\in\mathrm{H}^1(\mathfrak{M})$$

for some constant $\lambda>0$. Let us denote by λ_1 the first positive eigenvalue of $-\,\Delta_g$

Corollary

If $d=2$, then the MTO inequality holds with $\lambda=\Lambda:=\min\{4\,\pi,\lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than Λ

The flow

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

$$\mathcal{G}_\lambda[f] := \int_{\mathfrak{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla f, \nabla f)\,e^{-f/2}\,d\,v_g \\ - \lambda \int_{\mathfrak{M}} |\nabla f|^2\,e^{-f/2}\,d\,v_g$$

Then for any $\lambda \le \lambda_\star$ we have

$$\frac{d}{dt}\mathcal{F}_\lambda[f(t,\cdot)] = \int_{\mathfrak{M}} \left(-\tfrac{1}{2}\,\Delta_g f + \lambda\right)\left(\Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\right)d\,v_g \\ = -\,\mathcal{G}_\lambda[f(t,\cdot)]$$

Since $\mathcal{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathcal{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathcal{F}_\lambda[u] \ge \int_0^\infty \mathcal{G}_\lambda[f(t,\cdot)]\,dt$$

Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure μ does the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \lambda\left[\log\left(\int_{\mathbb{R}^d} e^u\,d\mu\right)-\int_{\mathbb{R}^d} u\,d\mu\right]$$

hold for some $\lambda > 0$? Let

$$\Lambda_\star := \inf_{x\in\mathbb{R}^2}\frac{-\,\Delta\log\mu}{8\,\pi\,\mu}$$

Theorem

Assume that μ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_\star$ and the inequality holds with $\lambda = \Lambda_\star$ if equality is achieved among radial functions

Euclidean space: Rényi entropy powers and fast diffusion

- The Euclidean space without weights

▷ Rényi entropy powers, the entropy approach without rescaling: (Savaré, Toscani): scalings, nonlinearity and a concavity property inspired by information theory

The fast diffusion equation in original variables

Consider the nonlinear diffusion equation in $\mathbb{R}^d$, $d \geq 1$

$$\frac{\partial v}{\partial t} = \Delta v^m$$

with initial datum $v(x, t = 0) = v_0(x) \geq 0$ such that $\int_{\mathbb{R}^d} v_0 \, dx = 1$ and $\int_{\mathbb{R}^d} |x|^2 \, v_0 \, dx < +\infty$. The large time behavior of the solutions is governed by the source-type Barenblatt solutions

$$\mathcal{U}_\star(t,x) := \frac{1}{\left(\kappa \, t^{1/\mu}\right)^d} \, \mathcal{B}_\star\left(\frac{x}{\kappa \, t^{1/\mu}}\right)$$

where

$$\mu := 2 + d\,(m-1)\,, \quad \kappa := \left|\frac{2\,\mu\,m}{m-1}\right|^{1/\mu}$$

and $\mathcal{B}_\star$ is the Barenblatt profile

$$\mathcal{B}_\star(x) := \begin{cases} \left(C_\star - |x|^2\right)_+^{1/(m-1)} & \text{if } m > 1 \\ \left(C_\star + |x|^2\right)^{1/(m-1)} & \text{if } m < 1 \end{cases}$$

The Rényi entropy power F

The *entropy* is defined by

$$\mathsf{E} := \int_{\mathbb{R}^d} v^m \, dx$$

and the *Fisher information* by

$$\mathsf{I} := \int_{\mathbb{R}^d} v \, |\nabla \mathsf{p}|^2 \, dx \quad \text{with} \quad \mathsf{p} = \frac{m}{m-1} \, v^{m-1}$$

If v solves the fast diffusion equation, then

$$\mathsf{E}' = (1-m) \, \mathsf{I}$$

To compute I', we will use the fact that

$$\frac{\partial \mathsf{p}}{\partial t} = (m-1) \, \mathsf{p} \, \Delta \mathsf{p} + |\nabla \mathsf{p}|^2$$

$$\mathsf{F} := \mathsf{E}^\sigma \quad \text{with} \quad \sigma = \frac{\mu}{d\,(1-m)} = 1 + \frac{2}{1-m} \left(\frac{1}{d} + m - 1 \right) = \frac{2}{d} \, \frac{1}{1-m} - 1$$

has a linear growth asymptotically as $t \to +\infty$

The variation of the Fisher information

Lemma

If v solves $\frac{\partial v}{\partial t} = \Delta v^m$ with $1 - \frac{1}{d} \le m < 1$, then

$$\mathsf{I}' = \frac{d}{dt} \int_{\mathbb{R}^d} v\, |\nabla \mathsf{p}|^2\, dx = -\,2 \int_{\mathbb{R}^d} v^m \left(\|\mathrm{D}^2 \mathsf{p}\|^2 + (m-1)\,(\Delta \mathsf{p})^2 \right) dx$$

Explicit arithmetic geometric inequality

$$\|\mathrm{D}^2 \mathsf{p}\|^2 - \frac{1}{d}\,(\Delta \mathsf{p})^2 = \left\| \mathrm{D}^2 \mathsf{p} - \frac{1}{d}\,\Delta \mathsf{p}\,\mathrm{Id} \right\|^2$$

.... there are no boundary terms in the integrations by parts ?

The concavity property

Theorem

[Toscani-Savaré] *Assume that $m \ge 1-\frac{1}{d}$ if $d>1$ and $m>0$ if $d=1$. Then $F(t)$ is increasing, $(1-m)\,\mathsf{F}''(t)\le 0$ and*

$$\lim_{t\to+\infty}\frac{1}{t}\,\mathsf{F}(t)=(1-m)\,\sigma\lim_{t\to+\infty}\mathsf{E}^{\sigma-1}\,\mathsf{I}=(1-m)\,\sigma\,\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

[Dolbeault-Toscani] The inequality

$$\mathsf{E}^{\sigma-1}\,\mathsf{I}\ge\mathsf{E}_\star^{\sigma-1}\,\mathsf{I}_\star$$

is equivalent to the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta}\ge\mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

if $1-\frac{1}{d}\le m<1$. Hint: $v^{m-1/2}=\frac{w}{\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}}$, $q=\frac{1}{2\,m-1}$

Euclidean space: self-similar variables and relative entropies

- In the Euclidean space, it is possible to characterize the optimal constants using a spectral gap property

Self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type *Barenblatt solutions*

$$v_\star(t,x) := \frac{1}{\kappa^d (\mu\, t)^{d/\mu}}\, \mathcal{B}_\star\left(\frac{x}{\kappa\,(\mu\, t)^{1/\mu}}\right) \quad \text{where} \quad \mu := 2 + d\,(m-1)$$

where $\mathcal{B}_\star$ is the Barenblatt profile (with appropriate mass)

$$\mathcal{B}_\star(x) := \left(1 + |x|^2\right)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t,x) = \frac{1}{\kappa^d\, R^d}\, u\left(\tau, \frac{x}{\kappa\, R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2}\,\log\left(\frac{R(t)}{R_0}\right)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[u\left(\nabla u^{m-1} - 2\,x\right)\right] = 0$$

Free energy and Fisher information

- The function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau}+\nabla\cdot\Big[u\left(\nabla u^{m-1}-\,2\,x\right)\Big]=0$$

- (Ralston, Newman, 1984) Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{E}[u]:=\int_{\mathbb{R}^d}\left(-\frac{u^m}{m}+|x|^2u\right)\,dx-\mathcal{E}_0$$

- Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{E}[u]=-\,\mathcal{I}[u]\,,\quad \mathcal{I}[u]:=\int_{\mathbb{R}^d}u\left|\nabla u^{m-1}+\,2\,x\right|^2\,dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|u_\infty\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} = M > 0$

$$u_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

- *Entropy – entropy production inequality* (del Pino, JD)

Theorem

$d \ge 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[u] \ge 4\,\mathcal{E}[u]$$

$p = \frac{1}{2m-1}$, $u = w^{2p}$: (GN) $\|\nabla w\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|w\|_{\mathrm{L}^{q+1}(\mathbb{R}^d)}^{1-\theta} \ge \mathsf{C}_{\mathrm{GN}}\,\|w\|_{\mathrm{L}^{2q}(\mathbb{R}^d)}$

Corollary

(del Pino, JD) *A solution u with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies* $\mathcal{E}[u(t,\cdot)] \le \mathcal{E}[u_0]\,e^{-\,4\,t}$

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \Big[u \left(\nabla u^{m-1} - 2\,x \right) \Big] = 0 \quad \tau > 0\,, \quad x \in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R > 0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1} - 2\,x \right) \cdot \frac{x}{|x|} = 0 \quad \tau > 0\,, \quad x \in \partial B_R\,.$$

With $z(\tau,x) := \nabla \mathrm{Q}(\tau,x) := \nabla u^{m-1} - 2\,x$, the *relative Fisher information* is such that

$$\begin{aligned} \frac{d}{d\tau} \int_{B_R} u\,|z|^2\,dx + 4 \int_{B_R} u\,|z|^2\,dx & \\ + 2\,\tfrac{1-m}{m} \int_{B_R} u^m \left(\left\| \mathrm{D}^2 \mathrm{Q} \right\|^2 - (1-m)\,(\Delta \mathrm{Q})^2 \right) dx & \\ = \int_{\partial B_R} u^m \left(\omega \cdot \nabla |z|^2 \right) d\sigma \le 0 & \quad \text{(by Grisvard's lemma)} \end{aligned}$$

Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

(Blanchet, Bonforte, JD, Grillo, Vázquez) *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_* := \frac{d-4}{d-2}$*, the entropy decays according to*

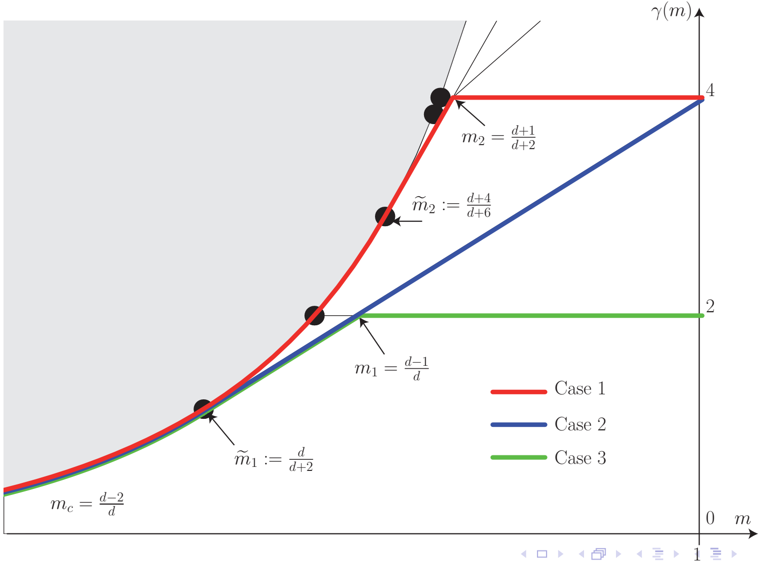
$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)\,, \int_{\mathbb{R}^d} f\, d\mu_{\alpha-1} = 0$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gap and best constants



Caffarelli-Kohn-Nirenberg, symmetry and symmetry breaking results, and weighted nonlinear flows

Joint work with M.J. Esteban and M. Loss

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p \left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

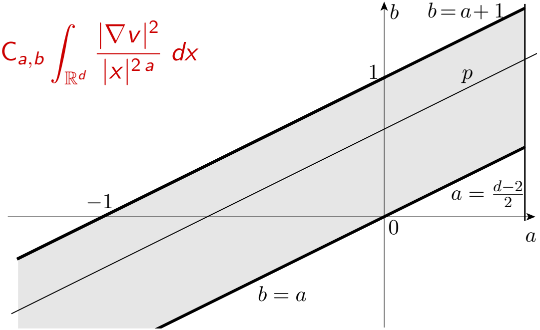
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d=3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$a < a_c := (d-2)/2$

$a \le b \le a+1$ if $d \ge 3$,

$a + 1/2 < b \le a+1$ if $d = 1$

and $a < b \le a+1 < 1$,

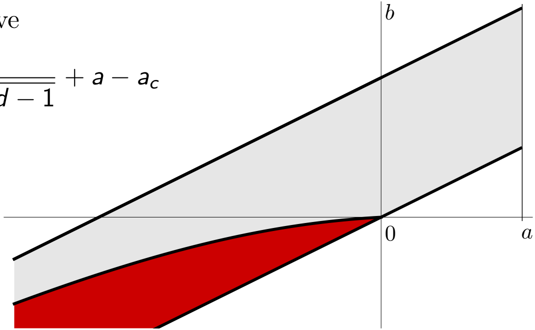
$p = 2/(b-a)$ if $d = 2$

[Il'in (1961)]
[Glaser, Martin, Grosse, Thirring (1976)]
[Caffarelli, Kohn, Nirenberg (1984)]
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$



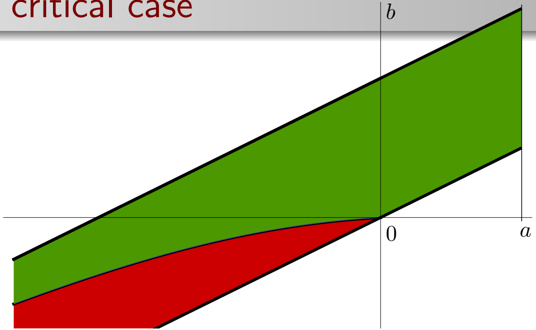
[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]

$$v \mapsto \mathsf{C}^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The symmetry proof in one slide

- A change of variables: $v(|x|^{\alpha-1}\,x)=w(x)$, $\mathsf{D}_\alpha v=\left(\alpha\,\frac{\partial v}{\partial s},\frac{1}{s}\,\nabla_\omega v\right)$

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)}\le\mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta}\quad\forall\,v\in\mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

- Concavity of the Rényi entropy power: with $\mathcal{L}_\alpha=-\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha=\alpha^2\left(u''+\frac{n-1}{s}\,u'\right)+\frac{1}{s^2}\,\Delta_\omega\,u$ and $\frac{\partial u}{\partial t}=\mathcal{L}_\alpha\,u^m$

$$\begin{aligned}
&-\frac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\left(\int_{\mathbb{R}^d}u^m\,d\mu\right)^{1-\sigma}\\
&\ge(1-m)\,(\sigma-1)\int_{\mathbb{R}^d}u^m\left|\mathcal{L}_\alpha\mathsf{P}-\frac{\int_{\mathbb{R}^d}u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d}u^m\,d\mu}\right|^2d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\frac{1}{n}\right)\left|\mathsf{P}''-\frac{\mathsf{P}'}{s}-\frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2+\frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}'-\frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right)u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\mathrm{FS}}^2-\alpha^2\right)|\nabla_\omega\mathsf{P}|^2+c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right)u^m\,d\mu
\end{aligned}$$

- Elliptic regularity and the Emden-Fowler transformation: justifying the integrations by parts

The variational problem on the cylinder

▷ *With the Emden-Fowler transformation*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad \omega=\frac{x}{r}$$

the variational problem becomes

$$\Lambda\mapsto\mu(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathcal{C})}\frac{\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

is a concave increasing function

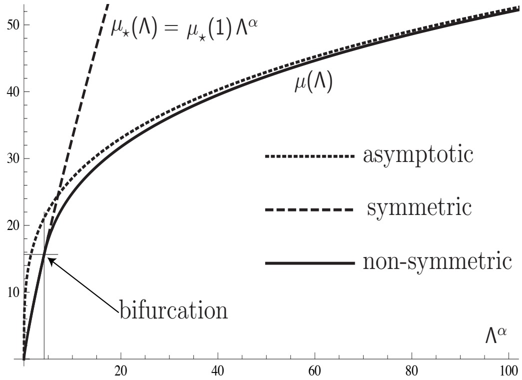
Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda):=\min_{\varphi\in\mathrm{H}^1(\mathbb{R})}\frac{\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\varphi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}=\mu_\star(1)\,\Lambda^\alpha$$

Symmetry means $\mu(\Lambda)=\mu_\star(\Lambda)$

Symmetry breaking means $\mu(\Lambda)<\mu_\star(\Lambda)$

Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as shown by F. Catrina and Z.-Q. Wang*

Three references

- Lecture notes on *Symmetry and nonlinear diffusion flows...* a course on entropy methods (see webpage)

- [JD, Maria J. Esteban, and Michael Loss] *Symmetry and symmetry breaking: rigidity and flows in elliptic PDEs* ... the elliptic point of view: Proc. Int. Cong. of Math., Rio de Janeiro, 3: 2279-2304, 2018.

- [JD, Maria J. Esteban, and Michael Loss] *Interpolation inequalities, nonlinear flows, boundary terms, optimality and linearization...* the parabolic point of view
Journal of elliptic and parabolic equations, 2: 267-295, 2016.

With magnetic fields (1/3) in dimensions 2 and 3

- Interpolation inequalities and spectral estimates
- Estimates, numerics; an open question on constant magnetic fields

Magnetic interpolation inequalities in the Euclidean space

▷ Three interpolation inequalities and their dual forms

▷ Estimates in dimension $d = 2$ for constant magnetic fields

- Lower estimates
- Upper estimates and numerical results
- A linear stability result (numerical) and an open question

Assumptions are not detailed: $\mathbf{A} \in \mathrm{L}^{d+\varepsilon}_{\mathrm{loc}}(\mathbb{R}^d)$, $\varepsilon > 0$ + integral conditions as in [Esteban, Lions, 1989]

Estimates are given (almost) only in the case $p > 2$ but similar estimates hold in the other cases

Joint work with M.J. Esteban, A. Laptev and M. Loss

Magnetic Laplacian and spectral gap

In dimensions $d=2$ and $d=3$: the *magnetic Laplacian* is

$$-\Delta_{\mathbf{A}}\,\psi=-\Delta\,\psi-\,2\,i\,\mathbf{A}\cdot\nabla\psi+|\mathbf{A}|^2\psi-\,i\,(\operatorname{div}\mathbf{A})\,\psi$$

where the magnetic potential (resp. field) is $\mathbf{A}$ (resp. $\mathbf{B}=\operatorname{curl}\mathbf{A}$) and

$$\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d):=\left\{\psi\in\mathrm{L}^2(\mathbb{R}^d)\,:\,\nabla_{\mathbf{A}}\psi\in\mathrm{L}^2(\mathbb{R}^d)\right\}\,,\quad\nabla_{\mathbf{A}}:=\nabla+\,i\,\mathbf{A}$$

Spectral gap inequality

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\geq\Lambda[\mathbf{B}]\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}\quad\forall\,\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

- Λ depends only on $\mathbf{B}=\operatorname{curl}\mathbf{A}$
- Assumption: *equality holds for some* $\psi\in\mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$
- If $\mathbf{B}$ is a constant magnetic field, $\Lambda[\mathbf{B}]=|\mathbf{B}|$
- If $d=2$, $\operatorname{spec}(-\Delta_{\mathbf{A}})=\{(2j+1)\,|\mathbf{B}|\,:\,j\in\mathbb{N}\}$ is generated by the *Landau levels*. The *Lowest Landau Level* corresponds to $j=0$

Magnetic interpolation inequalities

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mu_{\mathbf{B}}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

for any $\alpha \in (-\Lambda[\mathbf{B}], +\infty)$ and any $p \in (2, 2^*)$,

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \beta\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \ge \nu_{\mathbf{B}}(\beta)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\,\psi \in \mathrm{H}^1_{\mathbf{A}}(\mathbb{R}^d)$$

for any $\beta \in (0, +\infty)$ and any $p \in (1, 2)$

$$\|\nabla_{\mathbf{A}}\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \gamma \int_{\mathbb{R}^d} |\psi|^2 \log\left(\frac{|\psi|^2}{\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\right) dx + \xi_{\mathbf{B}}(\gamma)\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

(limit case corresponding to $p = 2$) for any $\gamma \in (0, +\infty)$

$$\mathsf{C}_p := \begin{cases} \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}} & \text{if} \quad p \in (2, 2^*) \\ \min_{u \in \mathrm{H}^1(\mathbb{R}^d)\setminus\{0\}} \dfrac{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} & \text{if} \quad p \in (1, 2) \end{cases}$$

$\mu_{\mathbf{0}}(1) = \mathsf{C}_p$ if $p \in (2, 2^*)$, $\nu_{\mathbf{0}}(1) = \mathsf{C}_p$ if $p \in (1, 2)$
$\xi_{\mathbf{0}}(\gamma) = \gamma \log\left(\pi\, e^2/\gamma\right)$ if $p = 2$

A statement

Theorem

$p\in(2,2^*)$: $\mu_{\mathbf B}$ *is monotone increasing on* $(-\Lambda[\mathbf B],+\infty)$*, concave and*

$$\lim_{\alpha\to(-\Lambda[\mathbf B])_+}\mu_{\mathbf B}(\alpha)=0\quad\text{and}\quad\lim_{\alpha\to+\infty}\mu_{\mathbf B}(\alpha)\,\alpha^{\frac{d-2}{2}-\frac dp}=\mathsf C_p$$

$p\in(1,2)$: $\nu_{\mathbf B}$ *is monotone increasing on* $(0,+\infty)$*, concave and*

$$\lim_{\beta\to0_+}\nu_{\mathbf B}(\beta)=\Lambda[\mathbf B]\quad\text{and}\quad\lim_{\beta\to+\infty}\nu_{\mathbf B}(\beta)\,\beta^{-\frac{2\,p}{2\,p+d\,(2-p)}}=\mathsf C_p$$

$\xi_{\mathbf B}$ *is continuous on* $(0,+\infty)$*, concave,* $\xi_{\mathbf B}(0)=\Lambda[\mathbf B]$ *and*

$$\xi_{\mathbf B}(\gamma)=\tfrac d2\,\gamma\,\log\left(\tfrac{\pi\,e^2}{\gamma}\right)(1+o(1))\quad\text{as}\quad\gamma\to+\infty$$

Constant magnetic fields: equality is achieved
Nonconstant magnetic fields: only partial answers are known

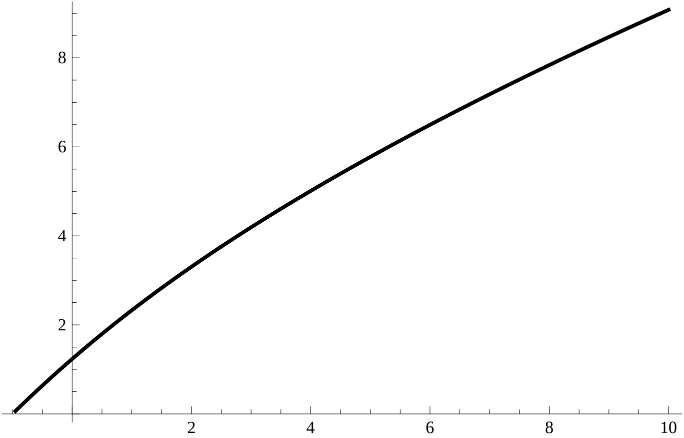


Figure: Case $d=2$, $p=3$, $B=1$: plot of $\alpha \mapsto (2\,\pi)^{\frac{2}{p}-1}\,\mu_{\mathbf{B}}(\alpha)$

Numerical results and the symmetry issue

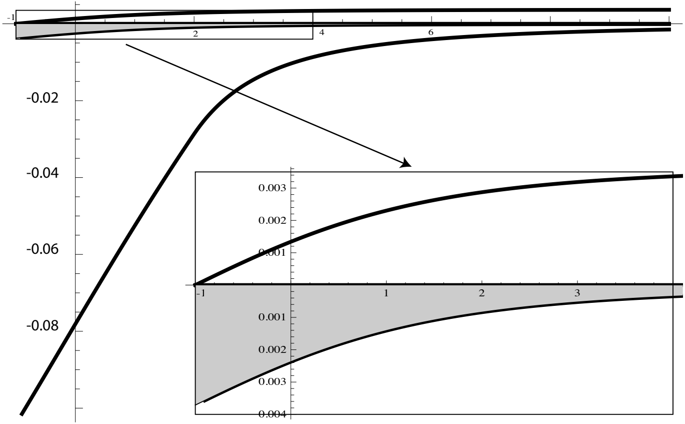


Figure: Case $d=2$, $p=3$, $B=1$
Upper estimates: $\alpha \mapsto \mu_{\rm Gauss}(\alpha)$, $\mu_{\rm EL}(\alpha)$
Lower estimates: $\alpha \mapsto \mu_{\rm interp}(\alpha)$, $\mu_{\rm LT}(\alpha)$
The exact value associated with $\mu_{\mathbf{B}}$ lies in the grey area.
Plots represent the curves $\log_{10}(\mu/\mu_{\rm EL})$

An open question of symmetry

- [Bonheure, Nys, Van Schaftingen, 2016] for a fixed $\alpha > 0$ and for $\mathbf{B}$ small enough, the optimal functions are radially symmetric functions, *i.e.*, belong to $\mathcal{C}_0$
  This regime is equivalent to the regime as $\alpha \to +\infty$ for a given $\mathbf{B}$, at least if the magnetic field is constant
- Numerically our upper and lower bounds are (in dimension $d = 2$, for a constant magnetic field) extremely close
- The optimal function in $\mathcal{C}_0$ is linearly stable with respect to perturbations in $\mathcal{C}_1$
- A reference: JD, M.J. Esteban, A. Laptev, M. Loss. Interpolation inequalities and spectral estimates for magnetic operators. Annales Henri Poincaré, 19 (5): 1439-1463, May 2018

▷ *Prove that the optimality case is achieved among radial function if* $d = 2$ *and* $\mathbf{B}$ *is a constant magnetic field*

With magnetic fields (2/3)

Magnetic rings: the case of $\mathbb{S}^1$

▷ A magnetic interpolation inequality on $\mathbb{S}^1$: with $p > 2$

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

▷ Consequences

- [A Keller-Lieb-Thirring inequality]
- A new Hardy inequality for Aharonov-Bohm magnetic fields in $\mathbb{R}^2$

Joint work with M.J. Esteban, A. Laptev and M. Loss

Magnetic flux, a reduction

Assume that $a : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function such that its restriction to $(-\pi,\pi] \approx \mathbb{S}^1$ is in $\mathrm{L}^1(\mathbb{S}^1)$ and define the space

$$X_a := \left\{\psi \in C_{\rm per}(\mathbb{R}) \,:\, \psi' + i\,a\,\psi \in \mathrm{L}^2(\mathbb{S}^1)\right\}$$

- A standard change of gauge (see *e.g.* [Ilyin, Laptev, Loss, Zelik, 2016])

$$\psi(s) \mapsto e^{i\int_{-\pi}^{s}(a(s)-\bar{a})\,\mathrm{d}\sigma}\,\psi(s)$$

where $\bar{a} := \int_{-\pi}^{\pi} a(s)\,\mathrm{d}\sigma$ is the *magnetic flux*, reduces the problem to

$$a \textit{ is a constant function}$$

- For any $k \in \mathbb{Z}$, ψ by $s \mapsto e^{iks}\,\psi(s)$ shows that $\mu_{a,p}(\alpha) = \mu_{k+a,p}(\alpha)$

$$a \in [0,1]$$

- $\mu_{a,p}(\alpha) = \mu_{1-a,p}(\alpha)$ because $|\psi' + i\,a\,\psi|^2 = |\chi' + i\,(1-a)\,\chi|^2 = \left|\overline{\psi}' - i\,a\,\overline{\psi}\right|^2$ if $\chi(s) = e^{-is}\,\overline{\psi(s)}$

$$a \in [0,1/2]$$

Optimal interpolation

We want to characterize the *optimal constant* in the inequality

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} \ge \mu_{a,p}(\alpha)\,\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}$$

written for any $p > 2$, $a \in (0, 1/2]$, $\alpha \in (-a^2, +\infty)$, $\psi \in X_a$

$$\mu_{a,p}(\alpha) := \inf_{\psi \in X_a \setminus \{0\}} \frac{\int_{-\pi}^{\pi} \left(|\psi' + i\,a\,\psi|^2 + \alpha\,|\psi|^2\right) d\sigma}{\|\psi\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

$p = -2 = 2\,d/(d-2)$ with $d = 1$ [Exner, Harrell, Loss, 1998]
$p = +\infty$ [Galunov, Olienik, 1995] [Ilyin, Laptev, Loss, Zelik, 2016]
$\lim_{\alpha \to -a^2} \mu_{a,p}(\alpha) = 0$ [JD, Esteban, Laptev, Loss, 2016]

Using a Fourier series $\psi(s) = \sum_{k \in \mathbb{Z}} \psi_k\, e^{iks}$, we obtain that

$$\|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} = \sum_{k \in \mathbb{Z}} (a+k)^2\,|\psi_k|^2 \ge a^2\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$$

$\psi \mapsto \|\psi' + i\,a\,\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|\psi\|^2_{\mathrm{L}^2(\mathbb{S}^1)}$ is coercive for any $\alpha > -a^2$

An interpolation result for the magnetic ring

Theorem

For any $p>2$, $a\in\mathbb{R}$, and $\alpha>-a^2$, $\mu_{a,p}(\alpha)$ is achieved and
(i) if $a\in[0,1/2]$ and $a^2\,(p+2)+\alpha\,(p-2)\le 1$, then $\mu_{a,p}(\alpha)=a^2+\alpha$ and equality is achieved only by the constant functions
(ii) if $a\in[0,1/2]$ and $a^2\,(p+2)+\alpha\,(p-2)>1$, then $\mu_{a,p}(\alpha)<a^2+\alpha$ and equality is not achieved among the constant functions
If $\alpha>-a^2$, $a\mapsto\mu_{a,p}(\alpha)$ is monotone increasing on $(0,1/2)$

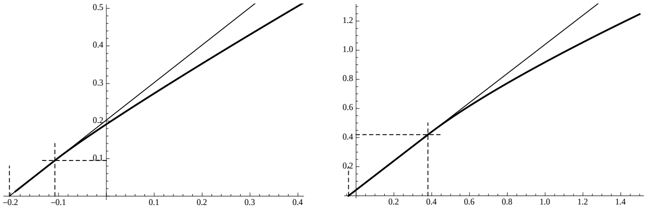


Figure: $\alpha\mapsto\mu_{a,p}(\alpha)$ with $p=4$ and (left) $a=0.45$ or (right) $a=0.2$

Elimination of the phase

Let us define

$$\mathcal{Q}_{a,p,\alpha}[u] := \frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{S}^1)} + a^2\,\|u^{-1}\|^{-2}_{\mathrm{L}^2(\mathbb{S}^1)} + \alpha\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^1)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^1)}}$$

Lemma

For any $a \in (0,1/2)$, $p > 2$, $\alpha > -\,a^2$,

$$\mu_{a,p}(\alpha) = \min_{u\in\mathrm{H}^1(\mathbb{S}^1)\setminus\{0\}} \mathcal{Q}_{a,p,\alpha}[u]$$

is achieved by a function $u > 0$

A new Hardy inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,dx \ge \tau \int_{\mathbb{R}^2} \frac{\varphi(\mathbf{x}/|\mathbf{x}|)}{|\mathbf{x}|^2}\,|\Psi|^2\,\mathrm{d}\mathbf{x} \quad \forall\,\varphi \in \mathrm{L}^q(\mathbb{S}^1)\,, \quad q \in (1,+\infty)$$

Corollary

Let $p > 2$, $a \in [0,1/2]$, $q = p/(p-2)$ and assume that φ is a non-negative function in $\mathrm{L}^q(\mathbb{S}^1)$. Then the inequality holds with $\tau > 0$ given by

$$\alpha_{a,p}\left(\tau\,\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\right) = 0$$

Moreover, $\tau = a^2/\|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}$ if $4\,a^2 + \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \le 1$

For any $a \in (0,1/2)$, by taking φ constant, small enough in order that $4\,a^2 + \|\varphi\|_{\mathrm{L}^q(\mathbb{S}^1)}\,(p-2) \le 1$, we recover the inequality

$$\int_{\mathbb{R}^2} |(i\,\nabla + \mathbf{a})\,\Psi|^2\,\mathrm{d}\mathbf{x} \ge a^2 \int_{\mathbb{R}^2} \frac{|\Psi|^2}{|\mathbf{x}|^2}\,\mathrm{d}\mathbf{x}$$

[Laptev, Weidl, 1999] constant magnetic fields; [Hoffmann-Ostenhof, Laptev, 2015] in $\mathbb{R}^d$, $d \ge 3$

With magnetic fields (3/3) Aharonov-Bohm magnetic fields in $\mathbb{R}^2$

- Aharonov-Bohm effect
- [Interpolation and Keller-Lieb-Thirring inequalities in $\mathbb{R}^2$]
- Aharonov-Symmetry and symmetry breaking

Joint work with D. Bonheure, M.J. Esteban, A. Laptev, & M. Loss

Aharonov-Bohm effect

A major difference between classical mechanics and quantum mechanics is that particles are described by a non-local object, the wave function. In 1959 Y. Aharonov and D. Bohm proposed a series of experiments intended to put in evidence such phenomena which are nowadays called *Aharonov-Bohm effects*

One of the proposed experiments relies on a long, thin solenoid which produces a magnetic field such that the region in which the magnetic field is non-zero can be approximated by a line in dimension $d = 3$ and by a point in dimension $d = 2$

▷ [Physics today, 2009] *"The notion, introduced 50 years ago, that electrons could be affected by electromagnetic potentials without coming in contact with actual force fields was received with a skepticism that has spawned a flourishing of experimental tests and expansions of the original idea."* Problem solved by considering appropriate weak solutions !

▷ Is the wave function a physical object or is its modulus ? Decisive experiments have been done only 20 years ago

The interpolation inequality

Let us consider an Aharonov-Bohm vector potential

$$\mathbf{A}(x) = \frac{a}{|x|^2}\,(x_2, -\,x_1)\;, \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}\,, \quad a \in \mathbb{R}$$

Magnetic Hardy inequality [Laptev, Weidl, 1999]

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx \ge \min_{k\in\mathbb{Z}}\,(a-k)^2 \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx$$

where $\nabla_{\mathbf{A}}\,\psi := \nabla\psi + i\,\mathbf{A}\,\psi$, so that, with $\psi = |\psi|\,e^{iS}$

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx = \int_{\mathbb{R}^2} \left[(\partial_r\,|\psi|)^2 + (\partial_r S)^2\,|\psi|^2 + \frac{1}{r^2}\,(\partial_\theta S + A)^2\,|\psi|^2\right] dx$$

Magnetic interpolation inequality

$$\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}}\,\psi|^2\,dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2}\,dx \ge \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2}\,dx\right)^{2/p}$$

▷ Symmetrization: [Erdös, 1996], [Boulenger, Lenzmann], [Lenzmann, Sok]

A magnetic Hardy-Sobolev inequality

Theorem

Let $a \in [0,1/2]$ and $p > 2$. For any $\lambda > -a^2$, there is an optimal, monotone increasing, concave function $\lambda \mapsto \mu(\lambda)$ which is such that

$$\boxed{\int_{\mathbb{R}^2} |\nabla_{\mathbf{A}} \psi|^2 \, dx + \lambda \int_{\mathbb{R}^2} \frac{|\psi|^2}{|x|^2} \, dx \geq \mu(\lambda) \left(\int_{\mathbb{R}^2} \frac{|\psi|^p}{|x|^2} \, dx \right)^{2/p}}$$

If $\lambda \leq \lambda_\star = 4\,\frac{1-4\,a^2}{p^2-4} - a^2$ equality is achieved by

$$\psi(x) = \left(|x|^\alpha + |x|^{-\alpha}\right)^{-\frac{2}{p-2}} \quad \forall\, x \in \mathbb{R}^2\,, \quad \textit{with} \quad \alpha = \frac{p-2}{2}\,\sqrt{\lambda + a^2}$$

If $\lambda > \lambda_\bullet$ with

$$\lambda_\bullet := \frac{8\left(\sqrt{p^4 - a^2\,(p-2)^2\,(p+2)\,(3\,p-2)} + 2\right) - 4\,p\,(p+4)}{(p-2)^3\,(p+2)} - a^2$$

there is symmetry breaking: optimal functions are not radially symmetric

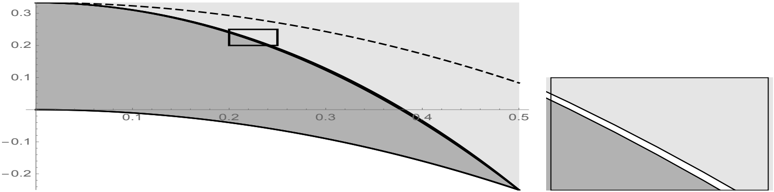


Figure: Case $p = 4$
Symmetry breaking region: $\lambda > \lambda_\bullet(a)$
Symmetry breaking region: $\lambda < \lambda_\star$

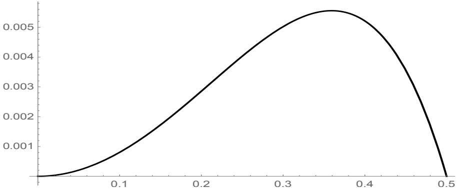


Figure: The curve $a \mapsto \lambda_\bullet(a) - \lambda_\star(a)$

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Thank you for your attention !