Sharp functional inequalities and nonlinear diffusions

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Prove inequalities with sharp constants on: the sphere, the line, compact manifolds, cylinders
- rigidity methods based on a nonlinear flow
- generalized entropies and generalized Fisher informations

We start with compact manifolds for which rigidity statements are easy and extend the method to non-compact settings which are much more difficult

- Interpolation inequalities on the sphere
- A nonlinear flow and improvements of the inequalities
- The line
- Compact manifolds
- The cylinder
- Symmetry breaking issues in Caffarelli-Kohn-Nirenberg inequalities

- Spectral estimates on the sphere
- Spectral consequences on Riemannian manifolds
- Spectral estimates on the cylinder
Interpolation inequalities on the sphere

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss
The following interpolation inequality holds on the sphere

\[
\frac{p - 2}{d} \int_{S^d} |\nabla u|^2 \, d\nu_g + \int_{S^d} |u|^2 \, d\nu_g \geq \left( \int_{S^d} |u|^p \, d\nu_g \right)^{2/p} \quad \forall \ u \in H^1(S^d, d\nu_g)
\]

for any \( p \in (2, 2^*] \) with \( 2^* = \frac{2d}{d-2} \) if \( d \geq 3 \)

for any \( p \in (2, \infty) \) if \( d = 2 \)

Here \( d\nu_g \) is the uniform probability measure: \( \nu_g(S^d) = 1 \)

1 is the optimal constant, equality achieved by constants

\( p = 2^* \) corresponds to Sobolev’s inequality...
The stereographic projection of $S^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho \phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1, 1]$, $\rho \geq 0$, $\phi \in S^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2 - 1}{r^2 + 1} = 1 - \frac{2}{r^2 + 1}, \quad \rho = \frac{2r}{r^2 + 1}$$

and transform any function $u$ on $S^d$ into a function $v$ on $\mathbb{R}^d$ using

$$u(y) = \left(\frac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\frac{r^2 + 1}{2}\right)^{\frac{d-2}{2}} v(x) = (1 - z)^{-\frac{d-2}{2}} v(x)$$

$p = 2^*$, $S_d = \frac{1}{4} d (d - 2) |S^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2 \, dx \geq S_d \left[ \int_{\mathbb{R}^d} |v|^{\frac{2d}{d-2}} \, dx \right]^{\frac{d-2}{d}} \quad \forall \, v \in D^{1,2}(\mathbb{R}^d)$$
Extended inequality

\[ \int_{S^d} |\nabla u|^2 \, d\nu_g \geq \frac{d}{2} \left[ \left( \int_{S^d} |u|^p \, d\nu_g \right)^{2/p} - \int_{S^d} |u|^2 \, d\nu_g \right] \quad \forall \ u \in H^1(S^d, d\mu) \]

is valid

- for any \( p \in (1, 2) \cup (2, \infty) \) if \( d = 1, 2 \)
- for any \( p \in (1, 2) \cup (2, 2^*] \) if \( d \geq 3 \)

Case \( p = 2 \): Logarithmic Sobolev inequality

\[ \int_{S^d} |\nabla u|^2 \, d\nu_g \geq \frac{d}{2} \int_{S^d} |u|^2 \log \left( \frac{|u|^2}{\int_{S^d} |u|^2 \, d\nu_g} \right) \, d\nu_g \quad \forall \ u \in H^1(S^d, d\mu) \]

Case \( p = 1 \): Poincaré inequality

\[ \int_{S^d} |\nabla u|^2 \, d\nu_g \geq d \int_{S^d} |u - \bar{u}|^2 \, d\nu_g \quad \text{with} \quad \bar{u} := \int_{S^d} u \, d\nu_g \quad \forall \ u \in H^1(S^d, d\mu) \]
For any $p \in (1, 2^*]$ if $d \geq 3$, any $p > 1$ if $d = 1$ or $2$, it is remarkable that

\[
Q[u] := \frac{(p - 2) \| \nabla u \|^2_{L^2(S^d)}}{\| u \|^2_{L^p(S^d)} - \| u \|^2_{L^2(S^d)}} \geq \inf_{u \in H^1(S^d, d\mu)} Q[u] = \frac{1}{d}
\]

is achieved in the limiting case

\[
Q[1 + \varepsilon \nu] \sim \frac{\| \nabla \nu \|^2_{L^2(S^d)}}{\| \nu \|^2_{L^2(S^d)}} \quad \text{as} \quad \varepsilon \to 0
\]

when $\nu$ is an eigenfunction associated with the first nonzero eigenvalue of $\Delta_g$, thus proving the optimality

$p < 2$: a proof by semi-groups using Nelson’s hypercontractivity lemma. $p > 2$: no simple proof based on spectral analysis is available: [Beckner], an approach based on Lieb’s duality, the Funk-Hecke formula and some (non-trivial) computations

elliptic methods / $\Gamma_2$ formalism of Bakry-Emery / nonlinear flows.
(ξ₀, ξ₁, . . . , ξₜ) ∈ S^d, ξₜ = z, ∑ᵢ₌₀^d |ξᵢ|^₂ = 1 [Smets-Willem]

**Lemma**

**Up to a rotation, any minimizer of Q depends only on ξₜ = z**

- Let \(dσ(θ) := \frac{(\sin θ)^{d-1}}{Z_d} dθ\), \(Z_d := \sqrt{\pi} \frac{Γ(\frac{d}{2})}{Γ(\frac{d+1}{2})}: ∀ ν ∈ H^1([0, π], dσ)\)

\[
\frac{p−2}{d} \int_0^π |ν'(θ)|^2 dσ + \int_0^π |ν(θ)|^2 dσ ≥ \left( \int_0^π |ν(θ)|^p dσ \right)^2\]

- Change of variables \(z = \cos θ\), \(ν(θ) = f(z)\)

\[
\frac{p−2}{d} \int_{−1}^1 |f'|^2 ν dν_d + \int_{−1}^1 |f|^2 dν_d ≥ \left( \int_{−1}^1 |f|^p dν_d \right)^2\]

where \(ν_d(z) dz = dν_d(z) := Z_d^{−1} ν_{\frac{d}{2}−1} dz\), \(ν(z) := 1 − z^2\)
The ultraspherical operator

With \( d\nu_d = \frac{Z_d^{-1}}{d^{\frac{d-1}{2}}} \, dz \), \( \nu(z) := 1 - z^2 \), consider the space \( L^2((-1,1), d\nu_d) \) with scalar product

\[
\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1 \, f_2 \, d\nu_d, \quad \|f\|_p = \left( \int_{-1}^{1} f^p \, d\nu_d \right)^{\frac{1}{p}}
\]

The self-adjoint \textit{ultraspherical} operator is

\[
\mathcal{L} f := (1 - z^2) f'' - dz f' = \nu f'' + \frac{d}{2} \nu' f'
\]

which satisfies \( \langle f_1, \mathcal{L} f_2 \rangle = -\int_{-1}^{1} f_1' \, f_2' \, \nu \, d\nu_d \)

**Proposition**

Let \( p \in [1, 2) \cup (2, 2^*], \ d \geq 1 \)

\[
-\langle f, \mathcal{L} f \rangle = \int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d \geq d \frac{\|f\|_p^2 - \|f\|_2^2}{p - 2} \quad \forall f \in H^1([-1,1], d\nu_d)
\]
Flows on the sphere

Heat flow and the Bakry-Emery method

Fast diffusion (porous media) flow and the choice of the exponents

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss
Heat flow and the Bakry-Emery method

With $g = f^p$, i.e. $f = g^\alpha$ with $\alpha = 1/p$

(Ieq.) $-\langle f, \mathcal{L} f \rangle = -\langle g^\alpha, \mathcal{L} g^\alpha \rangle =: \mathcal{I}[g] \geq d \frac{\|g\|_1^{2\alpha} - \|g^{2\alpha}\|_1}{p-2} =: \mathcal{F}[g]$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L} g$$

$$\frac{d}{dt} \|g\|_1 = 0, \quad \frac{d}{dt} \|g^{2\alpha}\|_1 = -2(p-2) \langle f, \mathcal{L} f \rangle = 2(p-2) \int_{-1}^{1} |f'|^2 \nu \, d\nu_d$$

which finally gives

$$\frac{d}{dt} \mathcal{F}[g(t, \cdot)] = -\frac{d}{p-2} \frac{d}{dt} \|g^{2\alpha}\|_1 = -2d \mathcal{I}[g(t, \cdot)]$$

Ineq. $\iff \frac{d}{dt} \mathcal{F}[g(t, \cdot)] \leq - 2d \mathcal{F}[g(t, \cdot)] \iff \frac{d}{dt} \mathcal{I}[g(t, \cdot)] \leq - 2d \mathcal{I}[g(t, \cdot)]$
The equation for $g = f^p$ can be rewritten in terms of $f$ as

$$
\frac{\partial f}{\partial t} = \mathcal{L} f + (p - 1) \frac{|f'|^2}{f} \nu
$$

$$
- \frac{1}{2} \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \, d\nu_d = \frac{1}{2} \frac{d}{dt} \langle f, \mathcal{L} f \rangle = \langle \mathcal{L} f, \mathcal{L} f \rangle + (p - 1) \langle \frac{|f'|^2}{f} \nu, \mathcal{L} f \rangle
$$

$$
\frac{d}{dt} \mathcal{I}[g(t, \cdot)] + 2 d \mathcal{I}[g(t, \cdot)] = \frac{d}{dt} \int_{-1}^{1} |f'|^2 \nu \, d\nu_d + 2 d \int_{-1}^{1} |f'|^2 \nu \, d\nu_d
$$

$$
= -2 \int_{-1}^{1} \left( |f''|^2 + (p - 1) \frac{d}{d + 2} \frac{|f'|^4}{f^2} - 2 (p - 1) \frac{d - 1}{d + 2} \frac{|f'|^2 f''}{f} \right) \nu^2 \, d\nu_d
$$

is nonpositive if

$$
|f''|^2 + (p - 1) \frac{d}{d + 2} \frac{|f'|^4}{f^2} - 2 (p - 1) \frac{d - 1}{d + 2} \frac{|f'|^2 f''}{f}
$$

is pointwise nonnegative, which is granted if

$$
\left[ (p - 1) \frac{d - 1}{d + 2} \right]^2 \leq (p-1) \frac{d}{d+2} \quad \iff \quad p \leq \frac{2 d^2 + 1}{(d - 1)^2} = 2^# < \frac{2 d}{d - 2} = 2^* \quad
$$
... up to the critical exponent: a proof in two slides

\[
\left[ \frac{d}{dz}, \mathcal{L} \right] u = (\mathcal{L} u)' - \mathcal{L} u' = -2 z u'' - d u' 
\]

\[
\int_{-1}^{1} (\mathcal{L} u)^2 \, d\nu_d = \int_{-1}^{1} |u''|^2 \, \nu^2 \, d\nu_d + d \int_{-1}^{1} |u'|^2 \, \nu \, d\nu_d
\]

\[
\int_{-1}^{1} (\mathcal{L} u) \frac{|u'|^2}{u} \nu \, d\nu_d = \frac{d}{d+2} \int_{-1}^{1} \frac{|u'|^4}{u^2} \nu^2 \, d\nu_d - 2 \frac{d-1}{d+2} \int_{-1}^{1} \frac{|u'|^2 u''}{u} \nu^2 \, d\nu_d
\]

On \((-1, 1)\), let us consider the porous medium (fast diffusion) flow

\[
 u_t = u^{2-2\beta} \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right)
\]

If \(\kappa = \beta (p - 2) + 1\), the \(L^p\) norm is conserved

\[
\frac{d}{dt} \int_{-1}^{1} u^{\beta p} \, d\nu_d = \beta \, p (\kappa - \beta (p - 2) - 1) \int_{-1}^{1} u^{\beta(p-2)} |u'|^2 \nu \, d\nu_d = 0
\]
\( f = u^\beta, \| f' \|^2_{L^2(S^d)} + \frac{d}{p-2} \left( \| f \|^2_{L^2(S^d)} - \| f \|^2_{L^p(S^d)} \right) \geq 0 ? \)

\[
A := \int_{-1}^{1} |u''|^2 \nu^2 \ d\nu_d - 2 \frac{d-1}{d+2} (\kappa + \beta - 1) \int_{-1}^{1} \frac{|u'|^2}{u} \nu^2 \ d\nu_d \\
+ \left[ \kappa (\beta - 1) + \frac{d}{d+2} (\kappa + \beta - 1) \right] \int_{-1}^{1} \frac{|u'|^4}{u^2} \nu^2 \ d\nu_d
\]

\( A \) is nonnegative for some \( \beta \) if

\[
\frac{8 \, d^2}{(d + 2)^2} (p - 1) (2^* - p) \geq 0
\]

\( A \) is a sum of squares if \( p \in (2, 2^*) \) for an arbitrary choice of \( \beta \) in a certain interval (depending on \( p \) and)

\[
A = \int_{-1}^{1} \left| u'' - \frac{p + 2}{6 - p} \frac{|u'|^2}{u} \right|^2 \nu^2 \ d\nu_d \geq 0 \quad \text{if} \quad p = 2^* \quad \text{and} \quad \beta = \frac{4}{6 - p}
\]
The rigidity point of view

Which computation have we done? \( u_t = u^{2-2\beta} \left( \mathcal{L} u + \kappa \frac{|u'|^2}{u} \nu \right) \)

\[
-\mathcal{L} u - (\beta - 1) \frac{|u'|^2}{u} \nu + \frac{\lambda}{p-2} u = \frac{\lambda}{p-2} u^\kappa
\]

Multiply by \( \mathcal{L} u \) and integrate

\[
\ldots \int_{-1}^{1} \mathcal{L} u \ u^\kappa \ d\nu_d = -\kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \ d\nu_d
\]

Multiply by \( \kappa \frac{|u'|^2}{u} \) and integrate

\[
\ldots = +\kappa \int_{-1}^{1} u^\kappa \frac{|u'|^2}{u} \ d\nu_d
\]

The two terms cancel and we are left only with the two-homogenous terms
Improvements of the inequalities (subcritical range)

- As long as the exponent is either in the range $(1, 2)$ or in the range $(2, 2^*)$, one can establish improved inequalities.

- An improvement automatically gives an explicit stability result of the optimal functions in the (non-improved) inequality.

- By duality, this provides a stability result for Keller-Lieb-Tirring inequalities.

Joint work with M. J. Esteban, M. Kowalczyk, and M. Loss.
What does “improvement” mean?

An improved inequality is

\[ d \Phi(e) \leq i \quad \forall \ u \in H^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|_{L^2(\mathbb{S}^d)}^2 = 1 \]

for some function \( \Phi \) such that \( \Phi(0) = 0 \), \( \Phi'(0) = 1 \), \( \Phi' > 0 \) and \( \Phi(s) > s \) for any \( s \). With \( \Psi(s) := s - \Phi^{-1}(s) \)

\[ i - d \ e \geq d \ (\Psi \circ \Phi)(e) \quad \forall \ u \in H^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|_{L^2(\mathbb{S}^d)}^2 = 1 \]

Lemma (Generalized Csiszár-Kullback inequalities)

\[
\| \nabla u \|^2_{L^2(\mathbb{S}^d)} - \frac{d}{p - 2} \left[ \|u\|^2_{L^p(\mathbb{S}^d)} - \|u\|^2_{L^2(\mathbb{S}^d)} \right] \\
\geq d \|u\|^2_{L^2(\mathbb{S}^d)} (\Psi \circ \Phi) \left( C \frac{\|u\|^{(1-r)}_{L^p(\mathbb{S}^d)}}{\|u\|_{L^2(\mathbb{S}^d)}^2} \|u' - \bar{u}'\|^2_{L^q(\mathbb{S}^d)} \right) \quad \forall \ u \in H^1(\mathbb{S}^d)
\]

\( s(p) := \max\{2, p\} \) and \( p \in (1, 2) \): \( q(p) := 2/p \), \( r(p) := p \); \( p \in (2, 4) \):
\( q = p/2 \), \( r = 2 \); \( p \geq 4 \): \( q = p/(p - 2) \), \( r = p - 2 \)
Linear flow: improved Bakry-Emery method

Cf. [Arnold, JD]

\[ w_t = \mathcal{L} w + \kappa \frac{|w'|^2}{w} \nu \]

With \( 2^\# := \frac{2d^2+1}{(d-1)^2} \)

\[ \gamma_1 := \left( \frac{d - 1}{d + 2} \right)^2 (p - 1) (2^\# - p) \quad \text{if} \quad d > 1, \quad \gamma_1 := \frac{p - 1}{3} \quad \text{if} \quad d = 1 \]

If \( p \in [1, 2) \cup (2, 2^\#] \) and \( w \) is a solution, then

\[ \frac{d}{dt} (i - d e) \leq -\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2} \, d\nu_d \leq -\gamma_1 \frac{|e'|^2}{1 - (p - 2) e} \]

Recalling that \( e' = -i \), we get a differential inequality

\[ e'' + d e' \geq \gamma_1 \frac{|e'|^2}{1 - (p - 2) e} \]

After integration: \( d \Phi(e(0)) \leq i(0) \)
Nonlinear flow: the H"older estimate of J. Demange

\[
\begin{align*}
    w_t &= w^{2-2\beta} \left( \mathcal{L} w + \kappa \frac{|w'|^2}{w} \right) \\
    \text{For all } p \in [1, 2^*], \kappa &= \beta (p - 2) + 1, \quad \frac{d}{dt} \int_{-1}^{1} w^p \, d\nu_d = 0 \\
    -\frac{1}{2\beta^2} \frac{d}{dt} \int_{-1}^{1} \left( |(w^\beta)'|^2 \nu + \frac{d}{p-2} (w^{2\beta} - \overline{w^{2\beta}}) \right) \, d\nu_d &\geq \gamma \int_{-1}^{1} \frac{|w'|^4}{w^2} \nu^2 \, d\nu_d \\
\end{align*}
\]

**Lemma**

*For all* \( w \in H^1((-1, 1), d\nu_d), \text{ such that } \int_{-1}^{1} w^p \, d\nu_d = 1 *

\[
\begin{align*}
    \int_{-1}^{1} \frac{|w'|^4}{w^2} \nu^2 \, d\nu_d &\geq \frac{1}{\beta^2} \frac{\int_{-1}^{1} |(w^\beta)'|^2 \nu \, d\nu_d \int_{-1}^{1} |w'|^2 \nu \, d\nu_d}{\left( \int_{-1}^{1} w^{2\beta} \, d\nu_d \right)^{\delta}} \\
\end{align*}
\]

.... but there are conditions on \( \beta \)
Admissible \((p, \beta)\) for \(d = 5\)
The line

A first example of a non-compact manifold

Joint work with M.J. Esteban, A. Laptev and M. Loss
One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

\[ \| f \|_{L^p(\mathbb{R})} \leq C_{GN}(p) \| f' \|_{L^2(\mathbb{R})}^{\theta} \| f \|_{L^2(\mathbb{R})}^{1-\theta} \quad \text{if} \quad p \in (2, \infty) \]

\[ \| f \|_{L^2(\mathbb{R})} \leq C_{GN}(p) \| f' \|_{L^2(\mathbb{R})}^{\eta} \| f \|_{L^p(\mathbb{R})}^{1-\eta} \quad \text{if} \quad p \in (1, 2) \]

with \( \theta = \frac{p-2}{2p} \) and \( \eta = \frac{2-p}{2+p} \)

The threshold case corresponding to the limit as \( p \to 2 \) is the logarithmic Sobolev inequality

\[ \int_{\mathbb{R}} u^2 \log \left( \frac{u^2}{\| u \|_{L^2(\mathbb{R})}^2} \right) \, dx \leq \frac{1}{2} \| u \|_{L^2(\mathbb{R})}^2 \log \left( \frac{2}{\pi e} \frac{\| u' \|_{L^2(\mathbb{R})}^2}{\| u \|_{L^2(\mathbb{R})}^2} \right) \]

If \( p > 2 \), \( u_*(x) = (\cosh x)^{-\frac{2}{p-2}} \) solves

\[ -(p-2)^2 u'' + 4 u - 2p |u|^{p-2} u = 0 \]

If \( p \in (1, 2) \) consider \( u_*(x) = (\cos x)^{2-p} \), \( x \in (-\pi/2, \pi/2) \)
Let us define on $H^1(\mathbb{R})$ the functional

$$
\mathcal{F}[v] := \|v'\|^2_{L^2(\mathbb{R})} + \frac{4}{(p-2)^2} \|v\|^2_{L^2(\mathbb{R})} - C \|v\|^2_{L^p(\mathbb{R})} \quad \text{s.t. } \mathcal{F}[u_\star] = 0
$$

With $z(x) := \tanh x$, consider the flow

$$
v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1-z^2}} \left[ v'' + \frac{2p}{p-2} z v' + \frac{p}{2} \frac{|v'|^2}{v} + \frac{2}{p-2} v \right]
$$

**Theorem (Dolbeault-Esteban-Laptev-Loss)**

Let $p \in (2, \infty)$. Then

$$
\frac{d}{dt} \mathcal{F}[v(t)] \leq 0 \quad \text{and} \quad \lim_{t \to \infty} \mathcal{F}[v(t)] = 0
$$

$$
\frac{d}{dt} \mathcal{F}[v(t)] = 0 \iff v_0(x) = u_\star(x - x_0)
$$

Similar results for $p \in (1, 2)$
The inequality \((p > 2)\) and the ultraspherical operator

The problem on the line is equivalent to the critical problem for the ultraspherical operator

\[
\int_{\mathbb{R}} |v'|^2 \, dx + \frac{4}{(p-2)^2} \int_{\mathbb{R}} |v|^2 \, dx \geq C \left( \int_{\mathbb{R}} |v|^p \, dx \right)^{\frac{2}{p}}
\]

With

\[z(x) = \tanh x, \quad v_\star = (1 - z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\star(x) f(z(x))\]

equality is achieved for \(f = 1\) and, if we let \(\nu(z) := 1 - z^2\), then

\[
\int_{-1}^{1} |f'|^2 \, \nu \, d\nu_d + \frac{2p}{(p-2)^2} \int_{-1}^{1} |f|^2 \, d\nu_d \geq \frac{2p}{(p-2)^2} \left( \int_{-1}^{1} |f|^p \, d\nu_d \right)^{\frac{2}{p}}
\]

where \(d\nu_p\) denotes the probability measure \(d\nu_p(z) := \frac{1}{\zeta_p} \nu_z^{\frac{2}{p-2}} \, dz\)

\[d = \frac{2p}{p-2} \quad \iff \quad p = \frac{2d}{d-2}\]
Change of variables = stereographic projection + Emden-Fowler
Compact Riemannian manifolds

no sign is required on the Ricci tensor and an improved integral criterion is established

the flow explores the energy landscape... and shows the non-optimality of the improved criterion
Riemannian manifolds with positive curvature

$(\mathcal{M}, g)$ is a smooth closed compact connected Riemannian manifold of dimension $d$, no boundary, $\Delta_g$ is the Laplace-Beltrami operator, $\text{vol}(\mathcal{M}) = 1$, $\mathcal{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathcal{M}} \inf_{\xi \in S^{d-1}} \mathcal{R}(\xi, \xi)$$

**Theorem (Licois-Véron, Bakry-Ledoux)**

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1 - \theta) \lambda_1 + \theta \frac{d \rho}{d - 1} \quad \text{where} \quad \theta = \frac{(d - 1)^2 (p - 1)}{d (d + 2) + p - 1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\Delta_g v + \frac{\lambda}{p - 2} (v - v^{p-1}) = 0$$

has a unique positive solution $v \in C^2(\mathcal{M})$: $v \equiv 1$
Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1, 2) \cup (2, 2^*)$

$$0 < \lambda < \lambda_* = \inf_{u \in H^2(M)} \frac{\int_M \left[ (1 - \theta) (\Delta_g u)^2 + \frac{\theta d}{d - 1} \mathcal{R}(\nabla u, \nabla u) \right] d v_g}{\int_M |\nabla u|^2 d v_g}$$

there is a unique positive solution in $C^2(M)$: $u \equiv 1$

$$\lim_{p \to 1^+} \theta(p) = 0 \implies \lim_{p \to 1^+} \lambda_*(p) = \lambda_1$$ if $\rho$ is bounded

$$\lambda_* = \lambda_1 = d \rho/(d - 1) = d$$ if $M = S^d$ since $\rho = d - 1$

$$(1 - \theta) \lambda_1 + \theta \frac{d \rho}{d - 1} \leq \lambda_* \leq \lambda_1$$
Riemannian manifolds: second improvement

\(H_g u\) denotes Hessian of \(u\) and \(\theta = \frac{(d - 1)^2 (p - 1)}{d (d + 2) + p - 1}\)

\[Q_g u := H_g u - \frac{g}{d} \Delta_g u - \frac{(d - 1)(p - 1)}{\theta (d + 3 - p)} \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right]\]

\[\Lambda_* := \inf_{u \in H^2(M) \setminus \{0\}} (1 - \theta) \int_M (\Delta u)^2 \, dv_g + \frac{\theta d}{d - 1} \int_M \left[ \|Q_g u\|^2 + \mathcal{K}(\nabla u, \nabla u) \right] \]
\[\int_M |\nabla u|^2 \, dv_g \]

**Theorem (Dolbeault-Esteban-Loss)**

*Assume that \(\Lambda_* > 0\). For any \(p \in (1, 2) \cup (2, 2^*)\), the equation has a unique positive solution in \(C^2(M)\) if \(\lambda \in (0, \Lambda_*): u \equiv 1\)*
For any $p \in (1, 2) \cup (2, 2^*)$ or $p = 2^*$ if $d \geq 3$

$$\|\nabla v\|_{L^2(\Omega)}^2 \geq \frac{\lambda}{p-2} \left[ \|v\|_{L^p(\Omega)}^2 - \|v\|_{L^2(\Omega)}^2 \right] \quad \forall v \in H^1(\Omega)$$

**Theorem (Dolbeault-Esteban-Loss)**

Assume $\Lambda_* > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_*, \lambda_1]$

If $\Lambda_* < \lambda_1$, then the optimal constant $\Lambda$ is such that

$$\Lambda_* < \Lambda \leq \lambda_1$$

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \epsilon \varphi$ as a test function where $\varphi$ we get $\lambda \leq \lambda_1$

A minimum of

$$v \mapsto \|\nabla v\|_{L^2(\Omega)}^2 - \frac{\lambda}{p-2} \left[ \|v\|_{L^p(\Omega)}^2 - \|v\|_{L^2(\Omega)}^2 \right]$$

under the constraint $\|v\|_{L^p(\Omega)} = 1$ is negative if $\lambda > \lambda_1$
The key tools the flow

\[ u_t = u^{2-2\beta} \left( \Delta_g u + \kappa \frac{\nabla u^2}{u} \right), \quad \kappa = 1 + \beta (p - 2) \]

If \( v = u^\beta \), then \( \frac{d}{dt} \| v \|_{L^p(\mathcal{M})} = 0 \) and the functional

\[ \mathcal{F}[u] := \int_\mathcal{M} |\nabla (u^\beta)|^2 d\nu_g + \frac{\lambda}{p - 2} \left[ \int_\mathcal{M} u^{2\beta} d\nu_g - \left( \int_\mathcal{M} u^{\beta p} d\nu_g \right)^{2/p} \right] \]

is monotone decaying

Let $d \geq 2$, $u \in C^2(\mathcal{M})$, and consider the trace free Hessian

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

**Lemma**

$$\int_{\mathcal{M}} (\Delta_g u)^2 \, d\nu_g = \frac{d}{d-1} \int_{\mathcal{M}} \|L_g u\|^2 \, d\nu_g + \frac{d}{d-1} \int_{\mathcal{M}} \mathcal{R}(\nabla u, \nabla u) \, d\nu_g$$

Based on the Bochner-Lichnerovicz-Weitzenböck formula

$$\frac{1}{2} \Delta |\nabla u|^2 = \|H_g u\|^2 + \nabla(\Delta_g u) \cdot \nabla u + \mathcal{R}(\nabla u, \nabla u)$$
Lemma

\[ \int_{\mathfrak{M}} \Delta_g u \frac{|\nabla u|^2}{u} \, d\nu_g \]
\[ = \frac{d}{d+2} \int_{\mathfrak{M}} \frac{|
abla u|^4}{u^2} \, d\nu_g - \frac{2d}{d+2} \int_{\mathfrak{M}} [L_g u] : \left[ \frac{\nabla u \otimes \nabla u}{u} \right] \, d\nu_g \]

Lemma

\[ \int_{\mathfrak{M}} (\Delta_g u)^2 \, d\nu_g \geq \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, d\nu_g \quad \forall \, u \in H^2(\mathfrak{M}) \]

and \( \lambda_1 \) is the optimal constant in the above inequality
The key estimates

\[ G[u] := \int_{\mathcal{M}} \left[ \theta (\Delta_g u)^2 + (\kappa + \beta - 1) \Delta_g u \frac{|\nabla u|^2}{u} + \kappa (\beta - 1) \frac{|\nabla u|^4}{u^2} \right] d\nu_g \]

**Lemma**

\[ \frac{1}{2\beta^2} \frac{d}{dt} \mathcal{F}[u] = -(1 - \theta) \int_{\mathcal{M}} (\Delta_g u)^2 d\nu_g - G[u] + \lambda \int_{\mathcal{M}} |\nabla u|^2 d\nu_g \]

\[ Q^\theta_g u := L_g u - \frac{1}{\theta} \frac{d-1}{d+2} (\kappa + \beta - 1) \left[ \frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d} \frac{|\nabla u|^2}{u} \right] \]

**Lemma**

\[ G[u] = \frac{\theta d}{d-1} \left[ \int_{\mathcal{M}} \| Q^\theta_g u \|^2 d\nu_g + \int_{\mathcal{M}} \Re(\nabla u, \nabla u) d\nu_g \right] - \mu \int_{\mathcal{M}} \frac{|\nabla u|^4}{u^2} d\nu_g \]

with \( \mu := \frac{1}{\theta} \left( \frac{d-1}{d+2} \right)^2 (\kappa + \beta - 1)^2 - \kappa (\beta - 1) - (\kappa + \beta - 1) \frac{d}{d+2} \)
Assume that \( d \geq 2 \). If \( \theta = 1 \), then \( \mu \) is nonpositive if

\[
\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall \ p \in (1, 2^*)
\]

where \( \beta_{\pm} := \frac{b \pm \sqrt{b^2 - a}}{2a} \) with 

\[
a = 2 - p + \left[ \frac{(d-1)(p-1)}{d+2} \right]^2 \quad \text{and} \quad b = \frac{d+3-p}{d+2}
\]

Notice that \( \beta_-(p) < \beta_+(p) \) if \( p \in (1, 2^*) \) and \( \beta_-(2^*) = \beta_+(2^*) \)

\[
\theta = \frac{(d-1)^2 (p-1)}{d (d+2) + p - 1} \quad \text{and} \quad \beta = \frac{d + 2}{d + 3 - p}
\]

**Proposition**

Let \( d \geq 2, \ p \in (1, 2) \cup (2, 2^*) \ (p \neq 5 \text{ or } d \neq 2) \)

\[
\frac{1}{2} \beta^2 \frac{d}{d t} \mathcal{F}[u] \leq (\lambda - \Lambda_*) \int_{\mathbb{R}^n} |\nabla u|^2 \ dv_g
\]
The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

Extension to compact Riemannian manifolds of dimension 2...
We shall also denote by $\mathcal{R}$ the Ricci tensor, by $H_g u$ the Hessian of $u$ and by

$$L_g u := H_g u - \frac{g}{d} \Delta_g u$$

the trace free Hessian. Let us denote by $M_g u$ the trace free tensor

$$M_g u := \nabla u \otimes \nabla u - \frac{g}{d} |\nabla u|^2$$

We define

$$\lambda_* := \inf_{u \in H^2(\mathcal{M}) \setminus \{0\}} \frac{\int_{\mathcal{M}} \left[ \| L_g u - \frac{1}{2} M_g u \|^2 + \mathcal{R}(\nabla u, \nabla u) \right] e^{-u/2} d\nu_g}{\int_{\mathcal{M}} |\nabla u|^2 e^{-u/2} d\nu_g}$$
Theorem

Assume that $d = 2$ and $\lambda_\star > 0$. If $u$ is a smooth solution to

$$
- \frac{1}{2} \Delta g u + \lambda = e^u
$$

then $u$ is a constant function if $\lambda \in (0, \lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathcal{M}$

$$
\frac{1}{4} \| \nabla u \|^2_{L^2(\mathcal{M})} + \lambda \int_{\mathcal{M}} u \, d\nu_g \geq \lambda \log \left( \int_{\mathcal{M}} e^u \, d\nu_g \right) \quad \forall u \in H^1(\mathcal{M})
$$

for some constant $\lambda > 0$. Let us denote by $\lambda_1$ the first positive eigenvalue of $-\Delta_g$

Corollary

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4\pi, \lambda_\star\}$. Moreover, if $\Lambda$ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than $\Lambda$
The flow

\[ \frac{\partial f}{\partial t} = \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2} \]

\[ G_\lambda[f] := \int_\mathcal{M} \| L_g f - \frac{1}{2} M_g f \|^2 e^{-f/2} \, d\nu_g + \int_\mathcal{M} \mathcal{K}(\nabla f, \nabla f) e^{-f/2} \, d\nu_g \]

\[ - \lambda \int_\mathcal{M} |\nabla f|^2 e^{-f/2} \, d\nu_g \]

Then for any \( \lambda \leq \lambda_* \) we have

\[ \frac{d}{dt} F_\lambda[f(t, \cdot)] = \int_\mathcal{M} \left( -\frac{1}{2} \Delta_g f + \lambda \right) \left( \Delta_g (e^{-f/2}) - \frac{1}{2} |\nabla f|^2 e^{-f/2} \right) \, d\nu_g \]

\[ = - G_\lambda[f(t, \cdot)] \]

Since \( F_\lambda \) is nonnegative and \( \lim_{t \to \infty} F_\lambda[f(t, \cdot)] = 0 \), we obtain that

\[ F_\lambda[u] \geq \int_0^\infty G_\lambda[f(t, \cdot)] \, dt \]
On the Euclidean space $\mathbb{R}^2$, given a general probability measure $\mu$ does the inequality
\[
\frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \geq \lambda \left[ \log \left( \int_{\mathbb{R}^2} e^u \, d\mu \right) - \int_{\mathbb{R}^2} u \, d\mu \right]
\]
hold for some $\lambda > 0$? Let
\[
\Lambda_* := \inf_{x \in \mathbb{R}^2} \frac{-\Delta \log \mu}{8\pi \mu}
\]

**Theorem**

Assume that $\mu$ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_*$ and the inequality holds with $\lambda = \Lambda_*$ if equality is achieved among radial functions.
Caffarelli-Kohn-Nirenberg inequalities

Work in progress with M.J. Esteban and M. Loss
Caffarelli-Kohn-Nirenberg inequalities and the symmetry breaking issue

Let \( D_{a,b} := \left\{ v \in L^p \left( \mathbb{R}^d, |x|^{-b} \, dx \right) : |x|^{-a} |\nabla v| \in L^2 \left( \mathbb{R}^d, dx \right) \right\} \)

\[
\left( \int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b \, p}} \, dx \right)^{2/p} \leq C_{a,b} \int_{\mathbb{R}^d} \frac{|
abla v|^2}{|x|^{2 \, a}} \, dx \quad \forall \, v \in D_{a,b}
\]

hold under the conditions that \( a \leq b \leq a + 1 \) if \( d \geq 3 \), \( a < b \leq a + 1 \) if \( d = 2 \), \( a + 1/2 < b \leq a + 1 \) if \( d = 1 \), and \( a < a_c := (d - 2)/2 \)

\[
p = \frac{2 \, d}{d - 2 + 2 \, (b - a)}
\]

\( \triangleright \) With

\[
v_*(x) = \left( 1 + |x|^{(p-2) \, (a_c-a)} \right)^{-\frac{2}{p-2}} \quad \text{and} \quad C_{a,b}^* = \frac{\| |x|^{-b} \, v_* \|_p^2}{\| |x|^{-a} \, \nabla v_* \|_2^2}
\]

do we have \( C_{a,b} = C_{a,b}^* \) (symmetry)

or \( C_{a,b} > C_{a,b}^* \) (symmetry breaking) ?
The Emden-Fowler transformation and the cylinder

\[ v(r, \omega) = r^{a-a_c} \varphi(s, \omega) \quad \text{with} \quad r = |x|, \quad s = -\log r \quad \text{and} \quad \omega = \frac{x}{r} \]

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

\[ \| \partial_s \varphi \|_{L^2(C_1)}^2 + \| \nabla \omega \varphi \|_{L^2(C_1)}^2 + \Lambda \| \varphi \|_{L^2(C_1)}^2 \geq \mu(\Lambda) \| \varphi \|_{L^p(C_1)}^2 \quad \forall \varphi \in H^1(C) \]

where \( \Lambda := (a_c - a)^2 \), \( C = \mathbb{R} \times \mathbb{S}^{d-1} \) and the optimal constant \( \mu(\Lambda) \) is

\[ \mu(\Lambda) = \frac{1}{C_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda} \]
Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$, $\theta = 1$. Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point computed by V. Felli and M. Schneider. The branch behaves for large values of $\Lambda$ as predicted by F. Catrina and Z.-Q. Wang.
The symmetry result

\[ b_{FS}(a) := \frac{d (a_c - a)}{2 \sqrt{(a_c - a)^2 + d - 1}} + a - a_c \]

**Theorem**

Let \( d \geq 2 \) and \( p \leq 4 \). If either \( a \in [0, a_c) \) and \( b > 0 \), or \( a < 0 \) and \( b \geq b_{FS}(a) \), then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric.
The Felli-Schneider region, or symmetry breaking region, appears in dark grey and is defined by \( a < 0, \ a \leq b < b_{\text{FS}}(a) \). We prove that symmetry holds in the light grey region defined by \( b \geq b_{\text{FS}}(a) \) when \( a < 0 \) and for any \( b \in [a, a + 1] \) if \( a \in [0, a_c) \).
Sketch of a proof
A change of variables

With \((r = |x|, \omega = x/r) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}\), the Caffarelli-Kohn-Nirenberg inequality is

\[
\left( \int_0^\infty \int_{\mathbb{S}^{d-1}} |v|^p r^{d-bp} \frac{dr}{r} d\omega \right)^{\frac{2}{p}} \leq C_{a,b} \int_0^\infty \int_{\mathbb{S}^{d-1}} |\nabla v|^2 r^{d-2a} \frac{dr}{r} d\omega
\]

Change of variables \(r \mapsto r^\alpha, v(r, \omega) = w(r^\alpha, \omega)\)

\[
\alpha^{1 - \frac{2}{p}} \left( \int_0^\infty \int_{\mathbb{S}^{d-1}} |w|^p r^{\frac{d-bp}{\alpha}} \frac{dr}{r} d\omega \right)^{\frac{2}{p}} \leq C_{a,b} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left( \alpha^2 \left| \frac{\partial w}{\partial r} \right|^2 + \frac{1}{r^2} |\nabla_\omega w|^2 \right) r^{\frac{d-2a-2}{\alpha}} + 2 \frac{dr}{r} d\omega
\]

Choice of \(\alpha\)

\[
n = \frac{d - b p}{\alpha} = \frac{d - 2a - 2}{\alpha} + 2
\]

Then \(p = \frac{2n}{n-2}\) is the critical Sobolev exponent associated with \(n\)
A Sobolev type inequality

The parameters $\alpha$ and $n$ vary in the ranges $0 < \alpha < \infty$ and $d < n < \infty$ and the *Felli-Schneider curve* in the $(\alpha, n)$ variables is given by

$$\alpha = \sqrt{\frac{d - 1}{n - 1}} =: \alpha_{FS}$$

With

$$Dw = \left( \alpha \frac{\partial w}{\partial r}, \frac{1}{r} \nabla \omega w \right), \quad d\mu := r^{n-1} dr d\omega$$

the inequality becomes

$$\alpha^{1 - \frac{2}{p}} \left( \int_{\mathbb{R}^d} |w|^p \, d\mu \right)^{\frac{2}{p}} \leq C_{a,b} \int_{\mathbb{R}^d} |Dw|^2 \, d\mu$$

**Proposition**

*Let $d \geq 4$. Optimality is achieved by radial functions and $C_{a,b} = C_{a,b}^*$ if $\alpha \leq \alpha_{FS}$*

The case of Gagliardo-Nirenberg inequalities on general cylinders is similar
When there is no ambiguity, we will omit the index $\omega$ and from now on write that $\nabla = \nabla_\omega$ denotes the gradient with respect to the angular variable $\omega \in S^{d-1}$ and that $\Delta$ is the Laplace-Beltrami operator on $S^{d-1}$. We define the self-adjoint operator $\mathcal{L}$ by

$$\mathcal{L} w := - D^* D w = \alpha^2 w'' + \alpha^2 \frac{n-1}{r} w' + \frac{\Delta w}{r^2}$$

The fundamental property of $\mathcal{L}$ is the fact that

$$\int_{\mathbb{R}^d} w_1 \mathcal{L} w_2 \, d\mu = - \int_{\mathbb{R}^d} D w_1 \cdot D w_2 \, d\mu \quad \forall \ w_1, w_2 \in \mathcal{D}(\mathbb{R}^d)$$

Heuristics: we look for a monotonicity formula along a well chosen nonlinear flow, based on the analogy with the decay of the Fisher information along the fast diffusion flow in $\mathbb{R}^d$.
Let \( u^{\frac{1}{2} - \frac{1}{n}} = |w| \iff u = |w|^p, \; p = \frac{2n}{n-2} \)

\[
\mathcal{I}[u] := \int_{\mathbb{R}^d} u |Dp|^2 \, d\mu, \quad p = \frac{m}{1 - m} u^{m-1} \quad \text{and} \quad m = 1 - \frac{1}{n}
\]

Here \( \mathcal{I} \) is the \textit{Fisher information} and \( p \) is the \textit{pressure function}.

**Proposition**

With \( \Lambda = 4 \alpha^2 / (p - 2)^2 \) and for some explicit numerical constant \( \kappa \), we have

\[
\kappa \mu(\Lambda) = \inf \{ \mathcal{I}[u] : \|u\|_{L^1(S^d, d\nu_n)} \}
\]
The fast diffusion equation

\[ \frac{\partial u}{\partial t} = \mathcal{L} u^m, \quad m = 1 - \frac{1}{n} \]

Barenblatt self-similar solutions

\[ u_\star(t, r, \omega) = t^{-n} \left( c_\star + \frac{r^2}{2(n-1) \alpha^2 t^2} \right)^{-n} \]

Lemma

\[ \kappa \mu_\star(\Lambda) = \mathcal{I}[u_\star(t, \cdot)] \quad \forall \ t > 0 \]

▷ Strategy:
1) prove that \( \frac{d}{dt} \mathcal{I}[u(t, \cdot)] \leq 0 \),
2) prove that \( \frac{d}{dt} \mathcal{I}[u(t, \cdot)] = 0 \) means that \( u = u_\star \) up to a time shift
Decay of the Fisher information along the flow?

\[ \frac{\partial p}{\partial t} = \frac{1}{n} p \mathcal{L} p - |Dp|^2 \]

\[ \mathcal{Q}[p] := \frac{1}{2} \mathcal{L} |Dp|^2 - Dp \cdot D\mathcal{L} p \]

\[ \mathcal{K}[p] := \int_{\mathbb{R}^d} \left( \mathcal{Q}[p] - \frac{1}{n} (\mathcal{L} p)^2 \right) p^{1-n} d\mu \]

**Lemma**

\[ \frac{d}{dt} \mathcal{I}[u(t, \cdot)] = -2 (n-1)^{n-1} \mathcal{K}[p] \]

If \( u \) is a critical point, then \( \mathcal{K}[p] = 0 \)

Boundary terms! Regularity!
Proving decay (1/2)

\[
k[p] := Q(p) - \frac{1}{n} (\mathcal{L} p)^2 = \frac{1}{2} \mathcal{L} |Dp|^2 - Dp \cdot D \mathcal{L} p - \frac{1}{n} (\mathcal{L} p)^2
\]

\[
k_{\text{mm}}[p] := \frac{1}{2} \Delta |\nabla p|^2 - \nabla p \cdot \nabla \Delta p - \frac{1}{n-1} (\Delta p)^2 - (n-2) \alpha^2 |\nabla p|^2
\]

**Lemma**

Let \( n \neq 1 \) be any real number, \( d \in \mathbb{N}, \ d \geq 2 \), and consider a function \( p \in C^3((0, \infty) \times \mathcal{M}) \), where \((\mathcal{M}, g)\) is a smooth, compact Riemannian manifold. Then we have

\[
k[p] = \alpha^4 \left( 1 - \frac{1}{n} \right) \left[ p'' - \frac{p'}{r} - \frac{\Delta p}{\alpha^2 (n - 1) r^2} \right]^2
\]

\[
+ 2 \alpha^2 \frac{1}{r^2} \left| \nabla p' - \frac{\nabla p}{r} \right|^2 + \frac{1}{r^4} k_{\text{mm}}[p]
\]
Lemma

Assume that $d \geq 3$, $n > d$ and $\mathcal{M} = S^{d-1}$. There is a positive constant $\zeta_\star$ such that

$$\int_{S^{d-1}} k_{\mathcal{M}}[p] p^{1-n} \, d\omega \geq (\lambda_\star - (n - 2) \alpha^2) \int_{S^{d-1}} |\nabla p|^2 p^{1-n} \, d\omega + \zeta_\star (n - d) \int_{S^{d-1}} |\nabla p|^4 p^{1-n} \, d\omega$$

Proof based on the Bochner-Lichnerowicz-Weitzenböck formula

Corollary

Let $d \geq 2$ and assume that $\alpha \leq \alpha_{FS}$. Then for any nonnegative function $u \in L^1(\mathbb{R}^d)$ with $\mathcal{I}[u] < +\infty$ and $\int_{\mathbb{R}^d} u \, d\mu = 1$, we have

$$\mathcal{I}[u] \geq \mathcal{I}_\star$$

When $\mathcal{M} = S^{d-1}$, $\lambda_\star = (n - 2) \frac{d-1}{n-1}$.
A perturbation argument

If $u$ is a critical point of $I$ under the mass constraint $\int_{\mathbb{R}^d} u \, d\mu = 1$, then

$$o(\varepsilon) = I[u + \varepsilon \mathcal{L} u^m] - I[u] = -2(n-1)^{n-1} \varepsilon K[p] + o(\varepsilon)$$

because $\varepsilon \mathcal{L} u^m$ is an admissible perturbation. Indeed, we know that

$$\int_{\mathbb{R}^d} (u + \varepsilon \mathcal{L} u^m) \, d\mu = \int_{\mathbb{R}^d} u \, d\mu = 1$$

and, as we take the limit as $\varepsilon \to 0$, $u + \varepsilon \mathcal{L} u^m$ makes sense and, in particular, is positive.

If $\alpha \leq \alpha_{FS}$, then $K[p] = 0$ implies that $u = u_*$.
A summary
the sphere: the flow tells us what to do, and provides a simple proof \(\text{(choice of the exponents / of the nonlinearity)}\) once the problem is reduced to the ultraspherical setting + improvements

[not presented here: Keller-Lieb-Thirring estimates] the spectral point of view on the inequality: how to measure the deviation with respect to the \emph{semi-classical} estimates, a nice example of bifurcation (and \emph{symmetry breaking})

\textbf{Riemannian manifolds:} no sign is required on the Ricci tensor and an improved integral criterion is established. We extend the theory from pointwise criteria to a non-local Schrödinger type estimate (Rayleigh quotient). The method generically shows the non-optimality of the improved criterion

the flow is a nice way of exploring an energy space: it explain how to produce a good test function at \emph{any} critical point. A \emph{rigidity} result tells you that a local result is actually global because otherwise the flow would relate (far away) extremal points while keeping the energy minimal


http://www.ceremade.dauphine.fr/~dolbeaul
▷ Preprints (or arxiv, or HAL)


J.D., Maria J. Esteban, Gaspard Jankowiak. Rigidity results for semilinear elliptic equation with exponential nonlinearities and Moser-Trudinger-Onofri inequalities on two-dimensional Riemannian manifolds, Preprint, 2014.


These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures
Thank you for your attention!