Sharp functional inequalities and nonlinear diffusions

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

April 25, 2015, Shanghai

International Conference on Local and Nonlocal Nonlinear Partial Differential Equations

April 24 to 26, 2015, at NYU Shanghai

Scope

Prove inequalities with sharp constants on: the sphere, the line, compact manifolds, cylinders
- rigidity methods based on a nonlinear flow
- generalized entropies and generalized Fisher informations

We start with compact manifolds for which rigidity statements are easy and extend the method to non-compact settings which are much more difficult

- Interpolation inequalities on the sphere
- A nonlinear flow and improvements of the inequalities
- The line
- Compact manifolds
- The cylinder
- Symmetry breaking issues in Caffarelli-Kohn-Nirenberg inequalities

- Spectral estimates on the sphere
- Spectral consequences on Riemannian manifolds
- Spectral estimates on the cylinder

Interpolation inequalities on the sphere

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss

A family of interpolation inequalities on the sphere

The following interpolation inequality holds on the sphere

$$\frac{p-2}{d}\int_{\mathbb{S}^d}|\nabla u|^2\,d\,v_g+\int_{\mathbb{S}^d}|u|^2\,d\,v_g\geq\left(\int_{\mathbb{S}^d}|u|^p\,d\,v_g\right)^{2/p}\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d,dv_g)$$

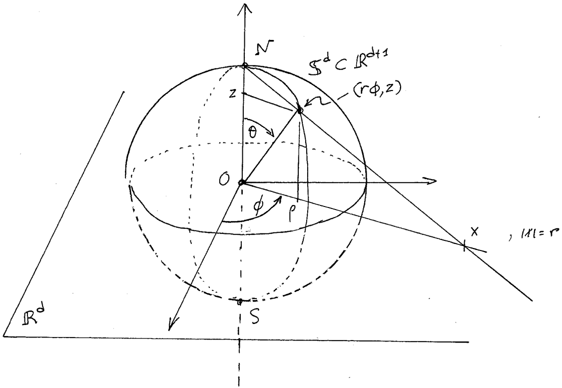
- for any $p\in(2,2^*]$ with $2^*=\frac{2\,d}{d-2}$ if $d\geq3$
- for any $p\in(2,\infty)$ if $d=2$

Here dv_g is the uniform probability measure: $v_g(\mathbb{S}^d)=1$

- 1 is the optimal constant, equality achieved by constants

- $p=2^*$ corresponds to Sobolev's inequality...

Stereographic projection



Sobolev inequality

The stereographic projection of $\mathbb{S}^d \subset \mathbb{R}^d \times \mathbb{R} \ni (\rho\,\phi, z)$ onto $\mathbb{R}^d$: to $\rho^2 + z^2 = 1$, $z \in [-1,1]$, $\rho \geq 0$, $\phi \in \mathbb{S}^{d-1}$ we associate $x \in \mathbb{R}^d$ such that $r = |x|$, $\phi = \frac{x}{|x|}$

$$z = \frac{r^2-1}{r^2+1} = 1 - \frac{2}{r^2+1}\,, \quad \rho = \frac{2\,r}{r^2+1}$$

and transform any function u on $\mathbb{S}^d$ into a function v on $\mathbb{R}^d$ using

$$u(y) = \left(\tfrac{r}{\rho}\right)^{\frac{d-2}{2}} v(x) = \left(\tfrac{r^2+1}{2}\right)^{\frac{d-2}{2}} v(x) = (1-z)^{-\frac{d-2}{2}}\, v(x)$$

- $p = 2^*$, $\mathsf{S}_d = \frac{1}{4}\, d\,(d-2)\,|\mathbb{S}^d|^{2/d}$: Euclidean Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla v|^2\, dx \geq \mathsf{S}_d \left[\int_{\mathbb{R}^d} |v|^{\frac{2\,d}{d-2}}\, dx\right]^{\frac{d-2}{d}} \quad \forall\, v \in \mathcal{D}^{1,2}(\mathbb{R}^d)$$

Extended inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq \frac{d}{p-2} \left[\left(\int_{\mathbb{S}^d} |u|^p \, d\,v_g \right)^{2/p} - \int_{\mathbb{S}^d} |u|^2 \, d\,v_g \right] \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

is valid

- for any $p \in (1,2) \cup (2,\infty)$ if $d = 1,\, 2$
- for any $p \in (1,2) \cup (2, 2^*]$ if $d \geq 3$

- Case $p = 2$: Logarithmic Sobolev inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq \frac{d}{2} \int_{\mathbb{S}^d} |u|^2 \log\left(\frac{|u|^2}{\int_{\mathbb{S}^d} |u|^2 \, d\,v_g} \right) d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

- Case $p = 1$: Poincaré inequality

$$\int_{\mathbb{S}^d} |\nabla u|^2 \, d\,v_g \geq d \int_{\mathbb{S}^d} |u - \bar{u}|^2 \, d\,v_g \quad \text{with} \quad \bar{u} := \int_{\mathbb{S}^d} u \, d\,v_g \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$$

Optimality: a perturbation argument

- For any $p \in (1, 2^*]$ if $d \geq 3$, any $p > 1$ if $d = 1$ or 2, it is remarkable that

$$\mathcal{Q}[u] := \frac{(p-2)\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \geq \inf_{u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)} \mathcal{Q}[u] = \frac{1}{d}$$

is achieved in the limiting case

$$\mathcal{Q}[1 + \varepsilon\, v] \sim \frac{\|\nabla v\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|v\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} \quad \text{as} \quad \varepsilon \to 0$$

when v is an eigenfunction associated with the first nonzero eigenvalue of Δ_g, thus proving the optimality

- $p < 2$: a proof by semi-groups using Nelson's hypercontractivity lemma. $p > 2$: no simple proof based on spectral analysis is available: [Beckner], an approach based on Lieb's duality, the Funk-Hecke formula and some (non-trivial) computations

- elliptic methods / Γ_2 formalism of Bakry-Emery / nonlinear flows

Schwarz symmetrization and the ultraspherical setting

$(\xi_0, \xi_1, \ldots \xi_d) \in \mathbb{S}^d$, $\xi_d = z$, $\sum_{i=0}^d |\xi_i|^2 = 1$ [Smets-Willem]

Lemma

Up to a rotation, any minimizer of $\mathcal{Q}$ depends only on $\xi_d = z$

- Let $d\sigma(\theta) := \frac{(\sin\theta)^{d-1}}{Z_d}\,d\theta$, $Z_d := \sqrt{\pi}\,\frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{d+1}{2})}$: $\forall\, v \in \mathrm{H}^1([0,\pi], d\sigma)$

$$\frac{p-2}{d}\int_0^\pi |v'(\theta)|^2\,d\sigma + \int_0^\pi |v(\theta)|^2\,d\sigma \ge \left(\int_0^\pi |v(\theta)|^p\,d\sigma\right)^{\frac{2}{p}}$$

- Change of variables $z = \cos\theta$, $v(\theta) = f(z)$

$$\frac{p-2}{d}\int_{-1}^1 |f'|^2\,\nu\,d\nu_d + \int_{-1}^1 |f|^2\,d\nu_d \ge \left(\int_{-1}^1 |f|^p\,d\nu_d\right)^{\frac{2}{p}}$$

where $\nu_d(z)\,dz = d\nu_d(z) := Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$

The ultraspherical operator

With $d\nu_d = Z_d^{-1}\,\nu^{\frac{d}{2}-1}\,dz$, $\nu(z) := 1 - z^2$, consider the space $\mathrm{L}^2((-1,1), d\nu_d)$ with scalar product

$$\langle f_1, f_2 \rangle = \int_{-1}^{1} f_1\, f_2\, d\nu_d\,, \quad \|f\|_p = \left(\int_{-1}^{1} f^p\, d\nu_d\right)^{\frac{1}{p}}$$

The self-adjoint *ultraspherical* operator is

$$\mathcal{L}\, f := (1-z^2)\, f'' - d\, z\, f' = \nu\, f'' + \frac{d}{2}\,\nu'\, f'$$

which satisfies $\langle f_1, \mathcal{L}\, f_2 \rangle = -\int_{-1}^{1} f_1'\, f_2'\, \nu\, d\nu_d$

Proposition

Let $p \in [1,2) \cup (2, 2^*]$, $d \geq 1$

$$-\langle f, \mathcal{L}\, f \rangle = \int_{-1}^{1} |f'|^2\, \nu\, d\nu_d \geq d\, \frac{\|f\|_p^2 - \|f\|_2^2}{p-2} \quad \forall\, f \in \mathrm{H}^1([-1,1], d\nu_d)$$

Flows on the sphere

- Heat flow and the Bakry-Emery method
- Fast diffusion (porous media) flow and the choice of the exponents

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss

Heat flow and the Bakry-Emery method

With $g = f^p$, *i.e.* $f = g^\alpha$ with $\alpha = 1/p$

$$\text{(Ineq.)} \qquad -\langle f, \mathcal{L} f\rangle = -\langle g^\alpha, \mathcal{L}\, g^\alpha\rangle =: \mathcal{I}[g] \geq d\, \frac{\|g\|_1^{2\,\alpha} - \|g^{2\,\alpha}\|_1}{p-2} =: \mathcal{F}[g]$$

Heat flow

$$\frac{\partial g}{\partial t} = \mathcal{L}\, g$$

$$\frac{d}{dt}\,\|g\|_1 = 0\,, \quad \frac{d}{dt}\,\|g^{2\,\alpha}\|_1 = -\,2\,(p-2)\,\langle f, \mathcal{L}\, f\rangle = 2\,(p-2)\int_{-1}^{1}|f'|^2\;\nu\;d\nu_d$$

which finally gives

$$\frac{d}{dt}\mathcal{F}[g(t,\cdot)] = -\frac{d}{p-2}\,\frac{d}{dt}\,\|g^{2\,\alpha}\|_1 = -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

$$\text{Ineq.} \iff \frac{d}{dt}\mathcal{F}[g(t,\cdot)] \leq -\,2\,d\,\mathcal{F}[g(t,\cdot)] \;\Longleftarrow\; \frac{d}{dt}\mathcal{I}[g(t,\cdot)] \leq -\,2\,d\,\mathcal{I}[g(t,\cdot)]$$

The equation for $g = f^p$ can be rewritten in terms of f as

$$\frac{\partial f}{\partial t} = \mathcal{L} f + (p-1)\,\frac{|f'|^2}{f}\,\nu$$

$$-\frac{1}{2}\,\frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d = \frac{1}{2}\,\frac{d}{dt}\,\langle f,\mathcal{L} f\rangle = \langle \mathcal{L} f,\mathcal{L} f\rangle + (p-1)\,\langle \frac{|f'|^2}{f}\,\nu,\mathcal{L} f\rangle$$

$$\frac{d}{dt}\mathcal{I}[g(t,\cdot)] + 2\,d\,\mathcal{I}[g(t,\cdot)] = \frac{d}{dt}\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d + 2\,d\int_{-1}^{1}|f'|^2\,\nu\,d\nu_d$$

$$= -\,2\int_{-1}^{1}\left(|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}\right)\nu^2\,d\nu_d$$

is nonpositive if

$$|f''|^2 + (p-1)\,\frac{d}{d+2}\,\frac{|f'|^4}{f^2} - 2\,(p-1)\,\frac{d-1}{d+2}\,\frac{|f'|^2\,f''}{f}$$

is pointwise nonnegative, which is granted if

$$\left[(p-1)\,\frac{d-1}{d+2}\right]^2 \le (p-1)\,\frac{d}{d+2} \iff p \le \frac{2\,d^2+1}{(d-1)^2} = 2^{\#} < \frac{2\,d}{d-2} = 2^*$$

... up to the critical exponent: a proof in two slides

$$\left[\frac{d}{dz}, \mathcal{L}\right] u = (\mathcal{L}\, u)' - \mathcal{L}\, u' = -2\, z\, u'' - d\, u'$$

$$\int_{-1}^{1} (\mathcal{L}\, u)^2 \, d\nu_d = \int_{-1}^{1} |u''|^2 \, \nu^2 \, d\nu_d + d \int_{-1}^{1} |u'|^2 \, \nu \, d\nu_d$$

$$\int_{-1}^{1} (\mathcal{L}\, u) \, \frac{|u'|^2}{u} \, \nu \, d\nu_d = \frac{d}{d+2} \int_{-1}^{1} \frac{|u'|^4}{u^2} \, \nu^2 \, d\nu_d - 2\, \frac{d-1}{d+2} \int_{-1}^{1} \frac{|u'|^2\, u''}{u} \, \nu^2 \, d\nu_d$$

On $(-1,1)$, let us consider the *porous medium (fast diffusion)* flow

$$u_t = u^{2-2\beta} \left(\mathcal{L}\, u + \kappa \, \frac{|u'|^2}{u} \, \nu \right)$$

If $\kappa = \beta\,(p-2) + 1$, the L^p norm is conserved

$$\frac{d}{dt} \int_{-1}^{1} u^{\beta p} \, d\nu_d = \beta\, p\, (\kappa - \beta\,(p-2) - 1) \int_{-1}^{1} u^{\beta(p-2)} \, |u'|^2 \, \nu \, d\nu_d = 0$$

$$f = u^\beta,\ \|f'\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \tfrac{d}{p-2}\left(\|f\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \|f\|^2_{\mathrm{L}^p(\mathbb{S}^d)}\right) \ge 0\ ?$$

$$\mathcal{A} := \int_{-1}^1 |u''|^2\,\nu^2\,d\nu_d - 2\,\frac{d-1}{d+2}\,(\kappa+\beta-1)\int_{-1}^1 u''\,\frac{|u'|^2}{u}\,\nu^2\,d\nu_d + \left[\kappa\,(\beta-1) + \frac{d}{d+2}\,(\kappa+\beta-1)\right]\int_{-1}^1 \frac{|u'|^4}{u^2}\,\nu^2\,d\nu_d$$

$\mathcal{A}$ is nonnegative for some β if

$$\frac{8\,d^2}{(d+2)^2}\,(p-1)\,(2^*-p) \ge 0$$

$\mathcal{A}$ is a sum of squares if $p \in (2, 2^*)$ for an arbitrary choice of β in a certain interval (depending on p and

$$\mathcal{A} = \int_{-1}^1 \left| u'' - \frac{p+2}{6-p}\,\frac{|u'|^2}{u} \right|^2 \nu^2\,d\nu_d \ge 0 \quad \text{if } p = 2^* \text{ and } \beta = \frac{4}{6-p}$$

The rigidity point of view

Which computation have we done ? $u_t = u^{2-2\beta}\left(\mathcal{L}\,u + \kappa\,\frac{|u'|^2}{u}\,\nu\right)$

$$-\,\mathcal{L}\;u - (\beta-1)\,\frac{|u'|^2}{u}\,\nu + \frac{\lambda}{p-2}\,u = \frac{\lambda}{p-2}\,u^\kappa$$

Multiply by $\mathcal{L}\,u$ and integrate

$$\ldots\int_{-1}^{1}\mathcal{L}\,u\,u^\kappa\;d\nu_d = -\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

Multiply by $\kappa\,\frac{|u'|^2}{u}$ and integrate

$$\ldots = +\,\kappa\int_{-1}^{1}u^\kappa\,\frac{|u'|^2}{u}\;d\nu_d$$

The two terms cancel and we are left only with the two-homogenous terms

Improvements of the inequalities (subcritical range)

- as long as the exponent is either in the range $(1,2)$ or in the range $(2,2^*)$, on can establish *improved inequalities*
- An improvement automatically gives an explicit stability result of the optimal functions in the (non-improved) inequality
- By duality, this provides a stability result for Keller-Lieb-Tirring inequalities

Joint work with M.J. Esteban, M. Kowalczyk and M. Loss

What does "improvement" mean ?

An *improved* inequality is

$$d\,\Phi\,(\mathsf{e}) \le \mathsf{i} \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

for some function Φ such that $\Phi(0) = 0$, $\Phi'(0) = 1$, $\Phi' > 0$ and $\Phi(s) > s$ for any s. With $\Psi(s) := s - \Phi^{-1}(s)$

$$\mathsf{i} - d\,\mathsf{e} \ge d\,(\Psi \circ \Phi)(\mathsf{e}) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d) \quad \text{s.t.} \quad \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} = 1$$

Lemma (Generalized Csiszár-Kullback inequalities)

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} - \frac{d}{p-2}\left[\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right]$$

$$\ge d\,\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,(\Psi \circ \Phi)\left(\mathsf{C}\,\frac{\|u\|^{2\,(1-r)}_{\mathrm{L}^s(\mathbb{S}^d)}}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\,\|u^r - \bar{u}^r\|^2_{\mathrm{L}^q(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d)$$

$s(p) := \max\{2, p\}$ and $p \in (1,2)$: $q(p) := 2/p$, $r(p) := p$; $p \in (2,4)$: $q = p/2$, $r = 2$; $p \ge 4$: $q = p/(p-2)$, $r = p-2$

Linear flow: improved Bakry-Emery method

Cf. [Arnold, JD]

$$w_t = \mathcal{L}\, w + \kappa\, \frac{|w'|^2}{w}\, \nu$$

With $2^\sharp := \frac{2\,d^2+1}{(d-1)^2}$

$$\gamma_1 := \left(\frac{d-1}{d+2}\right)^2 (p-1)\,(2^\# - p) \quad \text{if} \quad d>1\,, \qquad \gamma_1 := \frac{p-1}{3} \quad \text{if} \quad d=1$$

If $p \in [1,2) \cup (2, 2^\sharp]$ and w is a solution, then

$$\frac{d}{dt}\,(\mathsf{i} - \, d\,\mathsf{e}) \le -\,\gamma_1 \int_{-1}^{1} \frac{|w'|^4}{w^2}\, d\nu_d \le -\,\gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

Recalling that $\mathsf{e}' = -\,\mathsf{i}$, we get a differential inequality

$$\mathsf{e}'' + \, d\,\mathsf{e}' \ge \gamma_1\, \frac{|\mathsf{e}'|^2}{1 - \,(p-2)\,\mathsf{e}}$$

After integration: $d\,\Phi(\mathsf{e}(0)) \le \mathsf{i}(0)$

$$w_t = w^{2-2\beta}\left(\mathcal{L}\,w + \kappa\,\frac{|w'|^2}{w}\right)$$

For all $p \in [1, 2^*]$, $\kappa = \beta\,(p-2)+1$, $\frac{d}{dt}\int_{-1}^{1} w^{\beta p}\,d\nu_d = 0$

$$-\frac{1}{2\beta^2}\,\frac{d}{dt}\int_{-1}^{1}\left(|(w^\beta)'|^2\,\nu + \frac{d}{p-2}\,\left(w^{2\beta} - \overline{w}^{2\beta}\right)\right)d\nu_d \ge \gamma\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d$$

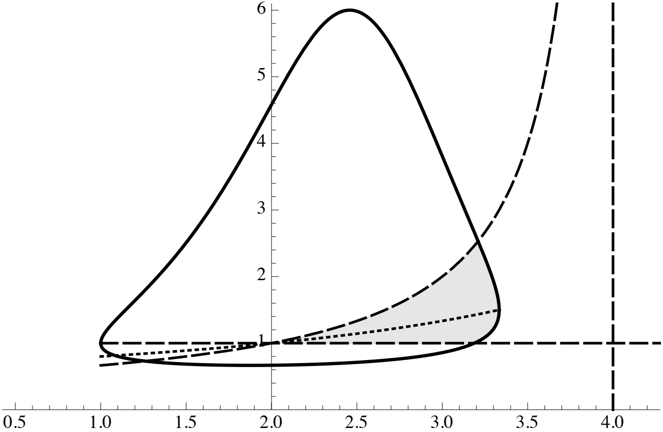
Lemma

For all $w \in \mathrm{H}^1\big((-1,1), d\nu_d\big)$*, such that* $\int_{-1}^{1} w^{\beta p}\,d\nu_d = 1$

$$\int_{-1}^{1}\frac{|w'|^4}{w^2}\,\nu^2\,d\nu_d \ge \frac{1}{\beta^2}\,\frac{\int_{-1}^{1}|(w^\beta)'|^2\,\nu\,d\nu_d\int_{-1}^{1}|w'|^2\,\nu\,d\nu_d}{\left(\int_{-1}^{1} w^{2\beta}\,d\nu_d\right)^\delta}$$

.... but there are conditions on β

Admissible (p, β) for $d = 5$



The line

- A first example of a non-compact manifold

Joint work with M.J. Esteban, A. Laptev and M. Loss

One-dimensional Gagliardo-Nirenberg-Sobolev inequalities

$$\|f\|_{\mathrm{L}^p(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|^{\theta}_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\theta}_{\mathrm{L}^2(\mathbb{R})} \quad \text{if} \quad p\in(2,\infty)$$

$$\|f\|_{\mathrm{L}^2(\mathbb{R})} \le \mathsf{C}_{\mathrm{GN}}(p)\,\|f'\|^{\eta}_{\mathrm{L}^2(\mathbb{R})}\,\|f\|^{1-\eta}_{\mathrm{L}^p(\mathbb{R})} \quad \text{if} \quad p\in(1,2)$$

with $\theta=\frac{p-2}{2\,p}$ and $\eta=\frac{2-p}{2+p}$

The threshold case corresponding to the limit as $p\to2$ is the logarithmic Sobolev inequality

$$\int_{\mathbb{R}} u^2\,\log\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{R})}}\right)dx \le \frac12\,\|u\|^2_{\mathrm{L}^2(\mathbb{R})}\,\log\left(\frac{2}{\pi\,e}\,\frac{\|u'\|^2_{\mathrm{L}^2(\mathbb{R})}}{\|u\|^2_{\mathrm{L}^2(\mathbb{R})}}\right)$$

If $p>2$, $u_\star(x)=(\cosh x)^{-\frac{2}{p-2}}$ solves

$$-(p-2)^2\,u''+4\,u-\,2\,p\,|u|^{p-2}\,u=0$$

If $p\in(1,2)$ consider $u_*(x)=(\cos x)^{\frac{2}{2-p}}$, $x\in(-\pi/2,\pi/2)$

Flow

Let us define on $\mathrm{H}^1(\mathbb{R})$ the functional

$$\mathcal{F}[v] := \|v'\|^2_{\mathrm{L}^2(\mathbb{R})} + \frac{4}{(p-2)^2}\,\|v\|^2_{\mathrm{L}^2(\mathbb{R})} - \mathsf{C}\,\|v\|^2_{\mathrm{L}^p(\mathbb{R})} \quad \text{s.t. } \mathcal{F}[u_\star] = 0$$

With $z(x) := \tanh x$, consider the *flow*

$$v_t = \frac{v^{1-\frac{p}{2}}}{\sqrt{1-z^2}}\left[v'' + \frac{2\,p}{p-2}\,z\,v' + \frac{p}{2}\,\frac{|v'|^2}{v} + \frac{2}{p-2}\,v\right]$$

Theorem (Dolbeault-Esteban-Laptev-Loss)

Let $p \in (2,\infty)$. Then

$$\frac{d}{dt}\mathcal{F}[v(t)] \le 0 \quad \textit{and} \quad \lim_{t\to\infty}\mathcal{F}[v(t)] = 0$$

$\frac{d}{dt}\mathcal{F}[v(t)] = 0 \quad \Longleftrightarrow \quad v_0(x) = u_\star(x - x_0)$

Similar results for $p \in (1,2)$

The inequality $(p > 2)$ and the ultraspherical operator

- *The problem on the line is equivalent to the critical problem for the ultraspherical operator*

$$\int_{\mathbb{R}} |v'|^2\,dx + \frac{4}{(p-2)^2}\int_{\mathbb{R}} |v|^2\,dx \ge \mathsf{C}\left(\int_{\mathbb{R}} |v|^p\,dx\right)^{\frac{2}{p}}$$

With

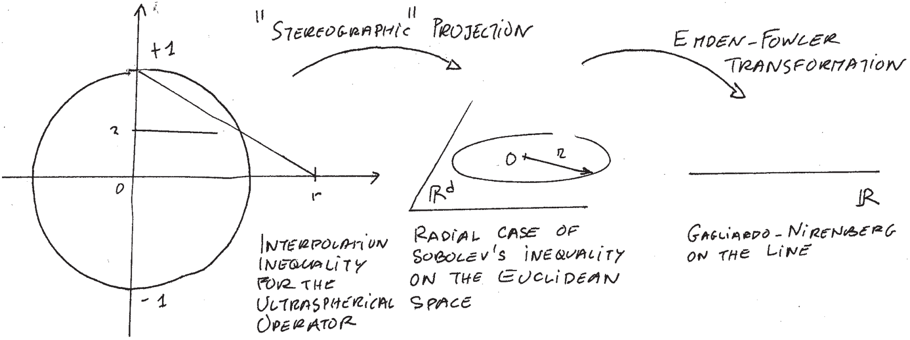
$$z(x) = \tanh x\,,\quad v_\star = (1-z^2)^{\frac{1}{p-2}} \quad \text{and} \quad v(x) = v_\star(x)\,f(z(x))$$

equality is achieved for $f = 1$ and, if we let $\nu(z) := 1 - z^2$, then

$$\int_{-1}^{1} |f'|^2\,\nu\,d\nu_d + \frac{2\,p}{(p-2)^2}\int_{-1}^{1} |f|^2\,d\nu_d \ge \frac{2\,p}{(p-2)^2}\left(\int_{-1}^{1} |f|^p\,d\nu_d\right)^{\frac{2}{p}}$$

where $d\nu_p$ denotes the probability measure $d\nu_p(z) := \frac{1}{\zeta_p}\,\nu^{\frac{2}{p-2}}\,dz$

$$d = \frac{2\,p}{p-2} \quad \Longleftrightarrow \quad p = \frac{2\,d}{d-2}$$



Change of variables = stereographic projection + Emden-Fowler

Compact Riemannian manifolds

- no sign is required on the Ricci tensor and an improved integral criterion is established
- the flow explores the energy landscape... and shows the non-optimality of the improved criterion

Riemannian manifolds with positive curvature

$(\mathfrak{M}, g)$ is a smooth closed compact connected Riemannian manifold dimension d, no boundary, Δ_g is the Laplace-Beltrami operator $\mathrm{vol}(\mathfrak{M}) = 1$, $\mathfrak{R}$ is the Ricci tensor, $\lambda_1 = \lambda_1(-\Delta_g)$

$$\rho := \inf_{\mathfrak{M}} \inf_{\xi \in \mathbb{S}^{d-1}} \mathfrak{R}(\xi\,,\xi)$$

Theorem (Licois-Véron, Bakry-Ledoux)

Assume $d \geq 2$ and $\rho > 0$. If

$$\lambda \leq (1-\theta)\,\lambda_1 + \theta\,\frac{d\,\rho}{d-1} \quad \textit{where} \quad \theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} > 0$$

then for any $p \in (2, 2^*)$, the equation

$$-\,\Delta_g v + \frac{\lambda}{p-2}\,\left(v - v^{p-1}\right) = 0$$

has a unique positive solution $v \in C^2(\mathfrak{M})$: $v \equiv 1$

Riemannian manifolds: first improvement

Theorem (Dolbeault-Esteban-Loss)

For any $p \in (1,2) \cup (2,2^*)$

$$0 < \lambda < \lambda_\star = \inf_{u \in \mathrm{H}^2(\mathfrak{M})} \frac{\int_{\mathfrak{M}} \left[(1-\theta)\,(\Delta_g u)^2 + \frac{\theta\, d}{d-1}\, \mathfrak{R}(\nabla u, \nabla u) \right] d\, v_g}{\int_{\mathfrak{M}} |\nabla u|^2\, d\, v_g}$$

there is a unique positive solution in $C^2(\mathfrak{M})$*:* $u \equiv 1$

$\lim_{p \to 1_+} \theta(p) = 0 \Longrightarrow \lim_{p \to 1_+} \lambda_\star(p) = \lambda_1$ if ρ is bounded
$\lambda_\star = \lambda_1 = d\,\rho/(d-1) = d$ if $\mathfrak{M} = \mathbb{S}^d$ since $\rho = d-1$

$$(1-\theta)\,\lambda_1 + \theta\, \frac{d\,\rho}{d-1} \le \lambda_\star \le \lambda_1$$

Riemannian manifolds: second improvement

$\mathrm{H}_g u$ denotes Hessian of u and $\theta = \dfrac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1}$

$$\mathrm{Q}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u - \frac{(d-1)\,(p-1)}{\theta\,(d+3-p)}\left[\frac{\nabla u \otimes \nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

$$\Lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M})\setminus\{0\}} \frac{(1-\theta)\displaystyle\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g + \frac{\theta\,d}{d-1}\int_{\mathfrak{M}}\left[\|\mathrm{Q}_g u\|^2 + \mathfrak{R}(\nabla u, \nabla u)\right]}{\displaystyle\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g}$$

Theorem (Dolbeault-Esteban-Loss)

Assume that $\Lambda_\star > 0$. For any $p \in (1,2)\cup(2,2^*)$, the equation has a unique positive solution in $C^2(\mathfrak{M})$ if $\lambda \in (0,\Lambda_\star)$: $u \equiv 1$

Optimal interpolation inequality

For any $p \in (1,2) \cup (2,2^*)$ or $p = 2^*$ if $d \geq 3$

$$\|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} \geq \frac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right] \quad \forall\, v \in \mathrm{H}^1(\mathfrak{M})$$

Theorem (Dolbeault-Esteban-Loss)

Assume $\Lambda_\star > 0$. The above inequality holds for some $\lambda = \Lambda \in [\Lambda_\star, \lambda_1]$
If $\Lambda_\star < \lambda_1$, then the optimal constant Λ is such that

$$\Lambda_\star < \Lambda \leq \lambda_1$$

If $p = 1$, then $\Lambda = \lambda_1$

Using $u = 1 + \varepsilon\,\varphi$ as a test function where φ we get $\lambda \leq \lambda_1$
A minimum of

$$v \mapsto \|\nabla v\|^2_{\mathrm{L}^2(\mathfrak{M})} - \tfrac{\lambda}{p-2} \left[\|v\|^2_{\mathrm{L}^p(\mathfrak{M})} - \|v\|^2_{\mathrm{L}^2(\mathfrak{M})}\right]$$

under the constraint $\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 1$ is negative if $\lambda > \lambda_1$

The flow

The key tools the flow

$$u_t = u^{2-2\beta}\left(\Delta_g u + \kappa\,\frac{|\nabla u|^2}{u}\right)\,,\quad \kappa = 1+\beta\,(p-2)$$

If $v = u^\beta$, then $\frac{d}{dt}\|v\|_{\mathrm{L}^p(\mathfrak{M})} = 0$ and the functional

$$\mathcal{F}[u] := \int_{\mathfrak{M}} |\nabla(u^\beta)|^2\,d\,v_g + \frac{\lambda}{p-2}\left[\int_{\mathfrak{M}} u^{2\,\beta}\,d\,v_g - \left(\int_{\mathfrak{M}} u^{\beta\,p}\,d\,v_g\right)^{2/p}\right]$$

is monotone decaying

- J. Demange, *Improved Gagliardo-Nirenberg-Sobolev inequalities on manifolds with positive curvature*, J. Funct. Anal., 254 (2008), pp. 593–611. Also see C. Villani, *Optimal Transport, Old and New*

Elementary observations (1/2)

Let $d \geq 2$, $u \in C^2(\mathfrak{M})$, and consider the trace free Hessian

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, d\, v_g = \frac{d}{d-1} \int_{\mathfrak{M}} \| \mathrm{L}_g u \|^2 \, d\, v_g + \frac{d}{d-1} \int_{\mathfrak{M}} \mathfrak{R}(\nabla u, \nabla u) \, d\, v_g$$

Based on the Bochner-Lichnerovicz-Weitzenböck formula

$$\frac{1}{2}\,\Delta\,|\nabla u|^2 = \|\mathrm{H}_g u\|^2 + \nabla(\Delta_g u)\cdot\nabla u + \mathfrak{R}(\nabla u, \nabla u)$$

Elementary observations (2/2)

Lemma

$$\int_{\mathfrak{M}} \Delta_g u \, \frac{|\nabla u|^2}{u} \, d v_g = \frac{d}{d+2} \int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2} \, d v_g - \frac{2\,d}{d+2} \int_{\mathfrak{M}} [\mathrm{L}_g u] : \left[\frac{\nabla u \otimes \nabla u}{u} \right] d v_g$$

Lemma

$$\int_{\mathfrak{M}} (\Delta_g u)^2 \, d v_g \ge \lambda_1 \int_{\mathfrak{M}} |\nabla u|^2 \, d v_g \quad \forall\, u \in \mathrm{H}^2(\mathfrak{M})$$

and λ_1 *is the optimal constant in the above inequality*

The key estimates

$$\mathcal{G}[u] := \int_{\mathfrak{M}} \left[\theta\,(\Delta_g u)^2 + (\kappa+\beta-1)\,\Delta_g u\,\frac{|\nabla u|^2}{u} + \kappa\,(\beta-1)\,\frac{|\nabla u|^4}{u^2}\right] d\,v_g$$

Lemma

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] = -\,(1-\theta)\int_{\mathfrak{M}} (\Delta_g u)^2\,d\,v_g - \mathcal{G}[u] + \lambda\int_{\mathfrak{M}} |\nabla u|^2\,d\,v_g$$

$$\mathrm{Q}_g^\theta u := \mathrm{L}_g u - \tfrac{1}{\theta}\,\tfrac{d-1}{d+2}\,(\kappa+\beta-1)\left[\frac{\nabla u\otimes\nabla u}{u} - \frac{g}{d}\,\frac{|\nabla u|^2}{u}\right]$$

Lemma

$$\mathcal{G}[u] = \frac{\theta\,d}{d-1}\left[\int_{\mathfrak{M}} \|\mathrm{Q}_g^\theta u\|^2\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla u,\nabla u)\,d\,v_g\right] - \mu\int_{\mathfrak{M}} \frac{|\nabla u|^4}{u^2}\,d\,v_g$$

with $\mu := \dfrac{1}{\theta}\left(\dfrac{d-1}{d+2}\right)^2(\kappa+\beta-1)^2 - \kappa\,(\beta-1) - (\kappa+\beta-1)\,\dfrac{d}{d+2}$

The end of the proof

Assume that $d \geq 2$. If $\theta = 1$, then μ is nonpositive if

$$\beta_-(p) \leq \beta \leq \beta_+(p) \quad \forall\, p \in (1, 2^*)$$

where $\beta_\pm := \frac{\mathsf{b} \pm \sqrt{\mathsf{b}^2 - \mathsf{a}}}{2\,\mathsf{a}}$ with $\mathsf{a} = 2 - p + \left[\frac{(d-1)\,(p-1)}{d+2}\right]^2$ and $\mathsf{b} = \frac{d+3-p}{d+2}$
Notice that $\beta_-(p) < \beta_+(p)$ if $p \in (1, 2^*)$ and $\beta_-(2^*) = \beta_+(2^*)$

$$\theta = \frac{(d-1)^2\,(p-1)}{d\,(d+2)+p-1} \quad \text{and} \quad \beta = \frac{d+2}{d+3-p}$$

Proposition

Let $d \geq 2$, $p \in (1,2) \cup (2, 2^*)$ ($p \neq 5$ or $d \neq 2$)

$$\frac{1}{2\,\beta^2}\,\frac{d}{dt}\mathcal{F}[u] \leq (\lambda - \Lambda_\star) \int_{\mathfrak{M}} |\nabla u|^2\, d\,v_g$$

The Moser-Trudinger-Onofri inequality on Riemannian manifolds

Joint work with G. Jankowiak and M.J. Esteban

- Extension to compact Riemannian manifolds of dimension 2...

We shall also denote by $\mathfrak{R}$ the Ricci tensor, by $\mathrm{H}_g u$ the Hessian of u and by

$$\mathrm{L}_g u := \mathrm{H}_g u - \frac{g}{d}\,\Delta_g u$$

the trace free Hessian. Let us denote by $\mathrm{M}_g u$ the trace free tensor

$$\mathrm{M}_g u := \nabla u \otimes \nabla u - \frac{g}{d}\,|\nabla u|^2$$

We define

$$\lambda_\star := \inf_{u \in \mathrm{H}^2(\mathfrak{M}) \setminus \{0\}} \frac{\int_{\mathfrak{M}} \left[\, \|\, \mathrm{L}_g u - \frac{1}{2}\,\mathrm{M}_g u \,\|^2 + \mathfrak{R}(\nabla u, \nabla u) \right] e^{-u/2}\, d\,v_g}{\int_{\mathfrak{M}} |\nabla u|^2\, e^{-u/2}\, d\,v_g}$$

Theorem

Assume that $d = 2$ and $\lambda_\star > 0$. If u is a smooth solution to

$$-\frac{1}{2}\,\Delta_g u + \lambda = e^u$$

then u is a constant function if $\lambda \in (0, \lambda_\star)$

The Moser-Trudinger-Onofri inequality on $\mathfrak{M}$

$$\frac{1}{4}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathfrak{M})} + \lambda \int_{\mathfrak{M}} u\,d\,v_g \geq \lambda\,\log\left(\int_{\mathfrak{M}} e^u\,d\,v_g\right) \quad \forall\,u \in \mathrm{H}^1(\mathfrak{M})$$

for some constant $\lambda > 0$. Let us denote by λ_1 the first positive eigenvalue of $-\,\Delta_g$

Corollary

If $d = 2$, then the MTO inequality holds with $\lambda = \Lambda := \min\{4\,\pi, \lambda_\star\}$. Moreover, if Λ is strictly smaller than $\lambda_1/2$, then the optimal constant in the MTO inequality is strictly larger than Λ

The flow

$$\frac{\partial f}{\partial t} = \Delta_g(e^{-f/2}) - \frac{1}{2}\,|\nabla f|^2\,e^{-f/2}$$

$$\mathcal{G}_\lambda[f] := \int_{\mathfrak{M}} \|\,\mathrm{L}_g f - \tfrac{1}{2}\,\mathrm{M}_g f\,\|^2\,e^{-f/2}\,d\,v_g + \int_{\mathfrak{M}} \mathfrak{R}(\nabla f,\nabla f)\,e^{-f/2}\,d\,v_g \\ - \lambda \int_{\mathfrak{M}} |\nabla f|^2\,e^{-f/2}\,d\,v_g$$

Then for any $\lambda \le \lambda_\star$ we have

$$\frac{d}{dt}\mathcal{F}_\lambda[f(t,\cdot)] = \int_{\mathfrak{M}} \left(-\tfrac{1}{2}\,\Delta_g f + \lambda\right)\left(\Delta_g(e^{-f/2}) - \tfrac{1}{2}\,|\nabla f|^2\,e^{-f/2}\right) d\,v_g \\ = -\,\mathcal{G}_\lambda[f(t,\cdot)]$$

Since $\mathcal{F}_\lambda$ is nonnegative and $\lim_{t\to\infty}\mathcal{F}_\lambda[f(t,\cdot)] = 0$, we obtain that

$$\mathcal{F}_\lambda[u] \ge \int_0^\infty \mathcal{G}_\lambda[f(t,\cdot)]\,dt$$

Weighted Moser-Trudinger-Onofri inequalities on the two-dimensional Euclidean space

On the Euclidean space $\mathbb{R}^2$, given a general probability measure μ does the inequality

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \lambda\left[\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)-\int_{\mathbb{R}^2}u\,d\mu\right]$$

hold for some $\lambda > 0$? Let

$$\Lambda_\star := \inf_{x\in\mathbb{R}^2}\frac{-\,\Delta\log\mu}{8\,\pi\,\mu}$$

Theorem

Assume that μ is a radially symmetric function. Then any radially symmetric solution to the EL equation is a constant if $\lambda < \Lambda_\star$ and the inequality holds with $\lambda = \Lambda_\star$ if equality is achieved among radial functions

Caffarelli-Kohn-Nirenberg inequalities

Work in progress with M.J. Esteban and M. Loss

Caffarelli-Kohn-Nirenberg inequalities and the symmetry breaking issue

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

hold under the conditions that $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$, and $a < a_c := (d-2)/2$

$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

▷ *With*

$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c - a)}\right)^{-\frac{2}{p-2}} \quad \textit{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

do we have $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry)*
or $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

The Emden-Fowler transformation and the cylinder

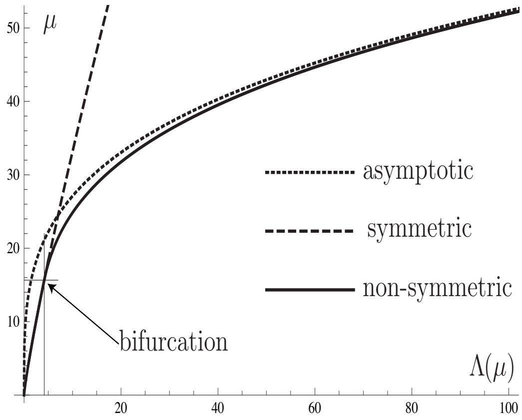
$$v(r,\omega) = r^{a-a_c}\,\varphi(s,\omega) \quad \text{with} \quad r=|x|\,, \quad s=-\log r \quad \text{and} \quad \omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_1)} + \|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_1)} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C}_1)} \ge \mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C}_1)} \quad \forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda := (a_c - a)^2$, $\mathcal{C} = \mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda) = \frac{1}{\mathsf{C}_{a,b}} \quad \text{with} \quad a = a_c \pm \sqrt{\Lambda} \quad \text{and} \quad b = \frac{d}{p} \pm \sqrt{\Lambda}$$



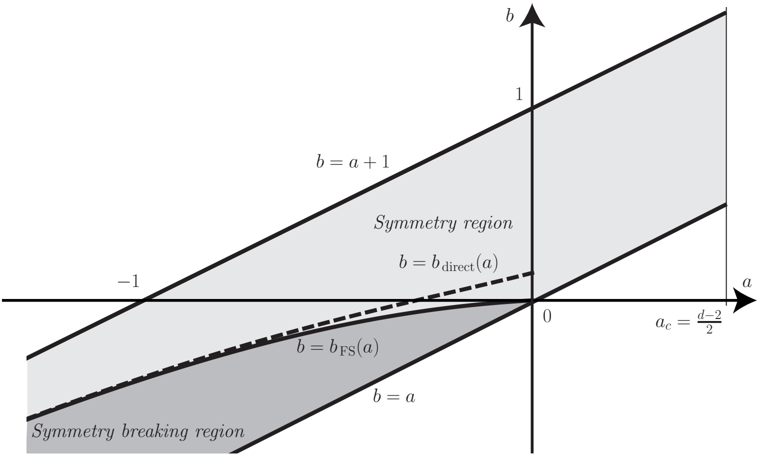
Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$, $\theta = 1$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

The symmetry result

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c-a)^2+d-1}} + a - a_c$$

Theorem

Let $d \geq 2$ and $p \leq 4$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\mathrm{FS}}(a)$, then the optimal functions for the Caffarelli-Kohn-Nirenberg inequalities are radially symmetric



The Felli-Schneider region, or symmetry breaking region, appears in dark grey and is defined by $a < 0$, $a \leq b < b_{\mathrm{FS}}(a)$. *We prove that symmetry holds in the light grey region defined by* $b \geq b_{\mathrm{FS}}(a)$ *when* $a < 0$ *and for any* $b \in [a, a+1]$ *if* $a \in [0, a_c)$

Sketch of a proof

A change of variables

With $(r = |x|,\, \omega = x/r) \in \mathbb{R}^+ \times \mathbb{S}^{d-1}$, the Caffarelli-Kohn-Nirenberg inequality is

$$\left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |v|^p \; r^{\,d-b\,p}\, \frac{dr}{r}\, d\omega\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b} \int_0^\infty \int_{\mathbb{S}^{d-1}} |\nabla v|^2 \; r^{\,d-2\,a}\, \frac{dr}{r}\, d\omega$$

Change of variables $r \mapsto r^\alpha$, $v(r,\omega) = w(r^\alpha,\omega)$

$$\alpha^{1-\frac{2}{p}} \left(\int_0^\infty \int_{\mathbb{S}^{d-1}} |w|^p \; r^{\frac{d-b\,p}{\alpha}}\, \frac{dr}{r}\, d\omega\right)^{\frac{2}{p}}$$
$$\le \mathsf{C}_{a,b} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left(\alpha^2 \left|\frac{\partial w}{\partial r}\right|^2 + \frac{1}{r^2}\, |\nabla_\omega w|^2\right) \; r^{\frac{d-2\,a-2}{\alpha}+2}\, \frac{dr}{r}\, d\omega$$

Choice of α

$$n = \frac{d-b\,p}{\alpha} = \frac{d-2\,a-2}{\alpha} + 2$$

Then $p = \frac{2\,n}{n-2}$ is the critical Sobolev exponent associated with n

A Sobolev type inequality

The parameters α and n vary in the ranges $0 < \alpha < \infty$ and $d < n < \infty$ and the *Felli-Schneider curve* in the (α, n) variables is given by

$$\alpha = \sqrt{\frac{d-1}{n-1}} =: \alpha_{\rm FS}$$

With

$$\mathsf{D} w = \left(\alpha \, \frac{\partial w}{\partial r}, \frac{1}{r} \, \nabla_\omega w\right) \,, \quad d\mu := r^{n-1} \, dr \, d\omega$$

the inequality becomes

$$\alpha^{1-\frac{2}{p}} \left(\int_{\mathbb{R}^d} |w|^p \, d\mu\right)^{\frac{2}{p}} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} |\mathsf{D} w|^2 \, d\mu$$

Proposition

Let $d \ge 4$. Optimality is achieved by radial functions and $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ if $\alpha \le \alpha_{\rm FS}$

The case of Gagliardo-Nirenberg inequalities on general cylinders is similar

Notations

When there is no ambiguity, we will omit the index $_\omega$ and from now on write that $\nabla = \nabla_\omega$ denotes the gradient with respect to the angular variable $\omega \in \mathbb{S}^{d-1}$ and that Δ is the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. We define the self-adjoint operator $\mathcal{L}$ by

$$\mathcal{L}\, w := -\,\mathsf{D}^*\,\mathsf{D}\, w = \alpha^2\, w'' + \alpha^2\, \frac{n-1}{r}\, w' + \frac{\Delta\, w}{r^2}$$

The fundamental property of $\mathcal{L}$ is the fact that

$$\int_{\mathbb{R}^d} w_1\, \mathcal{L}\, w_2\, d\mu = -\int_{\mathbb{R}^d} \mathsf{D} w_1 \cdot \mathsf{D} w_2\, d\mu \quad \forall\, w_1,\, w_2 \in \mathcal{D}(\mathbb{R}^d)$$

$\triangleright$ Heuristics: we look for a monotonicity formula along a well chosen nonlinear flow, based on the analogy with the decay of the Fisher information along the fast diffusion flow in $\mathbb{R}^d$

Fisher information

Let $u^{\frac{1}{2}-\frac{1}{n}} = |w| \quad \Longleftrightarrow \quad u = |w|^p,\ p = \frac{2\,n}{n-2}$

$$\mathcal{I}[u] := \int_{\mathbb{R}^d} u\,|\mathrm{Dp}|^2\,d\mu\,, \quad \mathsf{p} = \frac{m}{1-m}\,u^{m-1} \quad \text{and} \quad m = 1 - \frac{1}{n}$$

Here $\mathcal{I}$ is the *Fisher information* and p is the *pressure function*

Proposition

With $\Lambda = 4\,\alpha^2/(p-2)^2$ and for some explicit numerical constant κ, we have

$$\kappa\,\mu(\Lambda) = \inf\left\{\mathcal{I}[u]\ :\ \|u\|_{\mathrm{L}^1(\mathbb{S}^d,d\nu_n)}\right\}$$

The fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathcal{L}\, u^m\,, \quad m = 1 - \frac{1}{n}$$

Barenblatt self-similar solutions

$$u_\star(t,r,\omega) = t^{-n}\left(\mathsf{c}_\star + \frac{r^2}{2\,(n-1)\,\alpha^2\,t^2}\right)^{-n}$$

Lemma

$$\kappa\,\mu_\star(\Lambda) = \mathcal{I}[u_\star(t,\cdot)] \quad \forall\, t>0$$

▷ Strategy:

1) prove that $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] \le 0$,

2) prove that $\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = 0$ means that $u = u_\star$ up to a time shift

Decay of the Fisher information along the flow ?

$$\frac{\partial \mathsf{p}}{\partial t} = \frac{1}{n}\,\mathsf{p}\,\mathcal{L}\,\mathsf{p} - |\mathsf{Dp}|^2$$

$$\mathcal{Q}[\mathsf{p}] := \frac{1}{2}\,\mathcal{L}\,|\mathsf{Dp}|^2 - \mathsf{Dp}\cdot\mathsf{D}\mathcal{L}\,\mathsf{p}$$

$$\mathcal{K}[\mathsf{p}] := \int_{\mathbb{R}^d}\left(\mathcal{Q}[\mathsf{p}] - \frac{1}{n}\,(\mathcal{L}\,\mathsf{p})^2\right)\mathsf{p}^{1-n}\,d\mu$$

Lemma

$$\frac{d}{dt}\mathcal{I}[u(t,\cdot)] = -\,2\,(n-1)^{n-1}\,\mathcal{K}[\mathsf{p}]$$

If u is a critical point, then $\mathcal{K}[\mathsf{p}] = 0$

Boundary terms ! Regularity !

Proving decay (1/2)

$$\mathrm{k}[\mathrm{p}] := \mathcal{Q}(\mathrm{p}) - \frac{1}{n}\,(\mathcal{L}\,\mathrm{p})^2 = \frac{1}{2}\,\mathcal{L}\,|\mathrm{Dp}|^2 - \mathrm{Dp}\cdot\mathrm{D}\,\mathcal{L}\,\mathrm{p} - \frac{1}{n}\,(\mathcal{L}\,\mathrm{p})^2$$

$$\mathrm{k}_{\mathfrak{M}}[\mathrm{p}] := \frac{1}{2}\,\Delta\,|\nabla \mathrm{p}|^2 - \nabla \mathrm{p}\cdot\nabla\Delta\,\mathrm{p} - \tfrac{1}{n-1}\,(\Delta\,\mathrm{p})^2 - (n-2)\,\alpha^2\,|\nabla \mathrm{p}|^2$$

Lemma

Let $n \neq 1$ be any real number, $d \in \mathbb{N}$, $d \geq 2$, and consider a function $\mathrm{p} \in C^3((0,\infty)\times\mathfrak{M})$, where $(\mathfrak{M}, g)$ is a smooth, compact Riemannian manifold. Then we have

$$\mathrm{k}[\mathrm{p}] = \alpha^4\left(1-\frac{1}{n}\right)\left[\mathrm{p}'' - \frac{\mathrm{p}'}{r} - \frac{\Delta\,\mathrm{p}}{\alpha^2\,(n-1)\,r^2}\right]^2 + 2\,\alpha^2\,\frac{1}{r^2}\left|\nabla \mathrm{p}' - \frac{\nabla \mathrm{p}}{r}\right|^2 + \frac{1}{r^4}\,\mathrm{k}_{\mathfrak{M}}[\mathrm{p}]$$

Proving decay (2/2)

Lemma

Assume that $d \geq 3$, $n > d$ and $\mathfrak{M} = \mathbb{S}^{d-1}$. There is a positive constant $\zeta_\star$ such that

$$\int_{\mathbb{S}^{d-1}} \mathrm{k}_{\mathfrak{M}}[\mathrm{p}]\,\mathrm{p}^{1-n}\,d\omega \geq \left(\lambda_\star - (n-2)\,\alpha^2\right) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{p}|^2\,\mathrm{p}^{1-n}\,d\omega + \zeta_\star\,(n-d) \int_{\mathbb{S}^{d-1}} |\nabla \mathrm{p}|^4\,\mathrm{p}^{1-n}\,d\omega$$

Proof based on the Bochner-Lichnerowicz-Weitzenböck formula

Corollary

Let $d \geq 2$ and assume that $\alpha \leq \alpha_{\mathrm{FS}}$. Then for any nonnegative function $u \in \mathrm{L}^1(\mathbb{R}^d)$ with $\mathcal{I}[u] < +\infty$ and $\int_{\mathbb{R}^d} u\,d\mu = 1$, we have

$$\mathcal{I}[u] \geq \mathcal{I}_\star$$

When $\mathfrak{M} = \mathbb{S}^{d-1}$, $\lambda_\star = (n-2)\,\frac{d-1}{n-1}$

A perturbation argument

- If u is a critical point of $\mathcal{I}$ under the mass constraint $\int_{\mathbb{R}^d} u\,d\mu = 1$, then

$$o(\varepsilon) = \mathcal{I}[u + \varepsilon\,\mathcal{L}\,u^m] - \mathcal{I}[u] = -\,2\,(n-1)^{n-1}\,\varepsilon\,\mathcal{K}[\mathsf{p}] + o(\varepsilon)$$

because $\varepsilon\,\mathcal{L}\,u^m$ is an admissible perturbation. Indeed, we know that

$$\int_{\mathbb{R}^d} (u + \varepsilon\,\mathcal{L}\,u^m)\,d\mu = \int_{\mathbb{R}^d} u\,d\mu = 1$$

and, as we take the limit as $\varepsilon \to 0$, $u + \varepsilon\,\mathcal{L}\,u^m$ makes sense and, in particular, is positive

- If $\alpha \le \alpha_{\rm FS}$, then $\mathcal{K}[\mathsf{p}] = 0$ implies that $u = u_\star$

A summary

- the sphere: the flow tells us what to do, and provides a simple proof (*choice of the exponents / of the nonlinearity*) once the problem is reduced to the ultraspherical setting + improvements

- [not presented here: Keller-Lieb-Thirring estimates] the spectral point of view on the inequality: how to measure the deviation with respect to the *semi-classical* estimates, a nice example of bifurcation (and *symmetry breaking*)

- *Riemannian manifolds:* no sign is required on the Ricci tensor and an improved integral criterion is established. We extend the theory from pointwise criteria to a non-local Schrödinger type estimate (Rayleigh quotient). The method generically shows the non-optimality of the improved criterion

- the flow is a nice way of exploring an energy space: it explain how to produce a good test function at *any* critical point. A *rigidity* result tells you that a local result is actually global because otherwise the flow would relate (far away) extremal points while keeping the energy minimal

http://www.ceremade.dauphine.fr/∼dolbeaul
▷ Preprints (or arxiv, or HAL)

- J.D., Maria J. Esteban, Ari Laptev, and Michael Loss. Spectral properties of Schrödinger operators on compact manifolds: rigidity, flows, interpolation and spectral estimates, C.R. Math., 351 (11-12): 437–440, 2013.
- J.D., Maria J. Esteban, and Michael Loss. Nonlinear flows and rigidity results on compact manifolds. Journal of Functional Analysis, 267 (5): 1338-1363, 2014.
- J.D., Maria J. Esteban and Ari Laptev. Spectral estimates on the sphere. Analysis & PDE, 7 (2): 435-460, 2014.
- J.D., Maria J. Esteban, Michal Kowalczyk, and Michael Loss. Sharp interpolation inequalities on the sphere: New methods and consequences. Chinese Annals of Mathematics, Series B, 34 (1): 99-112, 2013.
- J.D., Maria J. Esteban, Ari Laptev, and Michael Loss. One-dimensional Gagliardo-Nirenberg-Sobolev inequalities: Remarks on duality and flows. Journal of the London Mathematical Society, 2014.

http://www.ceremade.dauphine.fr/∼dolbeaul
▷ Preprints (or arxiv, or HAL)

- J.D., Maria J. Esteban, Michal Kowalczyk, and Michael Loss. Improved interpolation inequalities on the sphere, Preprint, 2013. Discrete and Continuous Dynamical Systems Series S (DCDS-S), 7 (4): 695724, 2014.
- J.D., Maria J. Esteban, Gaspard Jankowiak.
The Moser-Trudinger-Onofri inequality, Preprint, 2014
- J.D., Maria J. Esteban, Gaspard Jankowiak. Rigidity results for semilinear elliptic equation with exponential nonlinearities and Moser-Trudinger-Onofri inequalities on two-dimensional Riemannian manifolds, Preprint, 2014
- J.D., Michal Kowalczyk. Uniqueness and rigidity in nonlinear elliptic equations, interpolation inequalities, and spectral estimates, Preprint, 2014
- J.D., and Maria J. Esteban. Branches of non-symmetric critical points and symmetry breaking in nonlinear elliptic partial differential equations. Nonlinearity, 27 (3): 435, 2014.
- J.D., Maria J. Esteban, and Michael Loss. Keller-Lieb-Thirring inequalities for Schrödinger operators on cylinders. Preprint, 2015

These slides can be found at

http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/
▷ Lectures

Thank you for your attention !