Symmetrization based on entropy methods and nonlinear flows

Jean Dolbeault

http://www.ceremade.dauphine.fr/∼dolbeaul

Ceremade, Université Paris-Dauphine

April 6, 2017

SPECTRAL DAYS 2017

Stuttgart, Germany, April 3–7, 2017

Outline

▷ **Symmetry breaking and linearization**

- The critical Caffarelli-Kohn-Nirenberg inequality
- Linearization and spectrum
- A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities

▷ **Without weights: Gagliardo-Nirenberg inequalities and fast diffusion flows**

- The Bakry-Emery method on the sphere
- Self-similar variables and relative entropies
- The role of the spectral gap

▷ **With weights: Caffarelli-Kohn-Nirenberg inequalities and weighted nonlinear flows**

- Large time asymptotics and spectral gaps
- A discussion of optimality cases

Collaborations

Collaboration with...

M.J. Esteban and M. Loss (symmetry, critical case)
M.J. Esteban, M. Loss and M. Muratori (symmetry, subcritical case)
M. Bonforte, M. Muratori and B. Nazaret (linearization and large time asymptotics for the evolution problem)
M. del Pino, G. Toscani (nonlinear flows and entropy methods)
A. Blanchet, G. Grillo, J.L. Vázquez (large time asymptotics and linearization for the evolution equations)

...and also

S. Filippas, A. Tertikas, G. Tarantello, M. Kowalczyk ...

Background references (partial)

- Rigidity methods, uniqueness in nonlinear elliptic PDE's: (B. Gidas, J. Spruck, 1981), (M.-F. Bidaut-Véron, L. Véron, 1991)
- Probabilistic methods (Markov processes), semi-group theory and *carré du champ* methods (Γ_2 theory): (D. Bakry, M. Emery, 1984), (Bentaleb), (Bakry, Ledoux, 1996), (Demange, 2008), (JD, Esteban, Kolwalczyk, Loss, 2014 & 2015) → *D. Bakry, I. Gentil, and M. Ledoux. Analysis and geometry of Markov diffusion operators (2014)*
- Entropy methods in PDEs
  ▷ Entropy-entropy production inequalities: Arnold, Carrillo, Desvillettes, JD, Jüngel, Lederman, Markowich, Toscani, Unterreiter, Villani..., (del Pino, JD, 2001), (Blanchet, Bonforte, JD, Grillo, Vázquez) → *A. Jüngel, Entropy Methods for Diffusive Partial Differential Equations (2016)*
  ▷ Mass transportation: (Otto) → *C. Villani, Optimal transport. Old and new (2009)*
  ▷ Rényi entropy powers (information theory) (Savaré, Toscani, 2014), (Dolbeault, Toscani)

Symmetry and symmetry breaking results

▷ The critical Caffarelli-Kohn-Nirenberg inequality

▷ Linearization and spectrum

▷ A family of sub-critical Caffarelli-Kohn-Nirenberg inequalities

Critical Caffarelli-Kohn-Nirenberg inequality

Let $\mathcal{D}_{a,b} := \left\{ v \in \mathrm{L}^p\left(\mathbb{R}^d, |x|^{-b}\,dx\right) \,:\, |x|^{-a}\,|\nabla v| \in \mathrm{L}^2\left(\mathbb{R}^d, dx\right) \right\}$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx \quad \forall\, v \in \mathcal{D}_{a,b}$$

holds under conditions on a and b

$$p = \frac{2\,d}{d-2+2\,(b-a)} \qquad \text{(critical case)}$$

▷ *An optimal function among radial functions:*

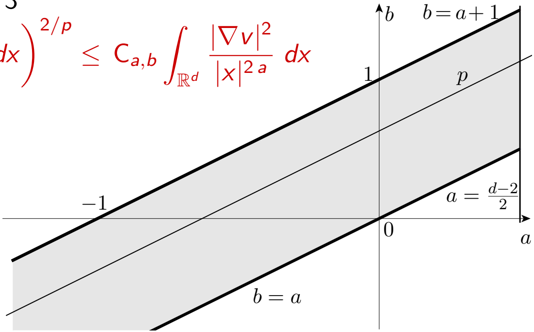
$$v_\star(x) = \left(1 + |x|^{(p-2)\,(a_c-a)}\right)^{-\frac{2}{p-2}} \quad \text{and} \quad \mathsf{C}^\star_{a,b} = \frac{\|\,|x|^{-b}\,v_\star\,\|_p^2}{\|\,|x|^{-a}\,\nabla v_\star\,\|_2^2}$$

Question: $\mathsf{C}_{a,b} = \mathsf{C}^\star_{a,b}$ *(symmetry) or* $\mathsf{C}_{a,b} > \mathsf{C}^\star_{a,b}$ *(symmetry breaking) ?*

Critical CKN: range of the parameters

Figure: $d = 3$

$$\left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le C_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx$$



$a \le b \le a+1$ if $d \ge 3$

$a < b \le a+1$ if $d = 2$, $a + 1/2 < b \le a+1$ if $d = 1$

and $a < a_c := (d-2)/2$

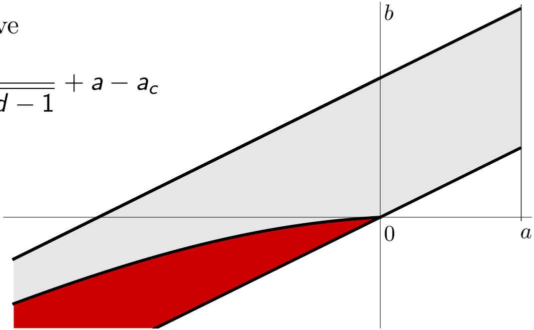
$$p = \frac{2\,d}{d - 2 + 2\,(b-a)}$$

(Glaser, Martin, Grosse, Thirring (1976))
(Caffarelli, Kohn, Nirenberg (1984))
[F. Catrina, Z.-Q. Wang (2001)]

Linear instability of radial minimizers: the Felli-Schneider curve

The Felli & Schneider curve

$$b_{\mathrm{FS}}(a) := \frac{d\,(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c$$



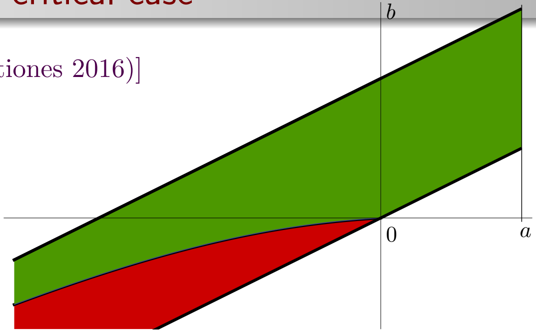
[Smets], [Smets, Willem], [Catrina, Wang], [Felli, Schneider]
The functional

$$\mathsf{C}^\star_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla v|^2}{|x|^{2\,a}}\,dx - \left(\int_{\mathbb{R}^d} \frac{|v|^p}{|x|^{b\,p}}\,dx\right)^{2/p}$$

is linearly instable at $v = v_\star$

Symmetry *versus* symmetry breaking: the sharp result in the critical case

[JD, Esteban, Loss (Inventiones 2016)]



Theorem

Let $d \geq 2$ and $p < 2^*$. If either $a \in [0, a_c)$ and $b > 0$, or $a < 0$ and $b \geq b_{\rm FS}(a)$, then the optimal functions for the critical Caffarelli-Kohn-Nirenberg inequalities are radially symmetric

The Emden-Fowler transformation and the cylinder

▷ *With an Emden-Fowler transformation, critical the Caffarelli-Kohn-Nirenberg inequality on the Euclidean space are equivalent to Gagliardo-Nirenberg inequalities on a cylinder*

$$v(r,\omega)=r^{a-a_c}\,\varphi(s,\omega)\quad\text{with}\quad r=|x|\,,\quad s=-\log r\quad\text{and}\quad \omega=\frac{x}{r}$$

With this transformation, the Caffarelli-Kohn-Nirenberg inequalities can be rewritten as the *subcritical* interpolation inequality

$$\|\partial_s\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\|\nabla_\omega\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}+\Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}\ge\mu(\Lambda)\,\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}\quad\forall\,\varphi\in\mathrm{H}^1(\mathcal{C})$$

where $\Lambda:=(a_c-a)^2$, $\mathcal{C}=\mathbb{R}\times\mathbb{S}^{d-1}$ and the optimal constant $\mu(\Lambda)$ is

$$\mu(\Lambda)=\frac{1}{\mathsf{C}_{a,b}}\quad\text{with}\quad a=a_c\pm\sqrt{\Lambda}\quad\text{and}\quad b=\frac{d}{p}\pm\sqrt{\Lambda}$$

Linearization around symmetric critical points

Up to a normalization and a scaling

$$\varphi_\star(s,\omega) = (\cosh s)^{-\frac{1}{p-2}}$$

is a critical point of

$$\mathrm{H}^1(\mathcal{C}) \ni \varphi \mapsto \|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}$$

under a constraint on $\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}$

$\varphi_\star$ *is not* optimal for (CKN) if the Pöschl-Teller operator

$$-\partial_s^2 - \Delta_\omega + \Lambda - \varphi_\star^{p-2} = -\partial_s^2 - \Delta_\omega + \Lambda - \frac{1}{(\cosh s)^2}$$

has *a negative eigenvalue*, i.e., for $\Lambda > \Lambda_1$ (explicit)

The variational problem on the cylinder

$$\Lambda \mapsto \mu(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathcal{C})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \|\nabla_\omega \varphi\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathcal{C})}}{\|\varphi\|^2_{\mathrm{L}^p(\mathcal{C})}}$$

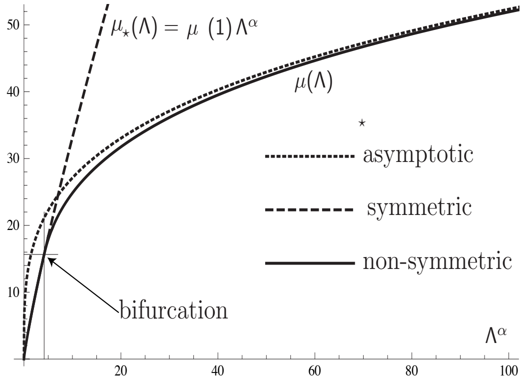
is a concave increasing function

Restricted to symmetric functions, the variational problem becomes

$$\mu_\star(\Lambda) := \min_{\varphi \in \mathrm{H}^1(\mathbb{R})} \frac{\|\partial_s \varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)} + \Lambda\,\|\varphi\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}{\|\varphi\|^2_{\mathrm{L}^p(\mathbb{R}^d)}} = \mu_\star(1)\,\Lambda^\alpha$$

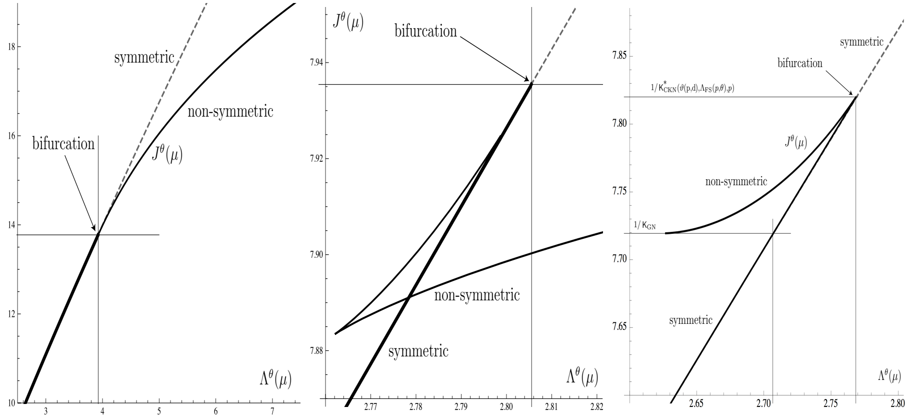
Symmetry means $\mu(\Lambda) = \mu_\star(\Lambda)$
Symmetry breaking means $\mu(\Lambda) < \mu_\star(\Lambda)$

Numerical results



Parametric plot of the branch of optimal functions for $p = 2.8$, $d = 5$. *Non-symmetric solutions bifurcate from symmetric ones at a bifurcation point* Λ_1 *computed by V. Felli and M. Schneider. The branch behaves for large values of* Λ *as predicted by F. Catrina and Z.-Q. Wang*

what we wan to discard...



When the local criterion (linear stability) differs from global results in a larger family of inequalities (center, right)...

Subcritical Caffarelli-Kohn-Nirenberg inequalities

Norms: $\|w\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |w|^q\,|x|^{-\gamma}\,dx\right)^{1/q}$, $\|w\|_{\mathrm{L}^q(\mathbb{R}^d)} := \|w\|_{\mathrm{L}^{q,0}(\mathbb{R}^d)}$
(some) *Caffarelli-Kohn-Nirenberg interpolation inequalities* (1984)

$$\|w\|_{\mathrm{L}^{2p,\gamma}(\mathbb{R}^d)} \le \mathsf{C}_{\beta,\gamma,p}\,\|\nabla w\|^{\vartheta}_{\mathrm{L}^{2,\beta}(\mathbb{R}^d)}\,\|w\|^{1-\vartheta}_{\mathrm{L}^{p+1,\gamma}(\mathbb{R}^d)} \qquad \text{(CKN)}$$

Here $\mathsf{C}_{\beta,\gamma,p}$ denotes the optimal constant, the parameters satisfy

$$d \ge 2\,, \quad \gamma-2 < \beta < \tfrac{d-2}{d}\,\gamma\,, \quad \gamma \in (-\infty, d)\,, \quad p \in (1, p_\star] \quad \text{with } p_\star := \tfrac{d-\gamma}{d-\beta-2}$$

and the exponent ϑ is determined by the scaling invariance, *i.e.*,

$$\vartheta = \frac{(d-\gamma)\,(p-1)}{p\,\big(d+\beta+2-2\,\gamma-p\,(d-\beta-2)\big)}$$

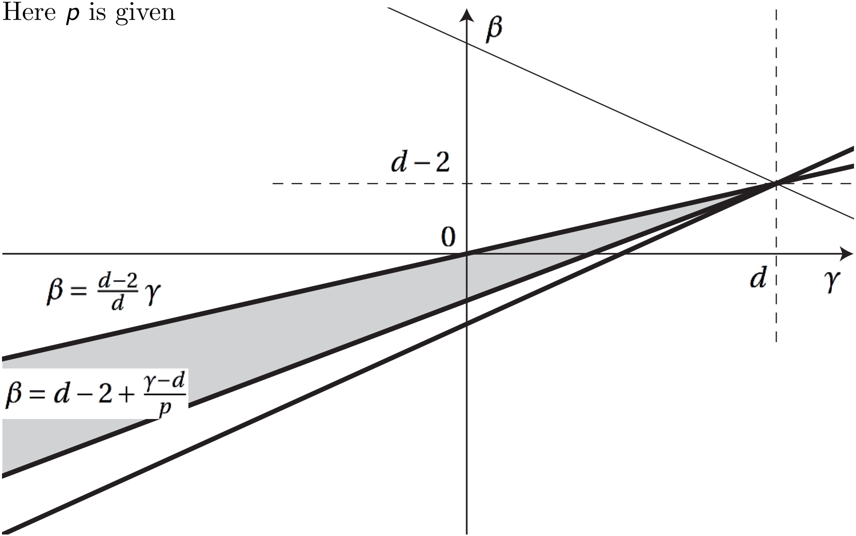
- Is the equality case achieved by the Barenblatt / Aubin-Talenti type function

$$w_\star(x) = \left(1 + |x|^{2+\beta-\gamma}\right)^{-1/(p-1)} \quad \forall\, x \in \mathbb{R}^d \quad ?$$

- Do we know (*symmetry*) that the equality case is achieved among radial functions ?

Range of the parameters

Here p is given



Symmetry and symmetry breaking

(M. Bonforte, J.D., M. Muratori and B. Nazaret, 2016) Let us define $\beta_{\rm FS}(\gamma) := d-2-\sqrt{(d-\gamma)^2-4\,(d-1)}$

Theorem

Symmetry breaking holds in (CKN) if

$$\gamma<0 \quad \text{and} \quad \beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$$

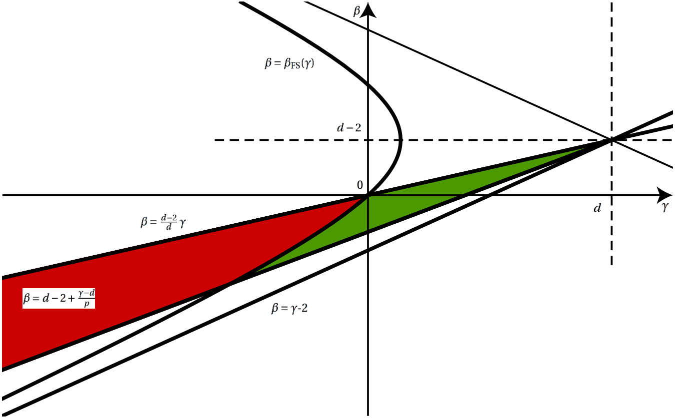
In the range $\beta_{\rm FS}(\gamma)<\beta<\frac{d-2}{d}\,\gamma$, $w_\star(x)=\left(1+|x|^{2+\beta-\gamma}\right)^{-1/(p-1)}$ *is not* optimal

(JD, Esteban, Loss, Muratori, 2016)

Theorem

Symmetry holds in (CKN) if

$$\gamma\ge0\,,\quad \text{or} \quad \gamma\le0 \quad \text{and} \quad \gamma-2\le\beta\le\beta_{\rm FS}(\gamma)$$



The green area is the region of symmetry, while the red area is the region of symmetry breaking. The threshold is determined by the hyperbola

$$(d-\gamma)^2-(\beta-d+2)^2-4\,(d-1)=0$$

Inequalities without weights and fast diffusion equations

- ▷ The Bakry-Emery method on the sphere
- ▷ Euclidean space: self-similar variables and relative entropies
- ▷ The role of the spectral gap

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2}\left[\int_{\mathbb{S}^d} \rho^{\frac{2}{p}}\,d\mu - \left(\int_{\mathbb{S}^d} \rho\,d\mu\right)^{\frac{2}{p}}\right] \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho\,\log\left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}}\right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2\,d\mu$$

Bakry-Emery (carré du champ) method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and compute $\frac{d}{dt}\mathcal{E}_p[\rho] = -\,\mathcal{I}_p[\rho]$ and $\frac{d}{dt}\mathcal{I}_p[\rho] \leq -\,d\,\mathcal{I}_p[\rho]$ to get

$$\frac{d}{dt}\left(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\right) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

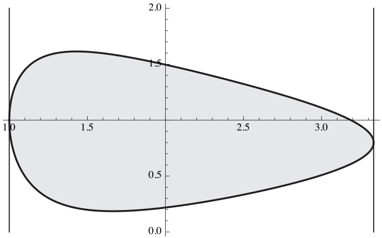
The evolution under the fast diffusion flow

To overcome the limitation $p \le 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m . \tag{1}$$

(Demange), (JD, Esteban, Kowalczyk, Loss): for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \le 0$$



(p, m) *admissible region*, $d = 5$

Euclidean space: self-similar variables and relative entropies

The large time behavior of the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ is governed by the source-type *Barenblatt solutions*

$$v_\star(t,x) := \frac{1}{\kappa^d (\mu\,t)^{d/\mu}}\,\mathcal{B}_\star\left(\frac{x}{\kappa\,(\mu\,t)^{1/\mu}}\right) \quad \text{where} \quad \mu := 2 + d\,(m-1)$$

where $\mathcal{B}_\star$ is the Barenblatt profile (with appropriate mass)

$$\mathcal{B}_\star(x) := \left(1 + |x|^2\right)^{1/(m-1)}$$

A time-dependent rescaling: self-similar variables

$$v(t,x) = \frac{1}{\kappa^d\,R^d}\,u\left(\tau, \frac{x}{\kappa\,R}\right) \quad \text{where} \quad \frac{dR}{dt} = R^{1-\mu}\,, \quad \tau(t) := \tfrac{1}{2}\,\log\left(\frac{R(t)}{R_0}\right)$$

Then the function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[u\left(\nabla u^{m-1} - 2\,x\right)\right] = 0$$

Free energy and Fisher information

- The function u solves a Fokker-Planck type equation

$$\frac{\partial u}{\partial \tau} + \nabla \cdot \left[u \left(\nabla u^{m-1} - 2\,x \right) \right] = 0$$

- (Ralston, Newman, 1984) Lyapunov functional: *Generalized entropy* or *Free energy*

$$\mathcal{E}[u] := \int_{\mathbb{R}^d} \left(-\frac{u^m}{m} + |x|^2 u \right) dx - \mathcal{E}_0$$

- Entropy production is measured by the *Generalized Fisher information*

$$\frac{d}{dt}\mathcal{E}[u] = -\mathcal{I}[u]\,, \quad \mathcal{I}[u] := \int_{\mathbb{R}^d} u \left| \nabla u^{m-1} + 2\,x \right|^2 dx$$

Without weights: relative entropy, entropy production

- *Stationary solution:* choose C such that $\|u_\infty\|_{\mathrm{L}^1} = \|u\|_{\mathrm{L}^1} = M > 0$

$$u_\infty(x) := \left(C + |x|^2\right)_+^{-1/(1-m)}$$

Relative entropy: Fix $\mathcal{E}_0$ so that $\mathcal{E}[u_\infty] = 0$

- *Entropy – entropy production inequality* (del Pino, J.D.)

Theorem

$d \geq 3$, $m \in [\frac{d-1}{d}, +\infty)$, $m > \frac{1}{2}$, $m \neq 1$

$$\mathcal{I}[u] \geq 4\,\mathcal{E}[u]$$

Corollary

(del Pino, J.D.) *A solution u with initial data $u_0 \in L^1_+(\mathbb{R}^d)$ such that $|x|^2\,u_0 \in L^1(\mathbb{R}^d)$, $u_0^m \in L^1(\mathbb{R}^d)$ satisfies*

$$\mathcal{E}[u(t,\cdot)] \leq \mathcal{E}[u_0]\,e^{-\,4\,t}$$

A computation on a large ball, with boundary terms

$$\frac{\partial u}{\partial \tau}+\nabla\cdot\Big[u\left(\nabla u^{m-1}-2\,x\right)\Big]=0\quad \tau>0\,,\quad x\in B_R$$

where B_R is a centered ball in $\mathbb{R}^d$ with radius $R>0$, and assume that u satisfies zero-flux boundary conditions

$$\left(\nabla u^{m-1}-2\,x\right)\cdot\frac{x}{|x|}=0\quad \tau>0\,,\quad x\in\partial B_R\,.$$

With $z(\tau,x):=\nabla Q(\tau,x):=\nabla u^{m-1}-2\,x$, the *relative Fisher information* is such that

$$\frac{d}{d\tau}\int_{B_R}u\,|z|^2\,dx+4\int_{B_R}u\,|z|^2\,dx$$
$$+\,2\,\tfrac{1-m}{m}\int_{B_R}u^m\left(\left\|\mathrm{D}^2Q\right\|^2-\,(1-m)\,(\Delta Q)^2\right)dx$$
$$=\int_{\partial B_R}u^m\left(\omega\cdot\nabla|z|^2\right)d\sigma\le 0\ \text{(by Grisvard's lemma)}$$

Entropy – entropy production, Gagliardo-Nirenberg ineq.

$$4\,\mathcal{E}[u] \le \mathcal{I}[u]$$

Rewrite it with $p = \frac{1}{2m-1}$, $u = w^{2p}$, $u^m = w^{p+1}$ as

$$\frac{1}{2}\left(\frac{2m}{2m-1}\right)^2 \int_{\mathbb{R}^d} |\nabla w|^2 dx + \left(\frac{1}{1-m} - d\right) \int_{\mathbb{R}^d} |w|^{1+p} dx - K \ge 0$$

- for some γ, $K = K_0 \left(\int_{\mathbb{R}^d} u\,dx = \int_{\mathbb{R}^d} w^{2p}\,dx\right)^\gamma$
- $w = w_\infty = v_\infty^{1/2p}$ is optimal

Theorem

[Del Pino, J.D.] *With* $1 < p \le \frac{d}{d-2}$ *(fast diffusion case) and* $d \ge 3$

$$\|w\|_{L^{2p}(\mathbb{R}^d)} \le \mathcal{C}^{\mathrm{GN}}_{p,d}\, \|\nabla w\|^\theta_{L^2(\mathbb{R}^d)}\, \|w\|^{1-\theta}_{L^{p+1}(\mathbb{R}^d)}$$

$$\mathcal{C}^{\mathrm{GN}}_{p,d} = \left(\frac{y(p-1)^2}{2\pi d}\right)^{\frac{\theta}{2}} \left(\frac{2y-d}{2y}\right)^{\frac{1}{2p}} \left(\frac{\Gamma(y)}{\Gamma(y-\frac{d}{2})}\right)^{\frac{\theta}{d}},\ \theta = \frac{d(p-1)}{p(d+2-(d-2)p)},\ y = \frac{p+1}{p-1}$$

Spectral gap: sharp asymptotic rates of convergence

Assumptions on the initial datum v_0

(H1) $V_{D_0} \le v_0 \le V_{D_1}$ *for some* $D_0 > D_1 > 0$

(H2) *if* $d \ge 3$ *and* $m \le m_*$, $(v_0 - V_D)$ *is integrable for a suitable* $D \in [D_1, D_0]$

Theorem

(Blanchet, Bonforte, J.D., Grillo, Vázquez) *Under Assumptions (H1)-(H2), if* $m < 1$ *and* $m \neq m_* := \frac{d-4}{d-2}$*, the entropy decays according to*

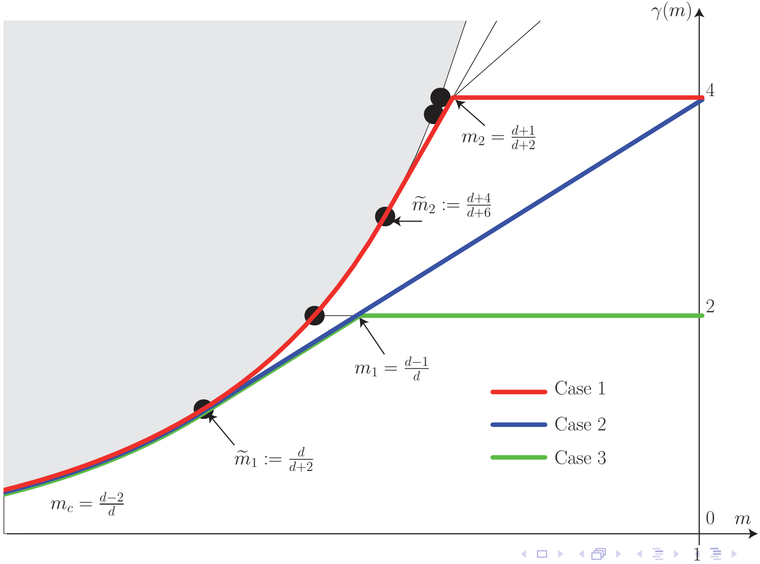
$$\mathcal{E}[v(t,\cdot)] \le C\, e^{-2\,(1-m)\,\Lambda_{\alpha,d}\,t} \quad \forall\, t \ge 0$$

where $\Lambda_{\alpha,d} > 0$ *is the best constant in the Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d} \int_{\mathbb{R}^d} |f|^2\, d\mu_{\alpha-1} \le \int_{\mathbb{R}^d} |\nabla f|^2\, d\mu_\alpha \quad \forall\, f \in H^1(d\mu_\alpha)$$

with $\alpha := 1/(m-1) < 0$, $d\mu_\alpha := h_\alpha\, dx$, $h_\alpha(x) := (1+|x|^2)^\alpha$

Spectral gap and best constants



Weighted nonlinear flows: Caffarelli-Kohn-Nirenberg inequalities

- ▷ Entropy and Caffarelli-Kohn-Nirenberg inequalities
- ▷ Large time asymptotics and spectral gaps
- ▷ Optimality cases

CKN and entropy – entropy production inequalities

When symmetry holds, (CKN) can be written as an *entropy – entropy production* inequality

$$\frac{1-m}{m}\,(2+\beta-\gamma)^2\,\mathcal{E}[v]\leq\mathcal{I}[v]$$

and equality is achieved by $\mathfrak{B}_{\beta,\gamma}(x):=\left(1+|x|^{2+\beta-\gamma}\right)^{\frac{1}{m-1}}$
Here the *free energy* and the *relative Fisher information* are defined by

$$\mathcal{E}[v]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left(v^m-\mathfrak{B}_{\beta,\gamma}^m-m\,\mathfrak{B}_{\beta,\gamma}^{m-1}\,(v-\mathfrak{B}_{\beta,\gamma})\right)\frac{dx}{|x|^\gamma}$$

$$\mathcal{I}[v]:=\int_{\mathbb{R}^d}v\left|\nabla v^{m-1}-\nabla\mathfrak{B}_{\beta,\gamma}^{m-1}\right|^2\frac{dx}{|x|^\beta}$$

If v solves the *Fokker-Planck type equation*

$$v_t+|x|^\gamma\,\nabla\cdot\left[|x|^{-\beta}\,v\,\nabla\left(v^{m-1}-|x|^{2+\beta-\gamma}\right)\right]=0\qquad\text{(WFDE-FP)}$$

then $\dfrac{d}{dt}\mathcal{E}[v(t,\cdot)]=-\dfrac{m}{1-m}\,\mathcal{I}[v(t,\cdot)]$

Proposition

Let $m=\frac{p+1}{2\,p}$ and consider a solution to (WFDE-FP) with nonnegative initial datum $u_0\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)$ such that $\|u_0^m\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d}u_0\,|x|^{2+\beta-2\gamma}\,dx$ are finite. Then

$$\mathcal{E}[v(t,\cdot)]\le\mathcal{E}[u_0]\,e^{-(2+\beta-\gamma)^2\,t}\quad\forall\,t\ge0$$

if one of the following two conditions is satisfied:
(i) either u_0 is a.e. radially symmetric
(ii) or symmetry holds in (CKN)

Proof of symmetry (1/3: changing the dimension)

We rephrase our problem in a space of higher, *artificial dimension* $n > d$ (here n is a dimension at least from the point of view of the scaling properties), or to be precise we consider a weight $|x|^{n-d}$ which is the same in all norms. With

$$v(|x|^{\alpha-1}\,x) = w(x)\,, \quad \alpha = 1 + \frac{\beta-\gamma}{2} \quad \text{and} \quad n = 2\,\frac{d-\gamma}{\beta+2-\gamma}\,,$$

we claim that Inequality (CKN) can be rewritten for a function $v(|x|^{\alpha-1}\,x) = w(x)$ as

$$\|v\|_{\mathrm{L}^{2p,d-n}(\mathbb{R}^d)} \le \mathsf{K}_{\alpha,n,p}\,\|\mathsf{D}_\alpha v\|_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}^{\vartheta}\,\|v\|_{\mathrm{L}^{p+1,d-n}(\mathbb{R}^d)}^{1-\vartheta} \quad \forall\,v \in \mathrm{H}^p_{d-n,d-n}(\mathbb{R}^d)$$

with the notations $s = |x|$, $\mathsf{D}_\alpha v = \left(\alpha\,\frac{\partial v}{\partial s}, \frac{1}{s}\,\nabla_\omega v\right)$ and

$$d \ge 2\,, \quad \alpha > 0\,, \quad n > d \quad \text{and} \quad p \in (1, p_\star]\,.$$

By our change of variables, $w_\star$ is changed into

$$v_\star(x) := \left(1+|x|^2\right)^{-1/(p-1)} \quad \forall\,x \in \mathbb{R}^d$$

The strategy of the proof (2/3: Rényi entropy)

The derivative of the generalized *Rényi entropy power* functional is

$$\mathcal{G}[u] := \left(\int_{\mathbb{R}^d} u^m \, d\mu\right)^{\sigma-1} \int_{\mathbb{R}^d} u \, |\mathsf{D}_\alpha \mathsf{P}|^2 \, d\mu$$

where $\sigma = \frac{2}{d} \frac{1}{1-m} - 1$. Here $d\mu = |x|^{n-d} \, dx$ and the pressure is

$$\mathsf{P} := \frac{m}{1-m} \, u^{m-1}$$

Looking for an optimal function in (CKN) is equivalent to minimize $\mathcal{G}$ under a mass constraint

With $\mathsf{L}_\alpha = -\,\mathsf{D}_\alpha^*\,\mathsf{D}_\alpha = \alpha^2\left(u'' + \frac{n-1}{s}\,u'\right) + \frac{1}{s^2}\,\Delta_\omega\,u$, we consider the fast diffusion equation

$$\frac{\partial u}{\partial t} = \mathsf{L}_\alpha\,u^m$$

in the subcritical range $1 - 1/n < m < 1$. The key computation is the proof that

$$\begin{aligned}
&-\tfrac{d}{dt}\,\mathcal{G}[u(t,\cdot)]\,\left(\textstyle\int_{\mathbb{R}^d} u^m\,d\mu\right)^{1-\sigma}\\
&\ge (1-m)\,(\sigma-1)\int_{\mathbb{R}^d} u^m\left|\mathsf{L}_\alpha\mathsf{P} - \frac{\int_{\mathbb{R}^d} u\,|\mathsf{D}_\alpha\mathsf{P}|^2\,d\mu}{\int_{\mathbb{R}^d} u^m\,d\mu}\right|^2 d\mu\\
&+2\int_{\mathbb{R}^d}\left(\alpha^4\left(1-\tfrac{1}{n}\right)\left|\mathsf{P}'' - \frac{\mathsf{P}'}{s} - \frac{\Delta_\omega\,\mathsf{P}}{\alpha^2\,(n-1)\,s^2}\right|^2 + \frac{2\,\alpha^2}{s^2}\left|\nabla_\omega\mathsf{P}' - \frac{\nabla_\omega\mathsf{P}}{s}\right|^2\right) u^m\,d\mu\\
&+2\int_{\mathbb{R}^d}\left((n-2)\left(\alpha_{\mathrm{FS}}^2 - \alpha^2\right)|\nabla_\omega\mathsf{P}|^2 + c(n,m,d)\,\frac{|\nabla_\omega\mathsf{P}|^4}{\mathsf{P}^2}\right) u^m\,d\mu =: \mathcal{H}[u]
\end{aligned}$$

for some numerical constant $c(n,m,d) > 0$. Hence if $\alpha \le \alpha_{\mathrm{FS}}$, the r.h.s. $\mathcal{H}[u]$ vanishes if and only if P is an affine function of $|x|^2$, which proves the symmetry result. *A quantifier elimination problem (Tarski, 1951) ?*

(3/3: elliptic regularity, boundary terms)

This method has a hidden difficulty: integrations by parts ! Hints:

- use elliptic regularity: Moser iteration scheme, Sobolev regularity, local Hölder regularity, Harnack inequality, and get global regularity using scalings

- use the Emden-Fowler transformation, work on a cylinder, truncate, evaluate boundary terms of high order derivatives using Poincaré inequalities on the sphere

Summary: if u solves the Euler-Lagrange equation, we test by $\mathsf{L}_\alpha u^m$

$$0=\int_{\mathbb{R}^d} \mathrm{d}\mathcal{G}[u]\cdot\mathsf{L}_\alpha u^m\,d\mu\geq\mathcal{H}[u]\geq 0$$

$\mathcal{H}[u]$ is the integral of a sum of squares (with nonnegative constants in front of each term)... or test by $|x|^\gamma\,\mathrm{div}\left(|x|^{-\beta}\,\nabla w^{1+p}\right)$ the equation

$$\frac{(p-1)^2}{p\,(p+1)}\,w^{1-3p}\,\mathrm{div}\left(|x|^{-\beta}\,w^{2p}\,\nabla w^{1-p}\right)+|\nabla w^{1-p}|^2+|x|^{-\gamma}\left(c_1\,w^{1-p}-c_2\right)=0$$

Fast diffusion equations with weights: large time asymptotics

- Relative uniform convergence
- Asymptotic rates of convergence
- From asymptotic to global estimates

Here v solves the *Fokker-Planck type equation*

$$v_t + |x|^\gamma \nabla \cdot \Big[|x|^{-\beta} \, v \, \nabla \big(v^{m-1} - |x|^{2+\beta-\gamma} \big) \Big] = 0 \qquad \text{(WFDE-FP)}$$

Joint work with M. Bonforte, M. Muratori and B. Nazaret

Relative uniform convergence

$$\zeta := 1 - \left(1 - \frac{2-m}{(1-m)\,q}\right)\left(1 - \frac{2-m}{1-m}\,\theta\right)$$

$$\theta := \frac{(1-m)\,(2+\beta-\gamma)}{(1-m)\,(2+\beta)+2+\beta-\gamma} \text{ is in the range } 0 < \theta < \frac{1-m}{2-m} < 1$$

Theorem

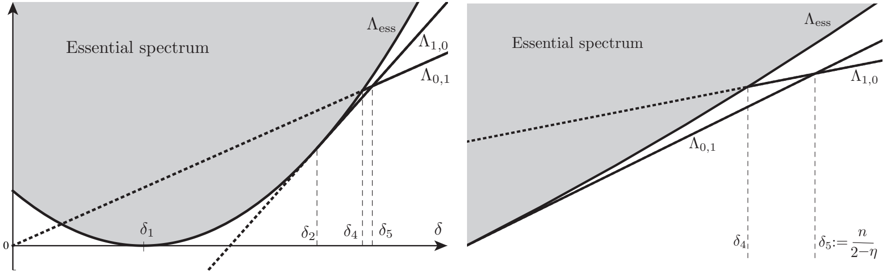
For "good" initial data, there exist positive constants $\mathcal{K}$ and t_0 such that, for all $q \in \left[\frac{2-m}{1-m}, \infty\right]$, the function $w = v/\mathfrak{B}$ satisfies

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,\zeta\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \in (0,d)$, and

$$\|w(t) - 1\|_{\mathrm{L}^{q,\gamma}(\mathbb{R}^d)} \le \mathcal{K}\, e^{-2\,\frac{(1-m)^2}{2-m}\,\Lambda\,(t-t_0)} \quad \forall\, t \ge t_0$$

in the case $\gamma \le 0$



The spectrum of $\mathcal{L}$ as a function of $\delta = \frac{1}{1-m}$, with $n = 5$. The essential spectrum corresponds to the grey area, and its bottom is determined by the parabola $\delta \mapsto \Lambda_{\rm ess}(\delta)$. The two eigenvalues $\Lambda_{0,1}$ and $\Lambda_{1,0}$ are given by the plain, half-lines, away from the essential spectrum. The spectral gap determines the asymptotic rate of convergence to the Barenblatt functions

Global vs. asymptotic estimates

- *Estimates on the global rates.* When symmetry holds (CKN) can be written as an *entropy – entropy production* inequality

$$(2+\beta-\gamma)^2\,\mathcal{E}[v]\le\frac{m}{1-m}\,\mathcal{I}[v]$$

so that

$$\mathcal{E}[v(t)]\le\mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda_\star\,t}\quad\forall\,t\ge0\quad\text{with}\quad\Lambda_\star:=\frac{(2+\beta-\gamma)^2}{2\,(1-m)}$$

- *Optimal global rates.* Let us consider again the *entropy – entropy production* inequality

$$\mathcal{K}(M)\,\mathcal{E}[v]\le\mathcal{I}[v]\quad\forall\,v\in\mathrm{L}^{1,\gamma}(\mathbb{R}^d)\quad\text{such that}\quad\|v\|_{\mathrm{L}^{1,\gamma}(\mathbb{R}^d)}=M\,,$$

where $\mathcal{K}(M)$ is the best constant: with $\Lambda(M):=\frac{m}{2}\,(1-m)^{-2}\,\mathcal{K}(M)$

$$\mathcal{E}[v(t)]\le\mathcal{E}[v(0)]\,e^{-\,2\,(1-m)\,\Lambda(M)\,t}\quad\forall\,t\ge0$$

Linearization and optimality

Joint work with M.J. Esteban and M. Loss

Linearization and scalar products

With u_ε such that

$$u_\varepsilon = \mathcal{B}_\star \left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right) \quad \text{and} \quad \int_{\mathbb{R}^d} u_\varepsilon\, dx = M_\star$$

at first order in $\varepsilon \to 0$ we obtain that f solves

$$\frac{\partial f}{\partial t} = \mathcal{L}\, f \quad \text{where} \quad \mathcal{L}\, f := (1-m)\, \mathcal{B}_\star^{m-2}\, |x|^\gamma\, \mathsf{D}_\alpha^* \left(|x|^{-\beta}\, \mathcal{B}_\star\, \mathsf{D}_\alpha f\right)$$

Using the scalar products

$$\langle f_1, f_2 \rangle = \int_{\mathbb{R}^d} f_1\, f_2\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx \quad \text{and} \quad \langle\!\langle f_1, f_2 \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f_1 \cdot \mathsf{D}_\alpha f_2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx$$

we compute

$$\frac{1}{2}\, \frac{d}{dt} \langle f, f \rangle = \langle f, \mathcal{L}\, f \rangle = \int_{\mathbb{R}^d} f\, (\mathcal{L}\, f)\, \mathcal{B}_\star^{2-m}\, |x|^{-\gamma}\, dx = -\int_{\mathbb{R}^d} |\mathsf{D}_\alpha f|^2\, \mathcal{B}_\star\, |x|^{-\beta}\, dx =$$

for any f smooth enough:

$$\frac{1}{2}\, \frac{d}{dt} \langle\!\langle f, f \rangle\!\rangle = \int_{\mathbb{R}^d} \mathsf{D}_\alpha f \cdot \mathsf{D}_\alpha (\mathcal{L}\, f)\, \mathcal{B}_\star\, |x|^{-\beta}\, dx = -\langle\!\langle f, \mathcal{L}\, f \rangle\!\rangle$$

Linearization of the flow, eigenvalues and spectral gap

Now let us consider an eigenfunction associated with the smallest positive eigenvalue λ_1 of $\mathcal{L}$

$$-\mathcal{L}\,f_1=\lambda_1\,f_1$$

so that f_1 realizes the equality case in the *Hardy-Poincaré inequality*

$$\langle\!\langle g,g\rangle\!\rangle=-\,\langle g,\mathcal{L}\,g\rangle\geq\lambda_1\,\|g-\bar{g}\|^2\,,\quad\bar{g}:=\langle g,1\rangle\,/\,\langle 1,1\rangle$$

$$-\,\langle\!\langle g,\mathcal{L}\,g\rangle\!\rangle\geq\lambda_1\,\langle\!\langle g,g\rangle\!\rangle$$

Proof: expansion of the square :
$-\,\langle\!\langle(g-\bar{g}),\mathcal{L}\,(g-\bar{g})\rangle\!\rangle=\langle\mathcal{L}\,(g-\bar{g}),\mathcal{L}\,(g-\bar{g})\rangle=\|\mathcal{L}\,(g-\bar{g})\|^2$

- Key observation:

$$\lambda_1\geq 4\quad\Longleftrightarrow\quad\alpha\leq\alpha_{\rm FS}:=\sqrt{\frac{d-1}{n-1}}$$

Why is this method optimal ?

- The condition $\lambda_1 < 4$ is sufficient for symmetry breaking
- With $\lambda_1 \geq 4$, we prove that

$$\mathcal{H}[v] := \frac{m}{1-m}\,\mathcal{I}[v] - (2+\beta-\gamma)^2\,\mathcal{E}[v]$$

is monotone decaying along the flow and that equality is achieved only on the stationary solution, which attracts all solutions. this has to do with the (formal) gradient flow structure of the problem

- The condition $\lambda_1 \geq 4$ is enough to prove that
  ▷ the Fisher information $\mathcal{I}[v]$ exponentially decays with rate e^{-4t}
  ▷ the functional $\mathcal{H}[v]$ is decreasing
- The decay of the Fisher information $\mathcal{I}[v]$ "has to be given" by the condition $\lambda_1 \geq 4$ because the problem degenerates into a sharp spectral gap problem in the asymptotic regime

Symmetry breaking in CKN inequalities

- Symmetry holds in (CKN) if $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$ with

$$\mathcal{J}[w] := \vartheta \log\left(\|\mathsf{D}_\alpha w\|_{\mathrm{L}^{2,\delta}(\mathbb{R}^d)}\right) + (1-\vartheta) \log\left(\|w\|_{\mathrm{L}^{p+1,\delta}(\mathbb{R}^d)}\right) - \log\left(\|w\|_{\mathrm{L}^{2p,\delta}(\mathbb{R}^d)}\right)$$

with $\delta := d - n$ and

$$\mathcal{J}[w_\star + \varepsilon\, g] = \varepsilon^2\, \mathcal{Q}[g] + o(\varepsilon^2)$$

where

$$\frac{2}{\vartheta}\, \|\mathsf{D}_\alpha w_\star\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)}\, \mathcal{Q}[g]$$

$$= \|\mathsf{D}_\alpha g\|^2_{\mathrm{L}^{2,d-n}(\mathbb{R}^d)} + \frac{p\,(2+\beta-\gamma)}{(p-1)^2} \left[d - \gamma - p\,(d-2-\beta)\right] \int_{\mathbb{R}^d} |g|^2\, \frac{|x|^{n-d}}{1+|x|^2}\, dx$$

$$- p\,(2\,p-1)\, \frac{(2+\beta-\gamma)^2}{(p-1)^2} \int_{\mathbb{R}^d} |g|^2\, \frac{|x|^{n-d}}{(1+|x|^2)^2}\, dx$$

is a nonnegative quadratic form if and only if $\alpha \le \alpha_{\mathrm{FS}}$

- Symmetry breaking holds if $\alpha > \alpha_{\mathrm{FS}}$

Information – production of information inequality

Let $\mathcal{K}[u]$ be such that

$$\frac{d}{d\tau}\mathcal{I}[u(\tau,\cdot)] = -\,\mathcal{K}[u(\tau,\cdot)] = -\,(\text{sum of squares})$$

If $\alpha \le \alpha_{\rm FS}$, then $\lambda_1 \ge 4$ and

$$u \mapsto \frac{\mathcal{K}[u]}{\mathcal{I}[u]} - 4$$

is a nonnegative functional
With $u_\varepsilon = \mathcal{B}_\star\,\big(1 + \varepsilon\,f\,\mathcal{B}_\star^{1-m}\big)$, we observe that

$$4 \le \mathcal{C}_2 := \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]} \le \liminf_{\varepsilon\to0}\,\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \inf_f \frac{\langle\!\langle f, \mathcal{L}\,f\rangle\!\rangle}{\langle\!\langle f, f\rangle\!\rangle} = \frac{\langle\!\langle f_1, \mathcal{L}\,f_1\rangle\!\rangle}{\langle\!\langle f_1, f_1\rangle\!\rangle} = \lambda_1$$

- if $\lambda_1 = 4$, that is, if $\alpha = \alpha_{\rm FS}$, then $\inf \mathcal{K}/\mathcal{I} = 4$ is achieved in the asymptotic regime as $u \to \mathcal{B}_\star$ and determined by the spectral gap of $\mathcal{L}$
- if $\lambda_1 > 4$, that is, if $\alpha < \alpha_{\rm FS}$, then $\mathcal{K}/\mathcal{I} > 4$

Symmetry in Caffarelli-Kohn-Nirenberg inequalities

If $\alpha \le \alpha_{\rm FS}$, the fact that $\mathcal{K}/\mathcal{I} \ge 4$ has an important consequence. Indeed we know that

$$\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{E}[u(\tau,\cdot)]\right) \le 0$$

so that

$$\mathcal{I}[u] - 4\,\mathcal{E}[u] \ge \mathcal{I}[\mathcal{B}_\star] - 4\,\mathcal{E}[\mathcal{B}_\star] = 0$$

This inequality is equivalent to $\mathcal{J}[w] \ge \mathcal{J}[w_\star]$, which establishes that optimality in (CKN) is achieved among symmetric functions. In other words, the linearized problem shows that for $\alpha \le \alpha_{\rm FS}$, the function

$$\tau \mapsto \mathcal{I}[u(\tau,\cdot)] - 4\,\mathcal{E}[u(\tau,\cdot)]$$

is monotone decreasing

- This explains why the method based on nonlinear flows provides the *optimal range for symmetry*

Entropy – production of entropy inequality

Using $\frac{d}{d\tau}\left(\mathcal{I}[u(\tau,\cdot)] - \mathcal{C}_2\,\mathcal{E}[u(\tau,\cdot)]\right) \le 0$, we know that

$$\mathcal{I}[u] - \mathcal{C}_2\,\mathcal{E}[u] \ge \mathcal{I}[\mathcal{B}_\star] - \mathcal{C}_2\,\mathcal{E}[\mathcal{B}_\star] = 0$$

As a consequence, we have that

$$\mathcal{C}_1 := \inf_u \frac{\mathcal{I}[u]}{\mathcal{E}[u]} \ge \mathcal{C}_2 = \inf_u \frac{\mathcal{K}[u]}{\mathcal{I}[u]}$$

With $u_\varepsilon = \mathcal{B}_\star\,\left(1 + \varepsilon\, f\, \mathcal{B}_\star^{1-m}\right)$, we observe that

$$\mathcal{C}_1 \le \lim_{\varepsilon\to 0}\inf_f \frac{\mathcal{I}[u_\varepsilon]}{\mathcal{E}[u_\varepsilon]} = \inf_f \frac{\langle f, \mathcal{L}\, f\rangle}{\langle f, f\rangle} = \frac{\langle f_1, \mathcal{L}\, f_1\rangle}{\langle f_1, f\rangle_1} = \lambda_1 = \lim_{\varepsilon\to 0}\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]}$$

- If $\lim_{\varepsilon\to 0}\inf_f \frac{\mathcal{K}[u_\varepsilon]}{\mathcal{I}[u_\varepsilon]} = \mathcal{C}_2$, then $\mathcal{C}_1 = \mathcal{C}_2 = \lambda_1$

This happens if $\alpha = \alpha_{\rm FS}$ and in particular in the case without weights (Gagliardo-Nirenberg inequalities)

These slides can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Conferences/
▷ Lectures

The papers can be found at

http://www.ceremade.dauphine.fr/~dolbeaul/Preprints/list/
▷ Preprints / papers

For final versions, use `Dolbeault` as login and `Jean` as password

Thank you for your attention !