Sobolev and Hardy-Littlewood-Sobolev inequalities: duality and fast diffusion

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Asymptotic dynamics driven by solitons and traveling fronts in nonlinear PDE Santiago, Chile (July 11-15, 2011)

1. Gagliardo-Nirenberg inequalities: old and new
2. Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows, and an improvement of Sobolev based on HLS
3. Gagliardo-Nirenberg inequalities: further improvements, in the spirit of [G. Bianchi and H. Egnell]

Improvements of Sobolev's inequality have been questioned by [H. Brezis and E. H. Lieb]

- [G. Bianchi and H. Egnell]: Compactness methods
- [A. Cianchi, N. Fusco, F. Maggi, and A. Pratelli]: Rearrangements techniques

Two new answers:

- [J.D]: A non-local estimate based on HLS
- [J.D., G. Toscani]: Entropy methods and matching Barenblatt approaches

Gagliardo-Nirenberg inequalities: old and new

Gagliardo-Nirenberg inequalities: old and new

Outline

- The fast diffusion equation and Gagliardo-Nirenberg inequalities
- Onofri's inequality as a limit case in dimension $d = 2$
- The Keller-Segel model: *relative entropies* or *free energies* and the logarithmic Hardy-Littlewood-Sobolev inequality
- A puzzling result of Carlen, Carrillo and Loss on Hardy-Littlewood-Sobolev inequalities

The fast diffusion equation

Consider the fast diffusion equation (FDE)

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with exponent $m \in (\frac{d-1}{d}, 1)$, $d \geq 3$, or its Fokker-Planck version

$$\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot (x\, u) \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with $u_0 \in \mathrm{L}^1_+(\mathbb{R}^d))$ such that $u_0^m \in \mathrm{L}^1_+(\mathbb{R}^d))$ and $|x|^2\, u_0 \in \mathrm{L}^1_+(\mathbb{R}^d))$

Any solution converges as $t \to \infty$ to the *Barenblatt profile*

$$u_\infty(x) = \left(C_M + \tfrac{1-m}{2m}\,|x|^2\right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

[A. Friedman, S. Kamin], [M. del Pino, J.D.]

Asymptotic behaviour of the solutions of FDE

[J. Ralston, W.I. Newman] Define the relative entropy by

$$\mathcal{F}[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} \left[u^m - u_\infty^m - m, u_\infty^{m-1}(u - u_\infty) \right] dx$$

and observe that

- $\frac{d}{dt}\mathcal{F}[u(t,\cdot)] = -\left(\frac{m}{m-1}\right)^2 \int_{\mathbb{R}^d} u \, |\nabla u^{m-1} - \nabla u_\infty^{m-1}|^2 \, dx =: -\mathcal{I}[u(t,\cdot)]$
- $\mathcal{F}[u(t,\cdot)] \le \frac{1}{2}\,\mathcal{I}[u(t,\cdot)]$

if m is in the range $(\frac{d-1}{d}, 1)$, thus showing that

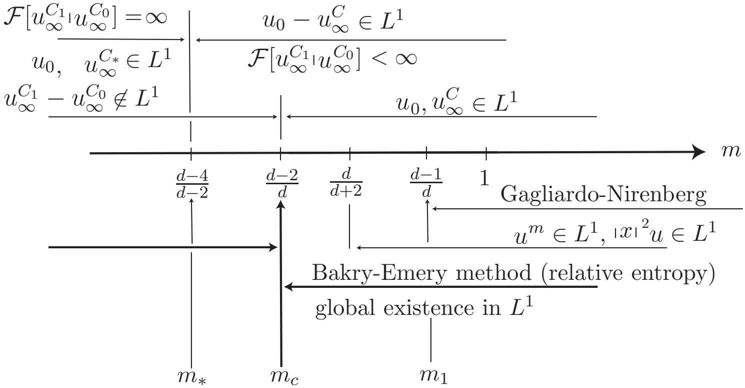
$$\mathcal{F}[u(t,\cdot)] \le \mathcal{F}[u_0]\, e^{-2t} \quad \forall\, t \ge 0$$

- With $p = \frac{1}{2m-1}$, the inequality $\mathcal{F}[u] \le \frac{1}{2}\,\mathcal{I}[u]$ can be rewritten in terms of $f = u^{m-1/2}$ as

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathsf{C}_{p,d}\, \|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\, \|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

- $f_\infty = u_\infty^{m-1/2}$ is optimal

FDE: old and new, and much more



$$u_\infty^C(x) = \left(C + \frac{1-m}{2\,m}\,|x|^2\right)^{\frac{1}{m-1}}$$

[A. Blanchet, M. Bonforte, J.D., G. Grillo, J.-L. Vázquez]

Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)} \le \mathsf{C}_{p,d}\,\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^d)}^{\theta}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

with $\theta = \theta(p) := \frac{p-1}{p}\,\frac{d}{d+2-p\,(d-2)}$

- $1 < p \le \frac{d}{d-2}$ if $d \ge 3$
- $1 < p < \infty$ if $d = 2$

[M. del Pino, J.D.] equality holds in if $f = F_p$ with

$$F_p(x) = (1+|x|^2)^{-\frac{1}{p-1}} \quad \forall\, x \in \mathbb{R}^d$$

and that all extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling.

- If $d \ge 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality [T. Aubin, G. Talenti]
- When $p \to 1$, we recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form [F. Weissler]
- If $d = 2$ and $p \to \infty$...

Onofri's inequality as a limit case

When $d=2$, Onofri's inequality can be seen as an endpoint case of the family of the Gagliardo-Nirenberg inequalities [J.D.]

Proposition

[J.D.] *Assume that* $g\in\mathcal{D}(\mathbb{R}^d)$ *is such that* $\int_{\mathbb{R}^2} g\,d\mu=0$ *and let*

$$f_p:=F_p\Big(1+\frac{g}{2p}\Big)$$

With $\mu(x):=\frac{1}{\pi}\,(1+|x|^2)^{-2}$, *and* $d\mu(x):=\mu(x)\,dx$, *we have*

$$1\le\lim_{p\to\infty}\mathsf{C}_{p,2}\,\frac{\|\nabla f\|_{\mathrm{L}^2(\mathbb{R}^2)}^{\theta(p)}\,\|f\|_{\mathrm{L}^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f\|_{\mathrm{L}^{2p}(\mathbb{R}^2)}}=\frac{e^{\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx}}{\int_{\mathbb{R}^2}e^{g}\,d\mu}$$

The standard form of the euclidean version of Onofri's inequality is

$$\log\left(\int_{\mathbb{R}^2}e^{g}\,d\mu\right)-\int_{\mathbb{R}^2}g\,d\mu\le\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla g|^2\,dx$$

The Keller-Segel model

With $t > 0$, $x \in \mathbb{R}^2$, consider the system

$$\frac{\partial v}{\partial t} = \Delta v - \nabla(v \nabla \varphi) \,, \quad -\Delta \varphi = v$$

The behaviour of the solution depends on the mass $M = \int_{\mathbb{R}^2} v \, dx$

- [W. Jäger, S. Luckhaus] if $M > 8\pi$ (under technical conditions), smooth solutions blow-up in finite time because

$$\frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \, v \, dx = 4\,M - \frac{M^2}{4\,\pi} < 0$$

- [J.D., B. Perthame] if $M \le 8\pi$, the entropy $\mathcal{H}[v]$ is bounded from below

$$\mathcal{H}[v] := \int_{\mathbb{R}^2} v \, \log v \, dx - \frac{1}{2} \int_{\mathbb{R}^2} v \, \varphi \, dx$$

- In the critical mass case $M = 8\pi$, $\mathcal{H}[v] \ge \mathcal{H}[\mu] > -\infty$ is given by the *logarithmic Hardy-Littlewood-Sobolev inequality*

$$\int_{\mathbb{R}^2} f \, \log\left(\frac{f}{M}\right) dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} f(x)\, f(y) \, \log|x-y| \, dx\, dy + M\,(1 + \log \pi) \ge 0$$

A puzzling result of Carlen, Carrillo and Loss ($d \geq 3$)

[E. Carlen, J.A. Carrillo and M. Loss] The fast diffusion equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d$$

with exponent $m = d/(d+2)$, when $d \geq 3$, is such that

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

obeys to

$$\frac{1}{2}\,\frac{d}{dt}\mathsf{H}_d[v(t,\cdot)] = \frac{1}{2}\,\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\;dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right]$$
$$= \frac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d\,\|u\|^{4/(d-1)}_{\mathrm{L}^{q+1}(\mathbb{R}^d)}\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^{2q}_{\mathrm{L}^{2q}(\mathbb{R}^d)}$$

with $u = v^{(d-1)/(d+2)}$ and $q = \frac{d+1}{d-1}$. If $\frac{d\,(d-2)}{(d-1)^2}\,\mathsf{S}_d = (\mathsf{C}_{q,\,d})^{2q}$, the r.h.s. is nonnegative. Optimality is achieved simultaneously in both functionals (Barenblatt regime): the Hardy-Littlewood-Sobolev inequalities can be improved by an integral remainder term

... and the two-dimensional case

Recall that $(-\Delta)^{-1} v = G_d * v$ with

- $G_d(x) = \frac{1}{d-2} \, |\mathbb{S}^{d-1}|^{-1} \, |x|^{2-d}$ if $d \geq 3$
- $G_2(x) = \frac{1}{2\,\pi} \, \log |x|$ if $d = 2$

Same computation in dimension $d = 2$ with $m = 1/2$ gives

$$\frac{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}}{8} \frac{d}{dt} \left[\frac{4\,\pi}{\|v\|_{\mathrm{L}^1(\mathbb{R}^2)}} \int_{\mathbb{R}^2} v \, (-\Delta)^{-1} v \; dx - \int_{\mathbb{R}^2} v \, \log v \; dx \right]$$
$$= \|u\|^4_{\mathrm{L}^4(\mathbb{R}^2)} \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^2)} - \pi \, \|v\|^6_{\mathrm{L}^6(\mathbb{R}^2)}$$

The r.h.s. is one of the Gagliardo-Nirenberg inequalities ($d = 2$, $q = 3$): $\pi \, (\mathrm{C}_{3,2})^6 = 1$
The l.h.s. is bounded from below by the logarithmic Hardy-Littlewood-Sobolev inequality and achieves its minimum if $v = \mu$ with

$$\mu(x) := \frac{1}{\pi \, (1 + |x|^2)^2} \quad \forall \, x \in \mathbb{R}^2$$

Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Sobolev and Hardy-Littlewood-Sobolev inequalities: duality, flows

Outline

- Legendre duality
- Sobolev and HLS inequalities can be related using a nonlinear flow *compatible with Legendre's duality*
- The asymptotic behaviour close to the *vanishing time* is determined by a solution with *separation of variables* based on the Aubin-Talenti solution
- The vanishing time T can be estimated using a priori estimates
- The entropy H is negative, concave, and we can relate $\mathsf{H}(0)$ with $\mathsf{H}'(0)$ by integrating estimates on $(0, T)$, which provides *an improvement of Sobolev's inequality* if $d \geq 5$

Legendre duality

To a convex functional F, we may associate the functional F^* defined by Legendre's duality as

$$F^*[v] := \sup \left(\int_{\mathbb{R}^d} u\, v\; dx - F[u] \right)$$

- To $F_1[u] = \frac{1}{2} \|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)}$, we associate $F_1^*[v] = \frac{1}{2} \|v\|^2_{\mathrm{L}^q(\mathbb{R}^d)}$ where p and q are Hölder conjugate exponents: $1/p + 1/q = 1$
- To $F_2[u] = \frac{1}{2} \mathsf{S}_d \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, we associate

$$F_2^*[v] = \frac{1}{2} \mathsf{S}_d^{-1} \int_{\mathbb{R}^d} v\, (-\Delta)^{-1} v\; dx$$

where $(-\Delta)^{-1} v = G_d * v$, $G_d(x) = \frac{1}{d-2} |\mathbb{S}^{d-1}|^{-1} |x|^{2-d}$ if $d \geq 3$

As a straightforward consequence of Legendre's duality, if we have a functional inequality of the form $F_1[u] \leq F_2[u]$, then we have the dual inequality $F_1^*[v] \geq F_2^*[v]$

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathcal{D}^{1,2}(\mathbb{R}^d) \tag{1}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v \; dx \quad \forall\, v \in \mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d) \tag{2}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2\,d}{d-2}$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0 \,, \quad x \in \mathbb{R}^d \tag{3}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\; dx - \mathsf{S}_d\, \|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\; dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\; dx \right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\; dx$$

where $v = v(t,\cdot)$ is a solution of (3). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2\,d}{d+2}$

A first statement

Proposition

[J.D.] *Assume that* $d \geq 3$ *and* $m = \frac{d-2}{d+2}$. *If* v *is a solution of* (3) *with nonnegative initial datum in* $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, *then*

$$\frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}\right]$$

$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left[\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right] \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (1) if v is optimal for (2)

Improved Sobolev inequality

By integrating along the flow defined by (3), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (1), but only when $d \geq 5$ for integrability reasons

Theorem

[J.D.] *Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that*

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left[\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right]$$

for any $w \in \mathcal{D}^{1,2}(\mathbb{R}^d)$

Solutions with *separation of variables*

Consider the solution vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}} \quad \forall\,(t,x) \in (0,T) \times \mathbb{R}^d$$

where $\alpha = (d+2)/4$, $c^{1-m} = 4\,m\,d$, $m = \frac{d-2}{d+2}$, $p = d/(d-2)$ and F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[M. delPino, M. Saez], [J. L. Vázquez, J. R. Esteban, A. Rodríguez]
For any solution v of (3) with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot) - 1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

A first set of *a priori* integral estimates

Let $\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$. Let $d \ge 3$ and $m = (d-2)/(d+2)$

Lemma

[J.D.] *If v is a solution of (3) vanishing at time $T>0$ with $v_0 \in \mathrm{L}^{2^*}_+(\mathbb{R}^d)$*

$$\left(\tfrac{4\,(T-t)}{(d+2)\,\mathsf{S}_d}\right)^{\frac d2} \le \mathsf{J}(t) \le \mathsf{J}(0)\,,\quad \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \mathsf{S}_d^{-1}\left(\tfrac{4\,(T-t)}{d+2}\right)^{\frac d2 -1}$$

$$T \le \tfrac14\,(d+2)\,\mathsf{S}_d\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{\frac 2d}$$

for any $t \in (0,T)$. Moreover, if $d \ge 5$, we also have

$$\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx \ge \int_{\mathbb{R}^d} v_0^{m+1}\,dx - \tfrac{2\,d}{d+2}\,t\,\|\nabla v_0^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$\|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \|\nabla v_0^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

$$T \ge \frac{d+2}{2\,d}\int_{\mathbb{R}^d} v_0^{m+1}\,dx\,\|\nabla v_0^m\|^{-2}_{\mathrm{L}^2(\mathbb{R}^d)}$$

Proofs (1/2)

$\mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1}\,dx$ satisfies

$$\mathsf{J}' = -(m+1)\,\|\nabla v^m\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{1-\frac{2}{d}}$$

If $d \ge 5$, then we also have

$$\mathsf{J}'' = 2\,m\,(m+1)\int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \ge 0$$

Such an estimate makes sense if $v = \overline{v}_T$. This is also true for any solution v as can be seen by rewriting the problem on $\mathbb{S}^d$: integrability conditions for v are exactly the same as for $\overline{v}_T$ □

Notice that

$$\frac{\mathsf{J}'}{\mathsf{J}} \le -\frac{m+1}{\mathsf{S}_d}\,\mathsf{J}^{-\frac{2}{d}} \le -\kappa \quad \text{with} \quad \kappa := \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-\frac{2}{d}} \le \frac{d}{2\,T}$$

Proofs (2/2)

By the Cauchy-Schwarz inequality, we have

$$\|\nabla v^m\|^4_{\mathrm{L}^2(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} v^{(m-1)/2}\,\Delta v^m \cdot v^{(m+1)/2}\,dx\right)^2 \le \int_{\mathbb{R}^d} v^{m-1}\,(\Delta v^m)^2\,dx \int_{\mathbb{R}^d} v^{m+1}\,dx$$

so that $\mathsf{Q}(t) := \|\nabla v^m(t,\cdot)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \left(\int_{\mathbb{R}^d} v^{m+1}(t,x)\,dx\right)^{-(d-2)/d}$ is monotone decreasing, and

$$\mathsf{H}' = 2\,\mathsf{J}\,(\mathsf{S}_d\,\mathsf{Q} - 1)\,, \quad \mathsf{H}'' = \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' + 2\,\mathsf{J}\,\mathsf{S}_d\,\mathsf{Q}' \le \frac{\mathsf{J}'}{\mathsf{J}}\,\mathsf{H}' \le 0$$

$$\mathsf{H}'' \le -\kappa\,\mathsf{H}' \quad \text{with} \quad \kappa = \frac{2\,d}{d+2}\,\frac{1}{\mathsf{S}_d}\left(\int_{\mathbb{R}^d} v_0^{m+1}\,dx\right)^{-2/d}$$

By writing that $-\mathsf{H}(0) = \mathsf{H}(T) - \mathsf{H}(0) \le \mathsf{H}'(0)\,(1 - e^{-\kappa\,T})/\kappa$ and using the estimate $\kappa\,T \le d/2$, we obtain our main result □

The two-dimensional case: Legendre duality

Onofri's inequality amounts to $F_1[u] \le F_2[u]$ with

$$F_1[u] := \log\left(\int_{\mathbb{R}^2} e^u \, d\mu\right) \quad \text{and} \quad F_2[u] := \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u\,\mu\,dx$$

Proposition

[E. Carlen, M. Loss], [V. Calvez, L. Corrias] *For any* $v \in \mathrm{L}^1_+(\mathbb{R}^2)$ *with* $\int_{\mathbb{R}^2} v\,dx = 1$*, such that* $v \log v$ *and* $(1+\log|x|^2)\,v \in \mathrm{L}^1(\mathbb{R}^2)$*, we have*

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right) dx - 4\,\pi \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx \ge 0$$

Notice that $-\Delta \log \mu = 8\,\pi\,\mu$ can be inverted as

$$(-\Delta)^{-1}\mu = \frac{1}{8\,\pi}\,\log(\pi\,\mu)$$

The two-dimensional case: log HLS and...

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v-\mu)\,(-\Delta)^{-1}(v-\mu)\,dx - \frac{1}{4\,\pi}\int_{\mathbb{R}^2} v\,\log\left(\frac{v}{\mu}\right)\,dx$$

Assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log\left(\frac{v}{\mu}\right) \quad t>0\,, \quad x\in\mathbb{R}^2$$

Proposition

[J.D.] *If v is a solution with nonnegative initial datum v_0 in $\mathrm{L}^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0\,dx = 1$, $v_0\,\log v_0 \in \mathrm{L}^1(\mathbb{R}^2)$ and $v_0\,\log\mu \in \mathrm{L}^1(\mathbb{R}^2)$, then*

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2} |\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

with $\log(v/\mu) = u/2$

The two-dimensional case: ...Onofri's inequality

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] = \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx - \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

The right hand side is nonnegative by Onofri's inequality:

$$\frac{d}{dt}\mathsf{H}_2[v(t,\cdot)] \ge \frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx + \int_{\mathbb{R}^2}u\,d\mu - \log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right) \ge 0$$

- If $\int_{\mathbb{R}^2}u\,d\mu = 1$, then

$$-\int_{\mathbb{R}^2}e^{\frac{u}{2}}u\,d\mu \ge -\log\left(\int_{\mathbb{R}^2}e^u\,d\mu\right)$$

- Corollary: for any $u\in\mathcal{D}(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^2}e^{\frac{u}{2}}\,d\mu = 1$, we have

$$\frac{1}{16\,\pi}\int_{\mathbb{R}^2}|\nabla u|^2\,dx \ge \int_{\mathbb{R}^2}\left(e^{\frac{u}{2}}-1\right)u\,d\mu$$

The two-dimensional case: the sphere setting

The image w of v by the inverse stereographic projection on the sphere $\mathbb{S}^2$, up to a scaling, solves the equation

$$\frac{\partial w}{\partial t} = \Delta_{\mathbb{S}^2} \log w \quad t > 0\,, \quad y \in \mathbb{S}^2$$

More precisely, if $x = (x_1, x_2) \in \mathbb{R}^2$, then u and w are related by

$$w(t,y) = \frac{u(t,x)}{4\,\pi\,\mu(x)}\,, \quad y = \left(\tfrac{2\,(x_1,x_2)}{1+|x|^2}, \tfrac{1-|x|^2}{1+|x|^2}\right) \in \mathbb{S}^2$$

The loss of mass of the solution of

$$\frac{\partial v}{\partial t} = \Delta \log v \quad t > 0\,, \quad x \in \mathbb{R}^2$$

is compensated in case of

$$\frac{\partial v}{\partial t} = \Delta \log \left(\frac{v}{\mu}\right) \quad t > 0\,, \quad x \in \mathbb{R}^2$$

by the source term $-\Delta \log \mu$

Gagliardo-Nirenberg inequalities: further improvements

Gagliardo-Nirenberg inequalities: further improvements

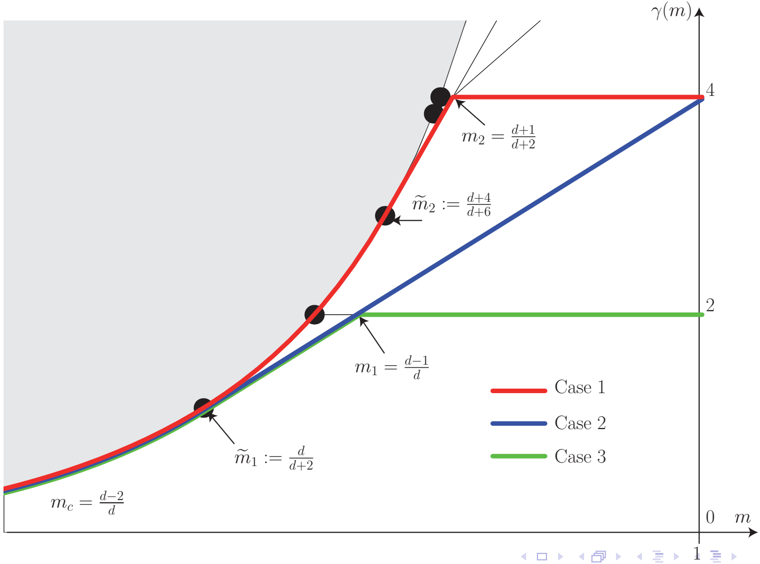
A brief summary of a strategy for further improvements

- Back to the basin of attraction of Barenblatt functions: improving the *asymptotic rates of convergence* for any m

$$\frac{\partial v}{\partial t} + \nabla \cdot \left(v \, \nabla v^{m-1} \right) = 0 \quad t > 0 \,, \quad x \in \mathbb{R}^d$$

  with $m \in (\frac{d-1}{d}, 1)$, $d \geq 3$... Refer to M. Bonforte's lecture
- The $\frac{1}{2}$ factor in the inequality $\mathcal{F}[u] \leq \frac{1}{2} \, \mathcal{I}[u]$ can be explained by *spectral gap* considerations
- This factor can be improved for well prepared initial data, if $m > \frac{d-1}{d}$
- *Global improvements* can be obtained using rescalings which depend on the second moment, even for $m = \frac{d-1}{d}$

Spectral gaps and best constants



Best matching Barenblatt profiles

Consider the *fast diffusion equation*

$$\frac{\partial u}{\partial t}+\nabla\cdot\left[u\,\left(\sigma^{\frac{d}{2}(m-m_c)}\,\nabla u^{m-1}-2\,x\right)\right]=0\quad t>0\,,\quad x\in\mathbb{R}^d$$

with a nonlocal, time-dependent diffusion coefficient

$$\sigma(t)=\frac{1}{K_M}\int_{\mathbb{R}^d}|x|^2\,u(x,t)\,dx\,,\quad K_M:=\int_{\mathbb{R}^d}|x|^2\,B_1(x)\,dx$$

where

$$B_\lambda(x):=\lambda^{-\frac{d}{2}}\left(C_M+\tfrac{1}{\lambda}\,|x|^2\right)^{\frac{1}{m-1}}\quad\forall\,x\in\mathbb{R}^d$$

and define the relative entropy

$$\mathcal{F}_\lambda[u]:=\frac{1}{m-1}\int_{\mathbb{R}^d}\left[u^m-B_\lambda^m-m\,B_\lambda^{m-1}\,(u-B_\lambda)\right]\,dx$$

Three ingredients for *global improvements*

1. $\inf_{\lambda>0} \mathcal{F}_\lambda[u(x,t)] = \mathcal{F}_{\sigma(t)}[u(x,t)]$ so that

$$\frac{d}{dt}\mathcal{F}_{\sigma(t)}[u(x,t)] = -\mathcal{J}_{\sigma(t)}[u(\cdot,t)]$$

where the relative Fisher information is

$$\mathcal{J}_\lambda[u] := \lambda^{\frac{d}{2}(m-m_c)} \frac{m}{1-m} \int_{\mathbb{R}^d} u \left|\nabla u^{m-1} - \nabla B_\lambda^{m-1}\right|^2 dx$$

2. In the *Bakry-Emery method*, there is an additional (good) term

$$4\left[1 + 2\, C_{m,d} \frac{\mathcal{F}_{\sigma(t)}[u(\cdot,t)]}{M^\gamma\, \sigma_0^{\frac{d}{2}(1-m)}}\right] \frac{d}{dt}\left(\mathcal{F}_{\sigma(t)}[u(\cdot,t)]\right) \geq \frac{d}{dt}\left(\mathcal{J}_{\sigma(t)}[u(\cdot,t)]\right)$$

3. The *Csiszár-Kullback inequality* is also improved

$$\mathcal{F}_\sigma[u] \geq \frac{m}{8\int_{\mathbb{R}^d} B_1^m\, dx} C_M^2 \|u - B_\sigma\|^2_{\mathrm{L}^1(\mathbb{R}^d)}$$

An improved Gagliardo-Nirenberg inequality (1/2)

Relative entropy functional

$$\mathcal{R}^{(p)}[f] := \inf_{g\in\mathfrak{M}_d^{(p)}} \int_{\mathbb{R}^d} \left[g^{1-p}\left(|f|^{2\,p}-g^{2\,p}\right)-\tfrac{2\,p}{p+1}\left(|f|^{p+1}-g^{p+1}\right)\right] dx$$

Theorem

Let $d\ge 2$, $p>1$ and assume that $p<d/(d-2)$ if $d\ge 3$. If

$$\frac{\int_{\mathbb{R}^d}|x|^2\,|f|^{2\,p}\,dx}{\left(\int_{\mathbb{R}^d}|f|^{2\,p}\,dx\right)^{\gamma}} = \frac{d\,(p-1)\,\sigma_*\,M_*^{\gamma-1}}{d+2-p\,(d-2)}\,,\quad \sigma_*(p) := \left(4\,\frac{d+2-p\,(d-2)}{(p-1)^2\,(p+1)}\right)^{\frac{4\,p}{d-p\,(d-4)}}$$

for any $f\in\mathrm{L}^{p+1}\cap\mathcal{D}^{1,2}(\mathbb{R}^d)$, then we have

$$\int_{\mathbb{R}^d}|\nabla f|^2\,dx+\int_{\mathbb{R}^d}|f|^{p+1}\,dx-\mathsf{K}_{p,d}\left(\int_{\mathbb{R}^d}|f|^{2\,p}\,dx\right)^{\gamma} \ge \mathsf{C}_{p,d}\,\frac{\left(\mathcal{R}^{(p)}[f]\right)^2}{\left(\int_{\mathbb{R}^d}|f|^{2\,p}\,dx\right)^{\gamma}}$$

An improved Gagliardo-Nirenberg inequality (2/2)

A Csiszár-Kullback inequality

$$\mathcal{R}^{(p)}[f] \ge \mathsf{C}_{\mathrm{CK}} \, \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-2)} \inf_{g \in \mathfrak{M}_d^{(p)}} \| |f|^{2p} - g^{2p} \|_{\mathrm{L}^1(\mathbb{R}^d)}^2$$

with $\mathsf{C}_{\mathrm{CK}} = \frac{p-1}{p+1} \, \frac{d+2-p\,(d-2)}{32\,p} \, \sigma_*^{\,d\,\frac{p-1}{4p}} \, M_*^{1-\gamma}$. Let

$$\mathfrak{C}_{p,d} := \mathsf{C}_{d,p} \, {\mathsf{C}_{\mathrm{CK}}}^2$$

Corollary

Under previous assumptions, we have

$$\int_{\mathbb{R}^d} |\nabla f|^2 \, dx + \int_{\mathbb{R}^d} |f|^{p+1} \, dx - \mathsf{K}_{p,d} \left(\int_{\mathbb{R}^d} |f|^{2p} \, dx \right)^{\gamma} \ge \mathfrak{C}_{p,d} \, \|f\|_{\mathrm{L}^{2p}(\mathbb{R}^d)}^{2p(\gamma-4)} \inf_{g \in \mathfrak{M}_d(p)} \| |f|^{2p} - g^{2p} \|_{\mathrm{L}^1(\mathbb{R}^d)}^4$$