

Sharp rates for the subcritical parabolic-elliptic Keller-Segel model in the plane

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July 18, 2013
Workshop on

Solitons, Vortices, Minimal Surfaces and their Dynamics
Mittag-Leffler Institute, Stockholm (July 15-20, 2013)

Outline

- ➊ Preliminaries: functional inequalities, fast diffusion, Onofri, etc.

- ➋ Keller-Segel model: a brief review
Measure valued solutions in the supercritical regime
[JD, C. Schmeiser]

- ➌ Subcritical and small mass results
[A. Blanchet, JD, M. Escobedo, J. Fernández]

- ➍ Keller-Segel model: inequalities and spectrum of the linearized operator
[J. Campos, JD]

- ➎ The nonlinear case, a functional analysis approach
[J. Campos, JD]

1 – Preliminaries: functional analysis

Flows, functional inequalities, functional spaces...

- Fast diffusion, Gagliardo-Nirenberg and Onofri inequalities
- Logarithmic Hardy-Littlewood-Sobolev and Onofri inequalities:
duality, flows
- Linearized fast diffusion and weighted L^2 spaces

Fast diffusion, Gagliardo-Nirenberg and Onofri inequalities

The fast diffusion equation

Consider the Fokker-Planck version of fast diffusion equation (FDE) with exponent $m \in (\frac{d-1}{d}, 1)$, $d \geq 3$

$$\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot (x u) \quad t > 0, \quad x \in \mathbb{R}^d$$

with $u_0 \in L^1_+(\mathbb{R}^d)$ such that $u_0^m \in L^1_+(\mathbb{R}^d)$ and $|x|^2 u_0 \in L^1_+(\mathbb{R}^d)$

Any solution converges as $t \rightarrow \infty$ to the *Barenblatt profile*

$$u_\infty(x) = \left(C_M + \frac{1-m}{2m} |x|^2 \right)^{\frac{1}{m-1}} \quad x \in \mathbb{R}^d$$

Asymptotic behaviour of the solutions of FDE

[J. Ralston, W.I. Newman 1984] Define the relative entropy (or free energy, or relative Tsallis entropy) by

$$\mathcal{F}[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} [u^m - u_\infty^m - m u_\infty^{m-1}(u - u_\infty)] dx$$

$$\frac{d}{dt} \mathcal{F}[u(t, \cdot)] = -\left(\frac{m}{m-1}\right)^2 \int_{\mathbb{R}^d} u |\nabla u^{m-1} - \nabla u_\infty^{m-1}|^2 dx =: -\mathcal{I}[u(t, \cdot)]$$

$$\mathcal{F}[u(t, \cdot)] \leq \frac{1}{2} \mathcal{I}[u(t, \cdot)]$$

if m is in the range $(\frac{d-1}{d}, 1)$, thus showing that

$$\mathcal{F}[u(t, \cdot)] \leq \mathcal{F}[u_0] e^{-2t} \quad \forall t \geq 0$$

With $p = \frac{1}{2m-1}$, the inequality $\mathcal{F}[u] \leq \frac{1}{2} \mathcal{I}[u]$ can be rewritten in terms of $f = u^{m-1/2}$ as

$$\|f\|_{L^{2p}(\mathbb{R}^d)} \leq C_{p,d} \|\nabla f\|_{L^2(\mathbb{R}^d)}^\theta \|f\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

$f_\infty = u_\infty^{m-1/2}$ is optimal [M. del Pino, JD 2002] [F. Otto 2001]
 [D. Cordero-Erausquin, B. Nazaret, C. Villani 2004]

Gagliardo-Nirenberg inequalities

Consider the following sub-family of Gagliardo-Nirenberg inequalities

$$\|f\|_{L^{2p}(\mathbb{R}^d)} \leq C_{p,d} \|\nabla f\|_{L^2(\mathbb{R}^d)}^\theta \|f\|_{L^{p+1}(\mathbb{R}^d)}^{1-\theta}$$

with $\theta = \theta(p) := \frac{p-1}{p} \frac{d}{d+2-p(d-2)}$, $p = \frac{1}{2m-1}$

- $1 < p \leq \frac{d}{d-2}$ if $d \geq 3$, $\frac{d-1}{d} \leq m < 1$
- $1 < p < \infty$ if $d = 2$, $\frac{1}{2} < m < 1$

[M. del Pino, JD 2002] equality holds in if $f = F_p$ with

$$F_p(x) = (1 + |x|^2)^{-\frac{1}{p-1}} \quad \forall x \in \mathbb{R}^d$$

All extremal functions are equal to F_p up to a multiplication by a constant, a translation and a scaling

- When $p \rightarrow 1$, we recover the euclidean logarithmic Sobolev inequality in optimal scale invariant form [F. Weissler 1978]
- If $d \geq 3$, the limit case $p = d/(d-2)$ corresponds to Sobolev's inequality [T. Aubin, G. Talenti 1976]
- If $d = 2$ and $p \rightarrow \infty$...

Onofri's inequality as a limit case

When $d = 2$, Onofri's inequality can be seen as an endpoint case of the family of the Gagliardo-Nirenberg inequalities [JD 2011], [del Pino, JD 2012]

Proposition (JD)

Assume that $g \in \mathcal{D}(\mathbb{R}^d)$ is such that $\int_{\mathbb{R}^2} g \, d\mu = 0$ and let

$f_p := F_p \left(1 + \frac{|g|}{2p}\right)$. With $\mu(x) := \frac{1}{\pi} (1 + |x|^2)^{-2}$, and $d\mu(x) := \mu(x) \, dx$, we have

$$1 \leq \lim_{p \rightarrow \infty} C_{p,2} \frac{\|\nabla f_p\|_{L^2(\mathbb{R}^2)}^{\theta(p)} \|f_p\|_{L^{p+1}(\mathbb{R}^2)}^{1-\theta(p)}}{\|f_p\|_{L^{2p}(\mathbb{R}^2)}} = \frac{e^{\frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla g|^2 \, dx}}{\int_{\mathbb{R}^2} e^g \, d\mu}$$

The standard form of the *Euclidean* version of Onofri's inequality is

$$\log \left(\int_{\mathbb{R}^2} e^g \, d\mu \right) - \int_{\mathbb{R}^2} g \, d\mu \leq \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla g|^2 \, dx$$

Logarithmic Hardy-Littlewood-Sobolev and Onofri inequalities: duality, flows

The logarithmic HLS inequality

The classical logarithmic Hardy-Littlewood-Sobolev (**logHLS**) in \mathbb{R}^2

$$\int_{\mathbb{R}^2} n \log \left(\frac{n}{M} \right) dx + \frac{2}{M} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log |x-y| dx dy + M (1 + \log \pi) \geq 0$$

Equality is achieved by

$$\mu(x) := \frac{1}{\pi (1 + |x|^2)^2} \quad \forall x \in \mathbb{R}^2$$

Notice that $-\Delta \log \mu = 8\pi \mu$ can be inverted as

$$(-\Delta)^{-1} \mu = \frac{1}{8\pi} \log(\pi \mu)$$

With $M = 8\pi$ and $n_\infty = 8\pi \mu$ (**logHLS**) can be rewritten as

$$\int_{\mathbb{R}^2} n \log \left(\frac{n}{n_\infty} \right) dx \geq \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty) (-\Delta)^{-1}(n - n_\infty) dx$$

The two-dimensional case: Legendre duality

Onofri's inequality

$$F_1[u] := \log \left(\int_{\mathbb{R}^2} e^u \, d\mu \right) \leq \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx + \int_{\mathbb{R}^2} u \mu \, dx =: F_2[u]$$

By duality: $F_i^*[v] = \sup \left(\int_{\mathbb{R}^2} v u \, d\mu - F_i[u] \right)$ we can relate Onofri's inequality with (logHLS)

Proposition (E. Carlen, M. Loss 1992 & V. Calvez, L. Corrias 2008)

For any $v \in L^1_+(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} v \, dx = 1$, such that $v \log v$ and $(1 + \log |x|^2)v \in L^1(\mathbb{R}^2)$, we have

$$F_1^*[v] - F_2^*[v] = \int_{\mathbb{R}^2} v \log \left(\frac{v}{\mu} \right) \, dx - 4\pi \int_{\mathbb{R}^2} (v - \mu)(-\Delta)^{-1}(v - \mu) \, dx \geq 0$$

The two-dimensional case: (logHLS) and flows

[E. Carlen, J. Carrillo, M. Loss 2010]

$$\mathsf{H}_2[v] := \int_{\mathbb{R}^2} (v - \mu) (-\Delta)^{-1}(v - \mu) \, dx - \frac{1}{4\pi} \int_{\mathbb{R}^2} v \log \left(\frac{v}{\mu} \right) \, dx$$

is related to Gagliardo-Nirenberg inequalities if $v_t = \Delta \sqrt{v}$

• Alternatively, assume that v is a positive solution of

$$\frac{\partial v}{\partial t} = \Delta \log \left(\frac{v}{\mu} \right) \quad t > 0, \quad x \in \mathbb{R}^2$$

Proposition (JD 2011)

If v is a solution with nonnegative initial datum v_0 in $L^1(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} v_0 \, dx = 1$, $v_0 \log v_0 \in L^1(\mathbb{R}^2)$ and $v_0 \log \mu \in L^1(\mathbb{R}^2)$, then

$$\frac{d}{dt} \mathsf{H}_2[v(t, \cdot)] = \frac{1}{16\pi} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx - \int_{\mathbb{R}^2} (e^{\frac{u}{2}} - 1) u \, d\mu \geq F_2[u] - F_1[u]$$

with $\log(v/\mu) = u/2$



Hierarchies of inequalities, improved inequalities

Proposition (JD, Jankowiak 2013)

If $d \geq 3$, with $q = \frac{d+2}{d-2}$

$$\begin{aligned} S_d \|u^q\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} u^q (-\Delta)^{-1} u^q \, dx \\ \leq S_d \|u\|_{L^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left[S_d \|\nabla u\|_{L^2(\mathbb{R}^d)}^2 - \|u\|_{L^{2^*}(\mathbb{R}^d)}^2 \right] \\ \forall u \in H^1(\mathbb{R}^d) \end{aligned}$$

and, when $d = 2$, if $\int_{\mathbb{R}^2} e^f \, d\mu = 1$

$$\begin{aligned} \frac{1}{4\pi} \int_{\mathbb{R}^2} f e^f \, d\mu - \int_{\mathbb{R}^d} (e^f - 1) \mu (-\Delta)^{-1} (e^f - 1) \mu \, dx \\ \leq \frac{1}{16\pi} \|\nabla f\|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^2} f \, d\mu \end{aligned}$$

Linearized fast diffusion and weighted L^2 spaces

Linearization and weighted L^2 spaces

Back to the fast diffusion equation $\frac{\partial u}{\partial t} = \Delta u^m + \nabla \cdot (x u)$ we investigate the linear regime $u = u_\infty(1 + \varepsilon f u_\infty^{1-m})$ as $\varepsilon \rightarrow 0$

$$\frac{\partial f}{\partial t} = -\mathcal{L}f := u_\infty^{m-2} \nabla \cdot (u_\infty \nabla f)$$

Use $L^2(\mathbb{R}^d, u_\infty^{2-m} dx)$ [A. Blanchet, M. Bonforte, JD, G. Grillo, J.L. Vázquez 2009] Also see: [J. Denzler & R. McCann 2005]

$$\mathcal{F}[u] := \frac{1}{m-1} \int_{\mathbb{R}^d} [u^m - u_\infty^m - m u_\infty^{m-1}(u - u_\infty)] dx \sim \frac{m}{2} \int_{\mathbb{R}^d} |f|^2 u_\infty^{2-m} dx$$

$$\frac{d}{dt} \mathcal{F}[u(t, \cdot)] = -\mathcal{I}[u(t, \cdot)] \sim - \int_{\mathbb{R}^d} |\nabla f|^2 u_\infty dx = -\langle f, \mathcal{L}f \rangle$$

$$\mathcal{F}[u(t, \cdot)] \leq \frac{1}{2} \mathcal{I}[u(t, \cdot)] \implies \int_{\mathbb{R}^d} |f|^2 u_\infty^{2-m} dx \leq \frac{1}{\Lambda} \int_{\mathbb{R}^d} |\nabla f|^2 u_\infty dx$$

for any $m < 1$, $m \neq (d-4)/(d-2)$ i.e. Hardy-Poincaré inequality (under zero average condition on f) and... $\Lambda = 1$ if $m \in [(d-1)/d, 1)$

Linearized fast diffusion: spectrum

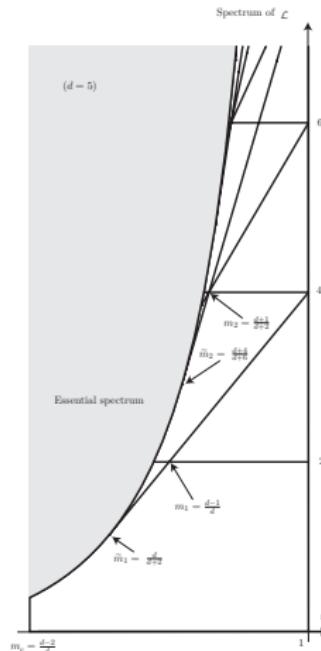


Figure: Spectrum of \mathcal{L} , appropriately normalized

2 – Keller-Segel model: a partial review

Warnings !

- ➊ Literature is huge
- ➋ Physics can be addressed in various ways: gravitation (Smoluchowski-Poisson) and statistics of gravitating systems, aggregation dynamics (sticky systems), biology (Patlak, Keller-Segel)
- ➌ Standard techniques have been reinvented many times: virial estimates, cumulated mass densities, matched asymptotics

⇒ my goal is to give some honest entry points

- ➌ do not specialize to radial solutions
- ➌ put emphasis on functional analysis
- ➌ insist on nonlinear evolution
- ➌ deal with the subcritical case: at least it gives some hint on how the *bubble* appears in the critical limit

The parabolic-elliptic Keller and Segel system

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2, t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2, t > 0 \\ u(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

We make the choice:

$$v(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x-y| u(t, y) dy$$

and observe that

$$\nabla v(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} u(t, y) dy$$

Mass conservation: $\frac{d}{dt} \int_{\mathbb{R}^2} u(t, x) dx = 0$

Blow-up: the virial computation

$M = \int_{\mathbb{R}^2} n_0 \, dx > 8\pi$ and $\int_{\mathbb{R}^2} |x|^2 n_0 \, dx < \infty$: blow-up in finite time
 a solution u of

$$\frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v)$$

satisfies

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 u(t, x) \, dx \\ = - \underbrace{\int_{\mathbb{R}^2} 2x \cdot \nabla u \, dx}_{-4M} + \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \underbrace{\frac{2x \cdot (y-x)}{|x-y|^2} u(t, x) u(t, y) \, dx \, dy}_{\frac{(x-y) \cdot (y-x)}{|x-y|^2} u(t, x) u(t, y) \, dx \, dy} \\ = 4M - \frac{M^2}{2\pi} < 0 \quad \text{if } M > 8\pi \end{aligned}$$

Blow-up and singular solutions: some results

- ① Formal asymptotic expansions in \mathbb{R}^2
[Herrero, Velázquez 1997], [Chavanis, Sire 2002-2005], [Campos, PhD thesis, 2012]
[Dejak, Lushnikov, Ovchinnikov, Sigal 2012], [Dejak, Egli, Lushnikov, Sigal 2013]
- ② Results in bounded domains: [Kavallaris, Souplet 2009]
- ③ A first rigorous result in \mathbb{R}^2 (radial case)
[Raphaël, Schweyer 2012-2013] stable chemotactic blow up, universality of the bubble
- ④ Other results in \mathbb{R}^2 : [Montaru 2012-2013]
- ⑤ Measure valued solutions: [Herrero, Velázquez 1997], [Luckhaus, Sugiyama, Velázquez 2012], [Seki, Sugiyama, Velázquez 2013]
[Haškovec, Schmeiser 2009] the particle system, Wasserstein's distance and free energy
[Bedrossian, Masmoudi 2012] spectral gap and free energy

The super-critical range: life after blow-up

Regularization

Regularize the Poisson kernel

$$(-\Delta)_\varepsilon^{-1} * \rho(x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y| + \varepsilon) \rho(y) dy$$

[F. Poupaud, Diagonal defect measures, adhesion dynamics and Euler equations, Meth. Appl. Anal. **9** (2002), pp. 533–561]

Proposition (JD, C. Schmeiser 2009)

For every $\varepsilon > 0$, the regularized problem has a global solution satisfying

$$\|\rho^\varepsilon(\cdot, t)\|_{L^1(\mathbb{R}^2)} = \|\rho_I\|_{L^1(\mathbb{R}^2)} := M$$

$$\|\rho^\varepsilon(t, \cdot)\|_{L^\infty(\mathbb{R}^2)} \leq c \left(1 + \frac{1}{\varepsilon^2} \right)$$

with an ε -independent constant c

The nonlinear term

$$m^\varepsilon(t, x) := \int_{\mathbb{R}^2} \mathcal{K}^\varepsilon(x - y) \rho^\varepsilon(t, x) \rho^\varepsilon(t, y) dy \quad \text{with } \mathcal{K}^\varepsilon(x) = \frac{x^{\otimes 2}}{|x|(|x| + \varepsilon)}$$

Lemma (Poupaud)

The families $\{\rho^\varepsilon(t)\}_{\varepsilon > 0}$ and $\{m^\varepsilon(t)\}_{\varepsilon > 0}$ are tightly bounded locally uniformly in t , and $\{\rho^\varepsilon(t)\}_{\varepsilon > 0}$ is tightly equicontinuous in t

Tight boundedness and equicontinuity of $\rho^\varepsilon(t) \implies$ compactness

$$\begin{aligned} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x, y) \rho^\varepsilon(t, x) \rho^\varepsilon(t, y) dx dy &\rightarrow \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \phi(x, y) \rho(t, x) \rho(t, y) dx dy \\ \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \phi(t, x) m^\varepsilon(t, x) dx dt &\rightarrow \int_{t_1}^{t_2} \int_{\mathbb{R}^2} \phi(t, x) m(t, x) dx dt \end{aligned} \quad \text{for all } \phi \in C_b([t_1, t_2] \times \mathbb{R}^2)$$

Defect measure

Atomic support

The limit is characterized by the pair (ρ, ν) , the atomic support of ρ is an at most countable set

Lemma (Poupaud 2002)

ν is symmetric, nonnegative, and satisfies

$$\text{tr}(\nu(t, x)) \leq \sum_{a \in S_{at}(\rho(t))} (\rho(t)(\{a\}))^2 \delta(x - a)$$

\mathcal{M} : spaces of Radon measures

\mathcal{M}_1^+ : subset of nonnegative bounded measures

For an interval $I \subset \mathbb{R}$, the set of time dependent admissible measures with diagonal defects is defined by

$$\mathcal{DM}^+(I; \mathbb{R}^2) = \left\{ (\rho, \nu) : \rho(t) \in \mathcal{M}_1^+(\mathbb{R}^2) \ \forall t \in I, \ \nu \in \mathcal{M}(I \times \mathbb{R}^2)^{2 \times 2} \right\}$$

ρ is tightly continuous with respect to t

ν is a nonnegative, symmetric, matrix valued measure

Limiting problem

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^2} \phi(t, x) j[\rho, \nu](t, x) \, dx \, dt \\ &= -\frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^4} (\phi(t, x) - \phi(t, y)) K(x - y) \rho(t, x) \rho(t, y) \, dx \, dy \, dt \\ & \quad - \frac{1}{4\pi} \int_0^T \int_{\mathbb{R}^2} \nu(t, x) \nabla \phi(t, x) \, dx \, dt \end{aligned}$$

for $\phi \in C_b^1((0, T) \times \mathbb{R}^2)$

Theorem (JD, C. Schmeiser 2009)

For every $T > 0$, ρ^ε converges tightly and uniformly in time to $\rho(t)$ and there exists $\nu(t)$ such that $(\rho, \nu) \in \mathcal{DM}^+((0, T); \mathbb{R}^2)$ is a generalized solution of

$$\partial_t \rho + \nabla \cdot (j[\rho, \nu] - \nabla \rho) = 0$$

$\rho(t = 0) = \rho_I$ holds in the sense of tight continuity



Strong formulation (formal) : an *ansatz*

- $\rho = \bar{\rho} + \hat{\rho}, \hat{\rho}(t, x) = \sum_{n \in N} M_n(t) \delta_n(t, x), \delta_n(t, x) = \delta(x - x_n(t))$
- $(\rho, \nu) \in \mathcal{DM}^+((0, T); \mathbb{R}^2)$
 $\implies \nu(t, x) = \sum_{n \in N} \nu_n(t) \delta_n(t, x), \text{tr}(\nu_n) \leq M_n^2$

$$j[\rho, \nu] = \bar{\rho} \nabla S_0[\bar{\rho} + \hat{\rho}] + \sum_n M_n \delta_n \nabla S_0 \left[\bar{\rho} + \sum_{m \neq n} M_m \delta_m \right] + \frac{1}{4\pi} \sum_n M_n \nu_n \nabla \delta_n$$

$$\begin{aligned} & \partial_t \bar{\rho} + \nabla \cdot (\bar{\rho} \nabla S_0[\bar{\rho}] - \nabla \bar{\rho}) + \nabla \bar{\rho} \cdot \nabla S_0[\hat{\rho}] \\ & \quad + \sum_n \delta_n (\dot{M}_n - \bar{\rho} M_n) \\ & - \sum_n M_n \nabla \delta_n \left(\dot{x}_n - \nabla S_0 \left[\bar{\rho} + \sum_{m \neq n} M_m \delta_m \right] \right) \\ & + \sum_n \left(\frac{1}{4\pi} \nu_n : \nabla^2 \delta_n - M_n \Delta \delta_n \right) = 0 \end{aligned}$$

The last row this gives

$$\nu_n = 4\pi M_n \text{id}$$

As a consequence of $\text{tr}(\nu_n) = 8\pi M_n \leq M_n^2$, point masses have to be at least 8π (there is only a finite number of them)

$$\partial_t \bar{\rho} + \nabla \cdot (\bar{\rho} \nabla S_0[\bar{\rho}] - \nabla \bar{\rho}) - \frac{1}{2\pi} \nabla \bar{\rho} \cdot \sum_n M_n \frac{x - x_n}{|x - x_n|^2} = 0$$

$$\dot{M}_n = \bar{\rho}(x = x_n) M_n$$

$$\dot{x}_n = \nabla S_0[\bar{\rho}](x = x_n) - \frac{1}{2\pi} \sum_{m \neq n} M_m \frac{x_n - x_m}{|x_n - x_m|^2}$$

Note that $\frac{d}{dt} \left(\int_{\mathbb{R}^2} \bar{\rho} dx + \sum_n M_n \right) = 0$

... Comparison with Velázquez' results 

Long time behaviour

Assume again

$$\nu(t, x) = 4\pi \operatorname{id} \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\}) \delta(x - a)$$

and

$$\int_{\mathbb{R}^2} |x|^2 \rho_I \, dx < \infty$$

With $\hat{M} = \sum_{a \in S_{at}(\rho(t))} \rho(t)(\{a\})$ and $\bar{M} = M - \hat{M}$

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} |x|^2 \rho \, dx &= 4M - \frac{1}{2\pi} \int_{\mathbb{R}^4} (1 - \chi_D) \rho \otimes \rho \, dy \, dx - \frac{1}{2\pi} \int_{\mathbb{R}^2} \operatorname{tr}(\nu) \, dx \\ &= \bar{M} \left(4 - \frac{M}{2\pi} - \frac{\hat{M}}{2\pi} \right) - \frac{1}{2\pi} \sum_{a \neq b, a, b \in S_{at}(\rho(t))} \rho(t)(\{a\}) \rho(t)(\{b\}) \end{aligned}$$

... compatible with Wasserstein's framework
 [Haškovec, Schmeiser 2009]

Local density profiles

For fixed t and $a \in S_{at}(\rho(t))$, let $\varepsilon\xi = x - a$ and $\varepsilon^2\rho^\varepsilon = R^\varepsilon$

$$\varepsilon^2\partial_t R^\varepsilon + \nabla_\xi \cdot (R^\varepsilon \nabla_\xi S_1[R^\varepsilon] - \nabla_\xi R^\varepsilon) = 0$$

R^ε is uniformly bounded, implying compactness of $\nabla_\xi S_1[R^\varepsilon]$. The L^∞ -weak* limit R of R^ε (take subsequences, formal) satisfies

$$\nabla_\xi \cdot (R \nabla_\xi S_1[R] - \nabla_\xi R) = 0$$

Observe that

$$\int_{\mathbb{R}^2} R(\xi) d\xi = \frac{1}{8\pi} \int_{\mathbb{R}^4} \frac{|\xi - \eta|}{|\xi - \eta| + 1} R(\xi) R(\eta) d\eta d\xi \leq \frac{1}{8\pi} \left(\int_{\mathbb{R}^2} R(\xi) d\xi \right)^2$$

This shows that either R vanishes or its mass is not smaller than 8π

Free energy

$$\begin{aligned} F_\varepsilon[\rho] &:= \int_{\mathbb{R}^2} \left(\rho \log \rho - \frac{1}{2} \rho S_\varepsilon[\rho] \right) dx \\ &= \int_{\mathbb{R}^2} \rho \log \rho dx + \frac{1}{4\pi} \int_{\mathbb{R}^4} \log(|x-y| + \varepsilon) \rho(x) \rho(y) dy dx \end{aligned}$$

and

$$\frac{d}{dt} F_\varepsilon[\rho^\varepsilon] = - \int_{\mathbb{R}^2} \rho^\varepsilon |\nabla(\log \rho^\varepsilon - S_\varepsilon[\rho^\varepsilon])|^2 dx$$

With an arbitrary $a \in \mathbb{R}^2$ and $R(\xi) = \varepsilon^2 \rho(a + \varepsilon \xi)$ we have

$$F_\varepsilon[\rho] = \left(2M - \frac{M^2}{4\pi} \right) \log \frac{1}{\varepsilon} + F_1[R]$$

Free energy

Lemma

Let $R \in L^1_+(\mathbb{R}^2)$ be radial, $\int_{\mathbb{R}^2} \log(1 + |x|) R(x) dx < \infty$, $M = \int_{\mathbb{R}^2} R dx$

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} \log(1 + |x - y|) R(y) dy \geq \frac{M}{4\pi} \log|x| \quad \forall x \in \mathbb{R}^2$$

$L^1_{+,M} := \{R \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} R d\xi = M\}$, $J_M := \inf_{R \in L^1_{+,M}} F_1[R] \geq -\infty$

Theorem

$J_M = -\infty$ for $M < 8\pi$, and $J_M > -\infty$ for $M \geq 8\pi$. If $M > 8\pi$, there exists a radial nonincreasing minimizer

The sub-critical range: how to get extinct

The sub-critical range: existence and free energy

$M = \int_{\mathbb{R}^2} n_0 \, dx \leq 8\pi$: global existence [W. Jäger, S. Luckhaus 1992],
[JD, B. Perthame 2004], [A. Blanchet, JD, B. Perthame 2006]

If u solves

$$\frac{\partial u}{\partial t} = \nabla \cdot [u (\nabla (\log u) - \nabla v)]$$

the free energy

$$F[u] := \int_{\mathbb{R}^2} u \log u \, dx - \frac{1}{2} \int_{\mathbb{R}^2} u v \, dx$$

satisfies

$$\frac{d}{dt} F[u(t, \cdot)] = - \int_{\mathbb{R}^2} u |\nabla (\log u) - \nabla v|^2 \, dx$$

(log HLS) inequality [E. Carlen, M. Loss 1992]:

F is bounded from below if $M \leq 8\pi$

... $M = 8\pi$ the critical case [A. Blanchet, J.A. Carrillo, N. Masmoudi 2008], [Blanchet et al.] cf. lecture of A. Blanchet

The existence setting for the subcritical regime

$$\begin{cases} \frac{\partial u}{\partial t} = \Delta u - \nabla \cdot (u \nabla v) & x \in \mathbb{R}^2, t > 0 \\ -\Delta v = u & x \in \mathbb{R}^2, t > 0 \\ u(\cdot, t = 0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

Initial conditions

$$n_0 \in L^1_+(\mathbb{R}^2, (1+|x|^2) dx), \quad n_0 \log n_0 \in L^1(\mathbb{R}^2, dx), \quad M := \int_{\mathbb{R}^2} n_0(x) dx < 8\pi$$

Global existence and mass conservation: $M = \int_{\mathbb{R}^2} u(x, t) dx$ for any $t \geq 0$, see [W. Jäger, S. Luckhaus], [A. Blanchet, JD, B. Perthame]

$$v = -\frac{1}{2\pi} \log |\cdot| * u$$

Time-dependent rescaling

$$u(x, t) = \frac{1}{R^2(t)} n\left(\frac{x}{R(t)}, \tau(t)\right) \quad \text{and} \quad v(x, t) = c\left(\frac{x}{R(t)}, \tau(t)\right)$$

with $R(t) = \sqrt{1 + 2t}$ and $\tau(t) = \log R(t)$

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n(\nabla c - x)) & x \in \mathbb{R}^2, t > 0 \\ c = -\frac{1}{2\pi} \log |\cdot| * n & x \in \mathbb{R}^2, t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

[A. Blanchet, JD, B. Perthame] Convergence in self-similar variables

$$\lim_{t \rightarrow \infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

means *intermediate asymptotics* in original variables:

$$\|u(x, t) - \frac{1}{R^2(t)} n_\infty\left(\frac{x}{R(t)}, \tau(t)\right)\|_{L^1(\mathbb{R}^2)} \searrow 0$$

The stationary solution in self-similar variables

$$n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} dx} = -\Delta c_\infty , \quad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty$$

- Radial symmetry [Y. Naito]
- Uniqueness [P. Biler, G. Karch, P. Laurençot, T. Nadzieja]
- As $|x| \rightarrow +\infty$, n_∞ is dominated by $e^{-(1-\epsilon)|x|^2/2}$ for any $\epsilon \in (0, 1)$ [A. Blanchet, JD, B. Perthame]
- Bifurcation diagram of $\|n_\infty\|_{L^\infty(\mathbb{R}^2)}$ as a function of M :

$$\lim_{M \rightarrow 0_+} \|n_\infty\|_{L^\infty(\mathbb{R}^2)} = 0$$

[D.D. Joseph, T.S. Lundgren] [JD, R. Stańczy]

The stationary solution when mass varies

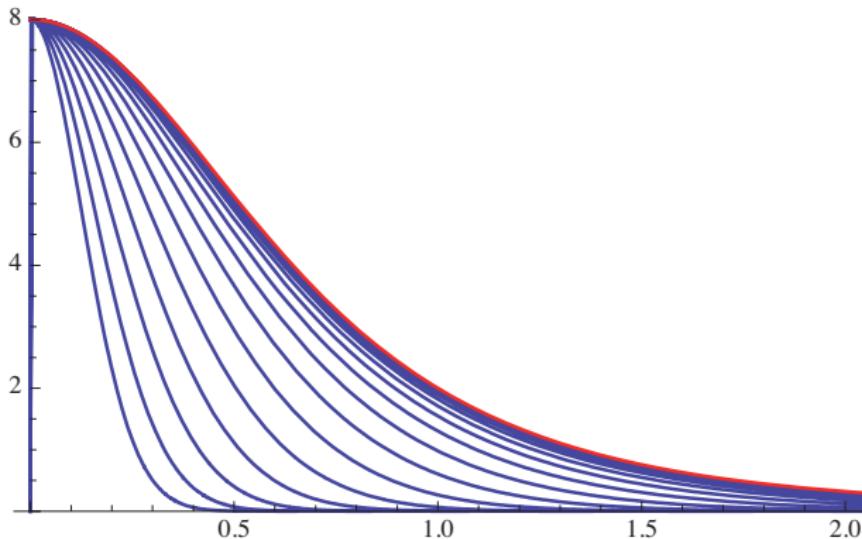


Figure: Representation of the solution appropriately scaled so that the 8π case appears as a limit (in red)

The free energy in self-similar variables

$$\frac{\partial n}{\partial t} = \nabla \left[n (\log n - x + \nabla c) \right]$$

$$F[n] := \int_{\mathbb{R}^2} n \log n \, dx + \int_{\mathbb{R}^2} \frac{1}{2} |x|^2 n \, dx - \frac{1}{2} \int_{\mathbb{R}^2} n c \, dx$$

satisfies

$$\frac{d}{dt} F[n(t, \cdot)] = - \int_{\mathbb{R}^2} n |\nabla(\log n) + x - \nabla c|^2 \, dx$$

A last remark on 8π and scalings: $n^\lambda(x) = \lambda^2 n(\lambda x)$

$$F[n^\lambda] = F[n] + \int_{\mathbb{R}^2} n \log(\lambda^2) \, dx + \int_{\mathbb{R}^2} \frac{\lambda^{-2}-1}{2} |x|^2 n \, dx + \frac{1}{4\pi} \int_{\mathbb{R}^2 \times \mathbb{R}^2} n(x) n(y) \log \frac{1}{\lambda} \, dx \, dy$$

$$F[n^\lambda] - F[n] = \underbrace{\left(2M - \frac{M^2}{4\pi} \right)}_{>0 \text{ if } M < 8\pi} \log \lambda + \frac{\lambda^{-2} - 1}{2} \int_{\mathbb{R}^2} |x|^2 n \, dx$$

Keller-Segel with subcritical mass in self-similar variables

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n(\nabla c - x)) & x \in \mathbb{R}^2, t > 0 \\ c = -\frac{1}{2\pi} \log |\cdot| * n & x \in \mathbb{R}^2, t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

$$\lim_{t \rightarrow \infty} \|n(\cdot, \cdot + t) - n_\infty\|_{L^1(\mathbb{R}^2)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\nabla c(\cdot, \cdot + t) - \nabla c_\infty\|_{L^2(\mathbb{R}^2)} = 0$$

$$n_\infty = M \frac{e^{c_\infty - |x|^2/2}}{\int_{\mathbb{R}^2} e^{c_\infty - |x|^2/2} dx} = -\Delta c_\infty, \quad c_\infty = -\frac{1}{2\pi} \log |\cdot| * n_\infty$$

First result: small mass case

Theorem (A. Blanchet, JD, M. Escobedo, J. Fernández)

There exists a positive constant M^ such that, for any initial data $n_0 \in L^2(n_\infty^{-1} dx)$ of mass $M < M^*$ satisfying the above assumptions, there is a unique solution $n \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty((\tau, \infty) \times \mathbb{R}^2)$ for any $\tau > 0$*

Moreover, there are two positive constants, C and δ , such that

$$\int_{\mathbb{R}^2} |n(t, x) - n_\infty(x)|^2 \frac{dx}{n_\infty} \leq C e^{-\delta t} \quad \forall t > 0$$

As a function of M , δ is such that $\lim_{M \rightarrow 0_+} \delta(M) = 1$

The condition $M \leq 8\pi$ is necessary and sufficient for the global existence of the solutions, but there are two extra smallness conditions in our proof:

- Uniform estimate: the *method of the trap*
- *Spectral gap* of a linearised operator \mathcal{L}

First step: the trap

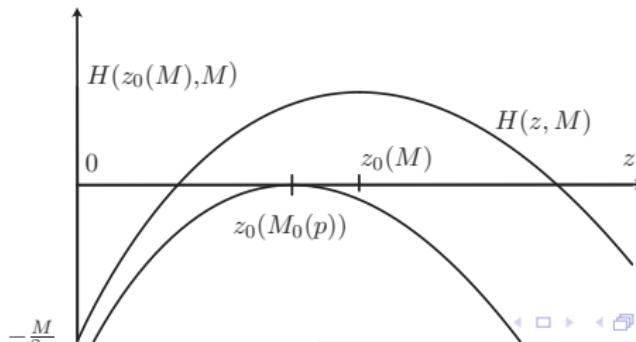
Lemma

For any $M < M_1$, there exists $C = C(M)$ such that, for any solution $u \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^\infty(\mathbb{R}_{\text{loc}}^+ \times \mathbb{R}^2)$

$$\|u(t)\|_{L^\infty(\mathbb{R}^2)} \leq C t^{-1} \quad \forall t > 0$$

The method of the trap... Duhamel's formula

$$H(t \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^2)}, M) \leq 0$$



Second step: L^p and H^1 estimates in the self-similar variables

$$\begin{cases} \frac{\partial n}{\partial t} = \Delta n - \nabla \cdot (n(\nabla c - x)) & x \in \mathbb{R}^2, t > 0 \\ c = -\frac{1}{2\pi} \log |\cdot| * n & x \in \mathbb{R}^2, t > 0 \\ n(\cdot, t=0) = n_0 \geq 0 & x \in \mathbb{R}^2 \end{cases}$$

With $K = K(x) = e^{-|x|^2/2}$, rewrite the equation for n as

$$\frac{\partial n}{\partial t} - \frac{1}{K} \nabla \cdot (K \nabla n) = -\nabla c \cdot \nabla n + 2n + n^2$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |n|^2 K dx + \int_{\mathbb{R}^2} |\nabla n|^2 K dx \\ = - \int_{\mathbb{R}^2} n \nabla c \cdot \nabla n K dx + 2 \int_{\mathbb{R}^2} n^2 K dx + \int_{\mathbb{R}^2} n^3 K dx \\ \leq \varepsilon \int_{\mathbb{R}^2} |\nabla n|^2 K dx + C \end{aligned}$$

Third step: linearization and spectral gap

Proposition

For any $M \in (0, M_2)$, for any $f \in H^1(n_\infty dx)$ such that

$$\int_{\mathbb{R}^2} f n_\infty dx = 0 \implies \int_{\mathbb{R}^2} |\nabla f|^2 n_\infty dx \geq \frac{1}{\Lambda(M)} \int_{\mathbb{R}^2} |f|^2 n_\infty dx$$

for some $\Lambda(M) > 0$ and $\lim_{M \rightarrow 0_+} \Lambda(M) = 1$

Corollary

If $M < M_2$, then any solution n is bounded in

$$L^\infty(\mathbb{R}^+, L^2(n_\infty^{-1} dx)) \cap L^\infty((\tau, \infty), H^1(n_\infty^{-1} dx))$$

for any $\tau > 0$

Fourth step: collecting the estimates

$$\begin{cases} \frac{\partial f}{\partial t} - \mathcal{L}(t, x, f, g) = -\frac{1}{n_\infty} \nabla \cdot [f n_\infty \nabla (g c_\infty)] & x \in \mathbb{R}^2, t > 0 \\ -\Delta(c_\infty g) = f n_\infty & x \in \mathbb{R}^2, t > 0 \end{cases}$$

Multiply by $f n_\infty$ and integrate by parts

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |f|^2 n_\infty \, dx + \int_{\mathbb{R}^2} |\nabla f|^2 n_\infty \, dx \\ &= \underbrace{\int_{\mathbb{R}^2} \nabla f \cdot \nabla (g c_\infty) n_\infty \, dx}_{=I} + \underbrace{\int_{\mathbb{R}^2} \nabla f \cdot \nabla (g c_\infty) f n_\infty \, dx}_{=II} \end{aligned}$$

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^2} |f|^2 n_\infty \, dx \leq -[1 - \gamma(M)] \int_{\mathbb{R}^2} |\nabla f|^2 n_\infty \, dx \leq -\frac{[1 - \gamma(M)]}{\Lambda(M)} \int_{\mathbb{R}^2} |f|^2 n_\infty \, dx$$

3 – Keller-Segel model: functional inequalities, spectrum of the linearized operator, symmetrization, nonlinear estimates, rates of convergence for subcritical masses

A parametrization of the solutions and the linearized operator

[J. Campos, JD]

$$-\Delta c = M \frac{e^{-\frac{1}{2}|x|^2+c}}{\int_{\mathbb{R}^2} e^{-\frac{1}{2}|x|^2+c} dx}$$

Solve

$$-\phi'' - \frac{1}{r} \phi' = e^{-\frac{1}{2}r^2+\phi}, \quad r > 0$$

with initial conditions $\phi(0) = a$, $\phi'(0) = 0$ and get with $r = |x|$

$$M(a) := 2\pi \int_{\mathbb{R}^2} e^{-\frac{1}{2}r^2+\phi_a} dx$$

$$n_a(x) = M(a) \frac{e^{-\frac{1}{2}r^2+\phi_a(r)}}{2\pi \int_{\mathbb{R}^2} r e^{-\frac{1}{2}r^2+\phi_a} dx} = e^{-\frac{1}{2}r^2+\phi_a(r)}$$

Mass

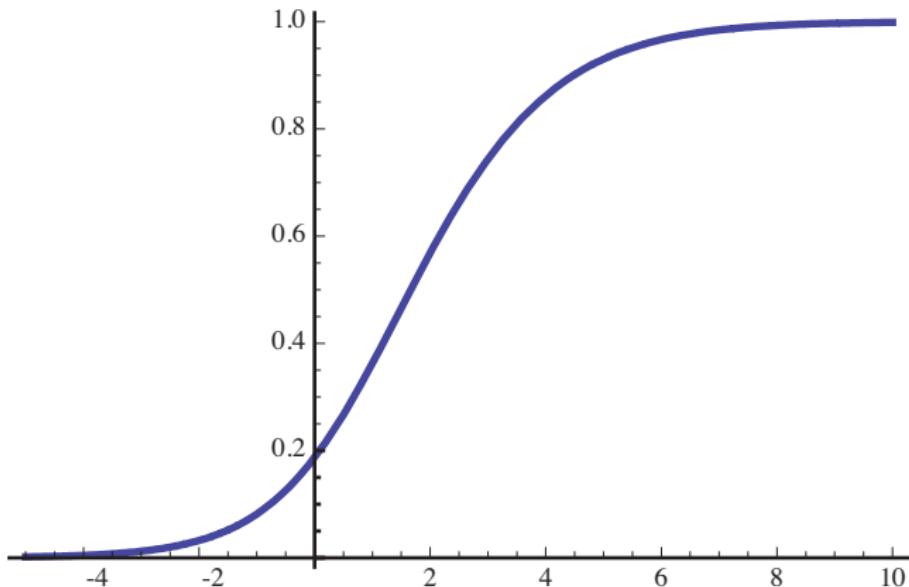


Figure: The mass can be computed as $M(a) = 2\pi \int_0^\infty n_a(r) r dr$. Plot of $a \mapsto M(a)/8\pi$

Bifurcation diagram

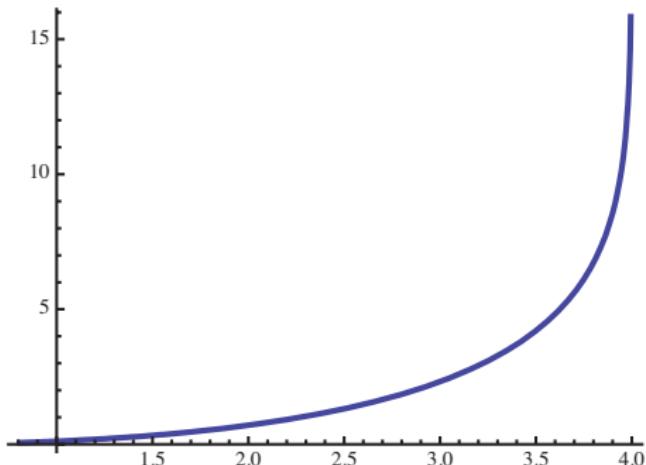


Figure: The bifurcation diagram can be parametrized by
 $a \mapsto (\frac{1}{2\pi} M(a), \|c_a\|_{L^\infty(\mathbb{R}^d)})$ with $\|c_a\|_{L^\infty(\mathbb{R}^d)} = c_a(0) = a - b(a)$
(cf. Keller-Segel system in a ball with no flux boundary conditions)

Spectrum of \mathcal{L} (lowest eigenvalues only)

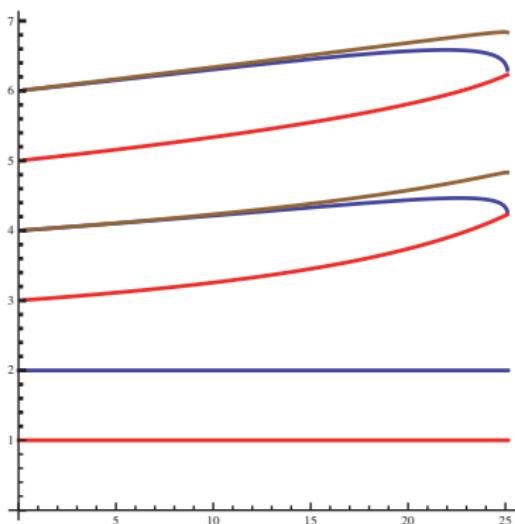


Figure: The lowest eigenvalues of $-\mathcal{L} = (-\Delta)^{-1}(n_a f)$ (shown as a function of the mass) are 0, 1 and 2, thus establishing that the spectral gap of $-\mathcal{L}$ is 1

[A. Blanchet, JD, M. Escobedo, J. Fernández], [J. Campos, JD],
[V. Calvez, J.A. Carrillo], [J. Bedrossian, N. Masmoudi]

Spectral analysis in the functional framework determined by the relative entropy method

Simple eigenfunctions

Kernel Let $f_0 = \frac{\partial}{\partial M} c_\infty$ be the solution of

$$-\Delta f_0 = n_\infty f_0$$

and observe that $g_0 = f_0/c_\infty$ is such that

$$\frac{1}{n_\infty} \nabla \cdot (n_\infty \nabla (f_0 - c_\infty g_0)) =: \mathcal{L} f_0 = 0$$

Lowest non-zero eigenvalues $f_1 := \frac{1}{n_\infty} \frac{\partial n_\infty}{\partial x_1}$ associated with
 $g_1 = \frac{1}{c_\infty} \frac{\partial c_\infty}{\partial x_1}$ is an eigenfunction of \mathcal{L} , such that $-\mathcal{L} f_1 = f_1$

With $D := x \cdot \nabla$, let $f_2 = 1 + \frac{1}{2} D \log n_\infty = 1 + \frac{1}{2n_\infty} D n_\infty$. Then

$$-\Delta(D c_\infty) + 2 \Delta c_\infty = D n_\infty = 2(f_2 - 1) n_\infty$$

and so $g_2 := \frac{1}{c_\infty} (-\Delta)^{-1}(n_\infty f_2)$ is such that $-\mathcal{L} f_2 = 2 f_2$

Functional setting...

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]

$$F[n] := \int_{\mathbb{R}^2} n \log \left(\frac{n}{n_\infty} \right) dx - \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty)(c - c_\infty) dx \geq 0$$

achieves its minimum for $n = n_\infty$

$$Q_1[f] = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} F[n_\infty(1 + \varepsilon f)] \geq 0$$

if $\int_{\mathbb{R}^2} f n_\infty dx = 0$. Notice that f_0 generates the kernel of Q_1

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty dx)$ such that

$\int_{\mathbb{R}^2} f n_\infty dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f n_\infty)|^2 n_\infty dx = \int_{\mathbb{R}^2} |\nabla(g c_\infty)|^2 n_\infty dx \leq \int_{\mathbb{R}^2} |f|^2 n_\infty dx$$



... and eigenvalues

With g such that $-\Delta(g c_\infty) = f n_\infty$, Q_1 determines a scalar product

$$\langle f_1, f_2 \rangle := \int_{\mathbb{R}^2} f_1 f_2 n_\infty dx - \int_{\mathbb{R}^2} f_1 n_\infty (g_2 c_\infty) dx$$

on the orthogonal space to f_0 in $L^2(n_\infty dx)$

$$Q_2[f] := \int_{\mathbb{R}^2} |\nabla(f - g c_\infty)|^2 n_\infty dx \quad \text{with} \quad g = -\frac{1}{c_\infty} \frac{1}{2\pi} \log |\cdot| * (f n_\infty)$$

is a positive quadratic form, whose polar operator is the self-adjoint operator \mathcal{L}

$$\langle f, \mathcal{L} f \rangle = Q_2[f] \quad \forall f \in \mathcal{D}(\mathcal{L}_2)$$

Lemma (J. Campos, JD)

\mathcal{L} has pure discrete spectrum and its lowest eigenvalue is 1

Linearized Keller-Segel theory



$$\mathcal{L} f = \frac{1}{n_\infty} \nabla \cdot (n_\infty \nabla (f - c_\infty g))$$

Corollary (J. Campos, JD)

$$\langle f, f \rangle \leq \langle \mathcal{L} f, f \rangle$$

The linearized problem takes the form

$$\frac{\partial f}{\partial t} = \mathcal{L} f$$

where \mathcal{L} is a self-adjoint operator on the orthogonal of f_0 equipped with $\langle \cdot, \cdot \rangle$. A solution of

$$\frac{d}{dt} \langle f, f \rangle = -2 \langle \mathcal{L} f, f \rangle$$

has therefore exponential decay

More functional inequalities

A subcritical logarithmic HLS inequality

Recall that

Lemma (A. Blanchet, JD, B. Perthame)

Sub-critical HLS inequality [A. Blanchet, JD, B. Perthame]

$$F[n] := \int_{\mathbb{R}^2} n \log \left(\frac{n}{n_\infty} \right) dx - \frac{1}{2} \int_{\mathbb{R}^2} (n - n_\infty)(c - c_\infty) dx \geq 0$$

achieves its minimum for $n = n_\infty$

Lemma (J. Campos, JD)

Poincaré type inequality For any $f \in H^1(\mathbb{R}^2, n_\infty dx)$ such that

$\int_{\mathbb{R}^2} f n_\infty dx = 0$, we have

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}(f n_\infty)|^2 n_\infty dx = \int_{\mathbb{R}^2} |\nabla(g c_\infty)|^2 n_\infty dx \leq \int_{\mathbb{R}^2} |f|^2 n_\infty dx$$

... Legendre duality

A new Onofri type inequality



Theorem (J. Campos, JD)

For any $M \in (0, 8\pi)$, if $n_\infty = M \frac{e^{c_\infty - \frac{1}{2} |x|^2}}{\int_{\mathbb{R}^2} e^{c_\infty - \frac{1}{2} |x|^2} dx}$ with $c_\infty = (-\Delta)^{-1} n_\infty$,
 $d\mu_M = \frac{1}{M} n_\infty dx$, we have the inequality

$$\log \left(\int_{\mathbb{R}^2} e^\phi d\mu_M \right) - \int_{\mathbb{R}^2} \phi d\mu_M \leq \frac{1}{2M} \int_{\mathbb{R}^2} |\nabla \phi|^2 dx \quad \forall \phi \in \mathcal{D}_0^{1,2}(\mathbb{R}^2)$$

Corollary (J. Campos, JD)

The following Poincaré inequality holds

$$\int_{\mathbb{R}^2} |\psi - \bar{\psi}|^2 n_\infty dx \leq \int_{\mathbb{R}^2} |\nabla \psi|^2 dx \quad \text{where} \quad \bar{\psi} = \int_{\mathbb{R}^2} \psi d\mu_M$$

An improved interpolation inequality (coercivity estimate)

Lemma (J. Campos, JD)

For any $f \in L^2(\mathbb{R}^2, n_\infty dx)$ such that $\int_{\mathbb{R}^2} f f_0 n_\infty dx = 0$ holds, we have

$$\begin{aligned} & -\frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f(x) n_\infty(x) \log|x-y| f(y) n_\infty(y) dx dy \\ & \leq (1-\varepsilon) \int_{\mathbb{R}^2} |f|^2 n_\infty dx \end{aligned}$$

for some $\varepsilon > 0$, where $g c_\infty = G_2 * (f n_\infty)$ and, if $\int_{\mathbb{R}^2} f n_\infty dx = 0$ holds,

$$\int_{\mathbb{R}^2} |\nabla(g c_\infty)|^2 dx \leq (1-\varepsilon) \int_{\mathbb{R}^2} |f|^2 n_\infty dx$$

The nonlinear Keller-Segel model, a functional analysis approach

Exponential convergence for any mass $M \leq 8\pi$



If $n_{0,*}(\sigma)$ stands for the symmetrized function associated to n_0 ,
assume that for any $s \geq 0$

$$(H) \quad \exists \varepsilon \in (0, 8\pi - M) \quad \text{such that} \quad \int_0^s n_{0,*}(\sigma) d\sigma \leq \int_{B(0, \sqrt{s/\pi})} n_{\infty, M+\varepsilon}(x) dx$$

Theorem (J. Campos, JD)

Under the above assumption, if $n_0 \in L^2_+(n_\infty^{-1} dx)$ and
 $M := \int_{\mathbb{R}^2} n_0 dx < 8\pi$, then any solution with initial datum n_0 is such that

$$\int_{\mathbb{R}^2} |n(t, x) - n_\infty(x)|^2 \frac{dx}{n_\infty(x)} \leq C e^{-2t} \quad \forall t \geq 0$$

for some positive constant C , where n_∞ is the unique stationary solution
with mass M

Sketch of the proof

- ➊ [J. Campos, JD] Uniform convergence of $n(t, \cdot)$ to n_∞ can be established for any $M \in (0, 8\pi)$ by an adaptation of the symmetrization techniques of [J.I. Díaz, T. Nagai, J.M. Rakotoson]
- ➋ Uniform estimates (with no rates) easily result
- ➌ Estimates based on Duhammel's formula inspired by [M. Escobedo, E. Zuazua] allow to prove uniform convergence
- ➍ Spectral estimates can be incorporated to the relative entropy approach
- ➎ Exponential convergence of the relative entropy follows

Step 1: symmetrization (1/2)

To any measurable function $u : \mathbb{R}^2 \mapsto [0, +\infty)$, we associate the distribution function defined by $\mu(t, \tau) := |\{u > \tau\}|$ and its decreasing rearrangement given by

$$u_* : [0, +\infty) \rightarrow [0, +\infty], \quad s \mapsto u_*(s) = \inf\{\tau \geq 0 : \mu(t, \tau) \leq s\}.$$

- 1 For every measurable function $F : \mathbb{R}^+ \mapsto \mathbb{R}^+$, we have

$$\int_{\mathbb{R}^2} F(u) dx = \int_{\mathbb{R}^+} F(u_*) ds$$

- 2 If $u \in W^{1,q}(0, T; L^p(\mathbb{R}^N))$ is a nonnegative function, with $1 \leq p < \infty$ and $1 \leq q \leq \infty$, then $u_* \in W^{1,q}(0, T; L^p(0, \infty))$ and the formula

$$\int_0^{\mu(t, \tau)} \frac{\partial u_*}{\partial t}(t, \sigma) d\sigma = \int_{\{u(t, \cdot) > \tau\}} \frac{\partial u}{\partial t}(t, x) dx$$

holds for almost every $t \in (0, T)$ [J.I. Díaz, T. Nagai, J.M. Rakotoson]

Step 1: symmetrization (2/2)

Lemma

If n is a solution, then the function

$$k(t, s) := \int_0^s n_*(t, \sigma) d\sigma$$

satisfies $k \in L^\infty([0, +\infty) \times (0, +\infty)) \cap H^1\left([0, +\infty); W_{\text{loc}}^{1,p}(0, +\infty)\right)$
 $\cap L^2\left([0, +\infty); W_{\text{loc}}^{2,p}(0, +\infty)\right)$ and

$$\begin{cases} \frac{\partial k}{\partial t} - 4\pi s \frac{\partial^2 k}{\partial s^2} - (k + 2s) \frac{\partial k}{\partial s} \leq 0 & \text{a.e. in } (0, +\infty) \times (0, +\infty) \\ k(t, 0) = 0, \quad k(t, +\infty) = \int_{\mathbb{R}^2} n_0 dx \quad \text{for } t \in (0, +\infty) \\ k(0, s) = \int_0^s (n_0)_* d\sigma & \text{for } s \geq 0 \end{cases}$$

Step 2: Uniform estimates

Proposition (J.I. Díaz, T. Nagai, J.M. Rakotoson)

Let f, g be two continuous functions on $Q = \mathbb{R}^+ \times (0, +\infty)$ such that ...

$$\begin{cases} \frac{\partial f}{\partial t} - 4\pi s \frac{\partial^2 f}{\partial s^2} - (f + 2s) \frac{\partial f}{\partial s} \leq \frac{\partial g}{\partial t} - 4\pi s \frac{\partial^2 g}{\partial s^2} - (g + 2s) \frac{\partial g}{\partial s} & \text{a.e. in } Q \\ f(t, 0) = 0 = g(t, 0) \quad \text{and} \quad f(t, +\infty) \leq g(t, +\infty) \text{ for any } t \in (0, +\infty) \\ f(0, s) \leq g(0, s) \text{ for } s \geq 0, \text{ and } g(t, s) \geq 0 \text{ in } Q \end{cases}$$

then $f \leq g$ on Q

Corollary

Assume that $n_0 \in L^2_+(n_\infty^{-1} dx)$ satisfies (H) and $M := \int_{\mathbb{R}^2} n_0 \, dx < 8\pi$. Then there exist positive constants $C_1 = C_1(M, p)$ and $C_2 = C_2(M, p)$ such that

$$\|n\|_{L^p(\mathbb{R}^2)} \leq C_1 \quad \text{and} \quad \|\nabla c\|_{L^\infty(\mathbb{R}^2)} \leq C_2$$



Step 3: Estimates based on Duhammel's formula

Consider the kernel associated to the Fokker-Planck equation

$$K(t, x, y) := \frac{1}{2\pi(1 - e^{-2t})} e^{-\frac{1}{2} \frac{|x - e^{-t}y|^2}{1 - e^{-2t}}} \quad x \in \mathbb{R}^2, \quad y \in \mathbb{R}^2, \quad t > 0$$

If n is a solution, then

$$n(t, x) = \int_{\mathbb{R}^2} K(t, x, y) n_0(y) dy + \int_0^t \int_{\mathbb{R}^2} \nabla_x K(t-s, x, y) \cdot n(s, y) \nabla c(s, y) dy ds$$

Corollary

Assume that n is a solution. Then

$$\lim_{t \rightarrow \infty} \|n(t, \cdot) - n_\infty\|_{L^p(\mathbb{R}^d)} = 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \|\nabla c(t, \cdot) - \nabla c_\infty\|_{L^q(\mathbb{R}^d)} = 0$$

for any $p \in [1, \infty]$ and any $q \in [2, \infty]$

Step 4: Spectral estimates can be incorporated

With $Q_1[f] = \langle f, f \rangle$ and $Q_2[f] = \langle f, \mathcal{L} f \rangle$

- ① For any function f in the orthogonal of the kernel of \mathcal{L} , we have

$$Q_1[f] \leq Q_2[f]$$

- ② For any radial function $f \in \mathcal{D}(L_2)$, we have

$$2 Q_1[f] \leq Q_2[f]$$

Step 5: Exponential convergence of the relative entropy

$$\frac{\partial f}{\partial t} = \mathcal{L} f - \frac{1}{n_\infty} \nabla [n_\infty f \nabla (g c_\infty)]$$

$$\frac{d}{dt} Q_1[f(t, \cdot)] = -2 Q_2[f(t, \cdot)] + \int_{\mathbb{R}^2} \nabla(f - g c_\infty) f n_\infty \cdot \nabla(g c_\infty) dx$$

$$\frac{d}{dt} Q_1[f(t, \cdot)] \leq -2 Q_2[f(t, \cdot)] + \delta(t, \varepsilon) \sqrt{Q_1[f(t, \cdot)] Q_2[f(t, \cdot)]}$$

$$Q_1[f(t, \cdot)] \leq \mathcal{Q} \quad \forall t \geq 0$$

$$\frac{d}{dt} Q_1[f(t, \cdot)] \leq -Q_1[f(t, \cdot)] \left[2 - \delta(t, \varepsilon) \left(Q_1[f(t, \cdot)]^{\frac{1-\varepsilon}{2-\varepsilon}} + Q_1[f(t, \cdot)]^{\frac{1}{2+\varepsilon}} \right) \right]$$

As a consequence, we finally get that

$$\limsup_{t \rightarrow \infty} e^{2t} Q_1[f(t, \cdot)] < \infty$$

Some key ideas

- ➊ Lyapunov / Entropy functionals and functional inequalities
 - ➋ Linearization and best constants
 - ➌ Functional framework for linearized operators can be deduced from the entropy functional
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- ➍ [G. Egaña, S. Mischler, personnal communication, 2013]
 - weak notion of solution (based on free energy estimates)
 - uniqueness, smoothing
 - linearized and nonlinear stability in rescaled variables and exponential convergence under weaker assumptions

Thank you for your attention !