Stability estimates in critical functional inequalities

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Geometric aspects of nonlinear partial differential equations

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Outline

An introduction to entropy methods on the sphere with application to stability results (subcritical inequalities)

An introduction to entropy methods on the sphere with application to stability results

Subcritical interpolation inequalities on the sphere

▷ Gagliardo-Nirenberg-Sobolev and logarithmic Sobolev inequalities
▷ The Bakry-Emery or *carré du champ* method; extension to nonlinear diffusions
▷ Improved inequalities (under an orthogonality constraint)
▷ Stability results
▷ Logarithmic Sobolev inequality:details

Subcritical interpolation inequalities on the sphere

With the uniform probability measure $d\mu$ on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, let us consider the following family of *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} \ge \frac{d}{p-2}\left(\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right) \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \quad \text{(GNS)}$$

- $d=1$ or $d=2$: $p \in [1,2) \cup (2,+\infty)$
- $d \ge 3$: $p \in [1,2) \cup (2,2^*]$ with $2^* = 2\,d/(d-2)$

$p=2$: Poincaré; $p=2^*$: Sobolev;
$p \to 2$: *logarithmic Sobolev inequality*

$$\int_{\mathbb{S}^d} |\nabla u|^2\, d\mu \ge \frac{d}{2}\int_{\mathbb{S}^d} u^2 \log\left(\frac{u^2}{\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \quad \forall\, u \in \mathrm{H}^1(\mathbb{S}^d, d\mu) \quad \text{(LSI)}$$

[Gidas, Spruck, 1981], [Bidaut-Véron, Véron, 1991]
[Bakry, Emery, 1985], [Demange, 2088], [JD, Esteban, Kowalczyk, Loss]

The Bakry-Emery method on the sphere

Entropy functional

$$\mathcal{E}_p[\rho] := \frac{1}{p-2} \left(\int_{\mathbb{S}^d} \rho^{\frac{2}{p}} \, d\mu - \left(\int_{\mathbb{S}^d} \rho \, d\mu \right)^{\frac{2}{p}} \right) \quad \text{if} \quad p \neq 2$$

$$\mathcal{E}_2[\rho] := \int_{\mathbb{S}^d} \rho \log \left(\frac{\rho}{\|\rho\|_{\mathrm{L}^1(\mathbb{S}^d)}} \right) d\mu$$

Fisher information functional

$$\mathcal{I}_p[\rho] := \int_{\mathbb{S}^d} |\nabla \rho^{\frac{1}{p}}|^2 \, d\mu$$

[Bakry, Emery, 1985] *carré du champ* method: use the heat flow

$$\frac{\partial \rho}{\partial t} = \Delta \rho$$

and observe that $\frac{d}{dt}\mathcal{E}_p[\rho] = -\mathcal{I}_p[\rho]$

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0 \quad \Longrightarrow \quad \mathcal{I}_p[\rho] \geq d\,\mathcal{E}_p[\rho]$$

with $\rho = |u|^p$, if $p \leq 2^\# := \frac{2\,d^2+1}{(d-1)^2}$

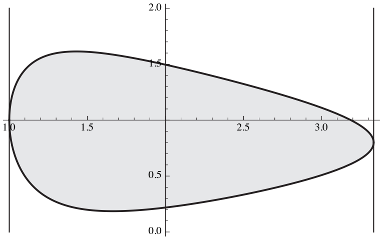
The evolution under the fast diffusion flow

To overcome the limitation $p \leq 2^{\#}$, one can consider a nonlinear diffusion of fast diffusion / porous medium type

$$\frac{\partial \rho}{\partial t} = \Delta \rho^m$$

[Demange], [JD, Esteban, Kowalczyk, Loss]: for any $p \in [1, 2^*]$

$$\mathcal{K}_p[\rho] := \frac{d}{dt}\Big(\mathcal{I}_p[\rho] - d\,\mathcal{E}_p[\rho]\Big) \leq 0$$



(p, m) *admissible region,* $d = 5$

Computation of the admissible region

With $\rho=|u|^{\beta\,p}$ and $m=1+\frac{2}{p}\left(\frac{1}{\beta}-1\right)$, $\kappa=\beta\,(p-2)+1$, with the *trace free Hessian*

$$\mathrm{L}u:=\mathrm{H}u-\frac{1}{d}\,(\Delta u)\,g_d$$

and the trace free tensor

$$\mathrm{M}u:=\frac{\nabla u\otimes\nabla u}{u}-\frac{1}{d}\,\frac{|\nabla u|^2}{u}\,g_d$$

we have

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho]-\,d\,\mathcal{E}_p[\rho]\Big)=-\frac{d}{d-1}\left(\mathsf{a}\,\|\mathrm{L}u\|^2-2\,\mathsf{b}\,\mathrm{L}u:\mathrm{M}u+\mathsf{c}\,\|\mathrm{M}u\|^2\right)$$

$$\mathsf{a}=1\,,\quad \mathsf{b}=(\kappa+\beta-1)\,\frac{d-1}{d+2}\,,\quad \mathsf{c}=(\kappa+\beta-1)\,\frac{d}{d+2}+\kappa\,(\beta-1)$$

so that the *admissible region* is defined by $\mathsf{b}^2-\mathsf{a}\,\mathsf{c}\le0$

Improved inequalities

▷ the monotonicity result

$$\frac{d}{dt}\Big(\mathcal{I}_p[\rho]-d\,\mathcal{E}_p[\rho]\Big)=-\frac{d}{d-1}\,\mathsf{a}\,\left\|\mathrm{L}u-\frac{\mathsf{b}}{\mathsf{a}}\,\mathrm{M}\right\|^2-\frac{d}{d-1}\,\left(\mathsf{c}-\frac{\mathsf{b}^2}{\mathsf{a}}\right)\|\mathrm{M}u\|^2$$

▷ improved inequalities [Arnold, JD, 2005], [JD, Nazaret, Savaré, 2008], [JD, Toscani, 2013], [JD, Esteban, Kowalczyk, Loss, 2014], [JD, Esteban, 2020]

$$\mathcal{I}_p[\rho]\ge d\,\Phi\Big(\mathcal{E}_p[\rho]\Big)$$

for some convex Φ with $\Phi(0)=0$ and $\Phi'(0)=1$

▷ Application: with $d\ge2$, $2-p\neq\gamma:=\left(\frac{d-1}{d+2}\right)^2(p-1)\,(2^\#-p)>0$, we have

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\ge\frac{d}{2-p-\gamma}\left(\|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\|u\|^{2-\frac{2\gamma}{2-p}}_{\mathrm{L}^p(\mathbb{S}^d)}\,\|u\|^{\frac{2\gamma}{2-p}}_{\mathrm{L}^2(\mathbb{S}^d)}\right)\quad\forall\,u\in\mathrm{H}^1(\mathbb{S}^d)$$

Subcritical interpolation inequalities on the sphere: stability

[Frank, 2022] *Degenerate stability of some Sobolev inequalities*
Annales IHP C (2022), arXiv:2107.11608

If $d \geq 2$ and $2 < p < 2^*$, there is $c_{d,p} > 0$ such that, if $\int_{\mathbb{S}^d} u \, d\mu = 1$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d \, \frac{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2} \geq c_{d,p} \, \frac{\left(\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2} \, \|u-1\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)^2}{\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \frac{d}{p-2} \, \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

An optimal result: take $u(x) = 1 + \varepsilon \, z$

Theorem

If $d \geq 2$ and $2 < p < 2^*$, there is $\mathscr{C}_{d,p} > 0$ such that for any $u \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$

$$\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + d \, \frac{\|u\|^2_{\mathrm{L}^p(\mathbb{S}^d)} - \|u\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{p-2} \geq \mathscr{C}_{d,p} \int_{\mathbb{S}^d} |\nabla u^\perp|^2 \, d\mu$$

with optimal constant $\mathscr{C}_{d,p} = \frac{2\,d - p\,(d-2)}{2\,d\,(d+p)}$ *[Brigati, JD, Simonov]*

LSI: improved inequality under constraint

Theorem

Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$ such that $\int_{\mathbb{S}^d} x\, F\, d\mu = 0$

$$\int_{\mathbb{S}^d} |\nabla F|^2\, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \geq \mathscr{C}_d \int_{\mathbb{S}^d} |\nabla F|^2\, d\mu$$

with optimal constant $\mathscr{C}_d = \frac{2}{d+2}$ if $d \geq 2$ and $\mathscr{C}_1 = \frac{3}{4}$

[Brigati, JD, Simonov]

LSI: without orthogonality constraint

Let

$$\gamma = \frac{4\,d-1}{(d+2)^2} \quad \text{if} \quad d \geq 2\,, \qquad \text{and} \quad \gamma = \frac{1}{3} \quad \text{if} \quad d = 1$$

$$\psi(t) := t - \frac{1}{\gamma}\,\log(1+\gamma\,t) \quad \forall\,t \geq 0$$

Proposition

For any $F \in \mathrm{H}^1(\mathbb{S}^d)$ we have

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu - \frac{d}{2}\int_{\mathbb{S}^d} F^2\,\log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \geq d\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\,\psi\left(\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right)$$

[Brigati, JD, Simonov]

$$\int_{\mathbb{S}^d} |\nabla F|^2\,d\mu - \frac{d}{2}\int_{\mathbb{S}^d} F^2\,\log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \geq \frac{d}{2}\,\frac{\gamma\,\|\nabla F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\gamma\,\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}$$

LSI: A general stability result

Theorem

Let $d \geq 1$. For any $F \in \mathrm{H}^1(\mathbb{S}^d, d\mu)$, we have

$$\int_{\mathbb{S}^d} |\nabla F|^2 \, d\mu - \frac{d}{2} \int_{\mathbb{S}^d} F^2 \log\left(\frac{F^2}{\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}}\right) d\mu \geq \mathscr{S}_d \left(\frac{\|\nabla \Pi F\|^4_{\mathrm{L}^2(\mathbb{S}^d)}}{\|\nabla \Pi F\|^2_{\mathrm{L}^2(\mathbb{S}^d)} + \|\Pi F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}} + \|\nabla F^\perp\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right)$$

for some stability constant $\mathscr{S}_d > 0$

[Brigati, JD, Simonov]

Sobolev and Hardy-Littlewood-Sobolev inequalities

▷ Stability in a weaker norm, with explicit constants

▷ From duality to improved estimates based on Yamabe's flow

Sobolev and HLS

As it has been noticed by E. Lieb, Sobolev's inequality in $\mathbb{R}^d$, $d \geq 3$,

$$\|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \leq \mathsf{S}_d \, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall \, u \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \qquad \text{(S)}$$

and the Hardy-Littlewood-Sobolev inequality

$$\mathsf{S}_d \, \|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} \geq \int_{\mathbb{R}^d} v \, (-\Delta)^{-1} v \, dx \quad \forall \, v \in \mathcal{L}^{\frac{2d}{d+2}}(\mathbb{R}^d) \qquad \text{(HLS)}$$

are dual of each other. Here S_d is the Aubin-Talenti constant and $2^* = \frac{2d}{d-2}$

Improved Sobolev inequality by duality

Theorem

[JD, Jankowiak] Assume that $d \geq 3$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} < 1$ such that

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx$$
$$\leq \mathcal{C} \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)$$

for any $w \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

Proof: the completion of a square

Integrations by parts show that

$$\int_{\mathbb{R}^d} |\nabla(-\Delta)^{-1}\,v|^2\,dx = \int_{\mathbb{R}^d} v\,(-\Delta)^{-1}\,v\,dx$$

and, if $v = u^q$ with $q = \frac{d+2}{d-2}$,

$$\int_{\mathbb{R}^d} \nabla u \cdot \nabla(-\Delta)^{-1}\,v\,dx = \int_{\mathbb{R}^d} u\,v\,dx = \int_{\mathbb{R}^d} u^{2^*}\,dx$$

Hence the expansion of the square

$$0 \le \int_{\mathbb{R}^d} \left| \mathsf{S}_d\,\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{4}{d-2}}\,\nabla u - \nabla(-\Delta)^{-1}\,v \right|^2 dx$$

shows that (with $\mathcal{C} = 1$)

$$0 \le \mathsf{S}_d\,\|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^{\frac{8}{d-2}} \left(\mathsf{S}_d\,\|\nabla u\|_{\mathrm{L}^2(\mathbb{R}^d)}^2 - \|u\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}^2 \right) - \left(\mathsf{S}_d\,\|u^q\|_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}^2 - \int_{\mathbb{R}^d} u^q\,(-\Delta)^{-1}\,u^q\,dx \right)$$

Using a nonlinear flow to relate Sobolev and HLS

Consider the *fast diffusion* equation

$$\frac{\partial v}{\partial t} = \Delta v^m \quad t > 0\,, \quad x \in \mathbb{R}^d \tag{FDE}$$

If we define $\mathsf{H}(t) := \mathsf{H}_d[v(t,\cdot)]$, with

$$\mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)}$$

then we observe that

$$\frac{1}{2}\,\mathsf{H}' = -\int_{\mathbb{R}^d} v^{m+1}\,dx + \mathsf{S}_d \left(\int_{\mathbb{R}^d} v^{\frac{2d}{d+2}}\,dx\right)^{\frac{2}{d}} \int_{\mathbb{R}^d} \nabla v^m \cdot \nabla v^{\frac{d-2}{d+2}}\,dx$$

where $v = v(t,\cdot)$ is a solution of (FDE). With the choice $m = \frac{d-2}{d+2}$, we find that $m+1 = \frac{2d}{d+2}$

A simple observation

Proposition

[JD] Assume that $d \geq 3$ and $m = \frac{d-2}{d+2}$. If v is a solution of (FDE) with nonnegative initial datum in $\mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, then

$$\frac{1}{2}\,\frac{d}{dt}\left(\int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}\right)$$
$$= \left(\int_{\mathbb{R}^d} v^{m+1}\,dx\right)^{\frac{2}{d}}\left(\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right) \geq 0$$

The HLS inequality amounts to $\mathsf{H} \leq 0$ and appears as a consequence of Sobolev, that is $\mathsf{H}' \geq 0$ if we show that $\limsup_{t>0} \mathsf{H}(t) = 0$
Notice that $u = v^m$ is an optimal function for (S) if v is optimal for (HLS)

Improved Sobolev inequality

By integrating along the flow defined by (FDE), we can actually obtain optimal integral remainder terms which improve on the usual Sobolev inequality (S), with $d \geq 5$ for integrability reasons

Theorem

[JD] Assume that $d \geq 5$ and let $q = \frac{d+2}{d-2}$. There exists a positive constant $\mathcal{C} \leq \left(1 + \frac{2}{d}\right)\left(1 - e^{-d/2}\right) \mathsf{S}_d$ such that

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \leq \mathcal{C} \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)$$

for any $w \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$

Proof: use the convexity properties of $t \mapsto \mathsf{J}(t) := \int_{\mathbb{R}^d} v(t,x)^{m+1} \, dx$ to get an estimate of the *extinction time* and combine with a differential inequality for $t \mapsto \mathsf{H}(t)$

Solutions with *separation of variables*

Consider the solution of $\frac{\partial v}{\partial t} = \Delta v^m$ vanishing at $t = T$:

$$\overline{v}_T(t,x) = c\,(T-t)^\alpha\,(F(x))^{\frac{d+2}{d-2}}$$

where F is the Aubin-Talenti solution of

$$-\Delta F = d\,(d-2)\,F^{(d+2)/(d-2)}$$

Let $\|v\|_* := \sup_{x\in\mathbb{R}^d}(1+|x|^2)^{d+2}\,|v(x)|$

Lemma

[del Pino, Saez], [Vázquez, Esteban, Rodriguez] For any solution v with initial datum $v_0 \in \mathrm{L}^{2d/(d+2)}(\mathbb{R}^d)$, $v_0 > 0$, there exists $T > 0$, $\lambda > 0$ and $x_0 \in \mathbb{R}^d$ such that

$$\lim_{t\to T_-}(T-t)^{-\frac{1}{1-m}}\,\|v(t,\cdot)/\overline{v}(t,\cdot)-1\|_* = 0$$

with $\overline{v}(t,x) = \lambda^{(d+2)/2}\,\overline{v}_T(t,(x-x_0)/\lambda)$

Another improvement

$$\mathsf{J}_d[v] := \int_{\mathbb{R}^d} v^{\frac{2\,d}{d+2}}\,dx \quad \text{and} \quad \mathsf{H}_d[v] := \int_{\mathbb{R}^d} v\,(-\Delta)^{-1} v\,dx - \mathsf{S}_d\,\|v\|^2_{\mathrm{L}^{\frac{2\,d}{d+2}}(\mathbb{R}^d)}$$

Theorem

[JD, Jankowiak] Assume that $d \ge 3$. Then we have

$$0 \le \mathsf{H}_d[v] + \mathsf{S}_d\,\mathsf{J}_d[v]^{1+\frac{2}{d}}\,\varphi\left(\mathsf{J}_d[v]^{\frac{2}{d}-1}\left(\mathsf{S}_d\,\|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \|u\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\right)\right)$$

$$\forall\, u \in \mathcal{D}\,,\quad v = u^{\frac{d+2}{d-2}}$$

where $\varphi(x) := \sqrt{\mathcal{C}^2 + 2\,\mathcal{C}\,x} - \mathcal{C}$ for any $x \ge 0$

Proof: $\mathsf{H}(t) = -\,\mathsf{Y}(\mathsf{J}(t))\ \forall\, t \in [0,T)$, $\kappa_0 := \frac{\mathsf{H}_0'}{\mathsf{J}_0}$ and consider the differential inequality

$$\mathsf{Y}'\left(\mathcal{C}\,\mathsf{S}_d\,s^{1+\frac{2}{d}} + \mathsf{Y}\right) \le \frac{d+2}{2\,d}\,\mathcal{C}\,\kappa_0\,\mathsf{S}_d^2\,s^{1+\frac{4}{d}}\,, \quad \mathsf{Y}(0) = 0\,, \quad \mathsf{Y}(\mathsf{J}_0) = -\,\mathsf{H}_0$$

$\mathcal{C} = 1$ is not optimal

$\mathcal{C} = 1$ is the constant in the expansion of the square method

Theorem

[JD, Jankowiak] In the inequality

$$\mathsf{S}_d \, \|w^q\|^2_{\mathrm{L}^{\frac{2d}{d+2}}(\mathbb{R}^d)} - \int_{\mathbb{R}^d} w^q \, (-\Delta)^{-1} w^q \, dx \\ \le \mathcal{C}_d \, \mathsf{S}_d \, \|w\|^{\frac{8}{d-2}}_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - \mathsf{S}_d \, \|w\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \right)$$

we have

$$\frac{d}{d+4} \le \mathcal{C}_d < 1$$

based on a (painful) linearization

Extensions:

- Moser-Trudinger-Onofri inequality
- fractional Laplacian operator [Jankowiak, Nguyen]

Constructive stability results in Gagliardo-Nirenberg-Sobolev inequalities

A joint project with M. Bonforte, B. Nazaret and N. Simonov
Stability in Gagliardo-Nirenberg-Sobolev inequalities: Flows, regularity and the entropy method
arXiv:2007.03674, to appear in *Memoirs of the AMS*

Fast diffusion equation and entropy methods

$$\frac{\partial u}{\partial t} = \Delta u^m \qquad \text{(FDE)}$$

Gagliardo-Nirenberg-Sobolev inequalities

$$\|\nabla f\|_2^{\theta}\,\|f\|_{p+1}^{1-\theta} \ge \mathcal{C}_{\mathrm{GNS}}(p)\,\|f\|_{2p} \qquad \text{(GNS)}$$

Range of exponents:

$$1 < p \le \frac{d}{d-2} \quad \Longleftrightarrow \quad \frac{d-1}{d} =: m_1 \le m < 1$$

- Sobolev inequality: $p = \frac{d}{d-2}$, $m = m_1$
- Logarithmic Sobolev inequality: $p = 1$, $m = 1$

Entropy – entropy production inequality

Fast diffusion equation (written in self-similar variables)

$$\frac{\partial v}{\partial \tau} + \nabla \cdot \left(v \left(\nabla v^{m-1} - 2\,x \right) \right) = 0 \qquad (r\,\mathrm{FDE})$$

Generalized entropy (free energy) and *Fisher information*

$$\mathcal{F}[v] := -\frac{1}{m} \int_{\mathbb{R}^d} \left(v^m - \mathcal{B}^m - m\,\mathcal{B}^{m-1}\,(v - \mathcal{B}) \right)\,dx$$

$$\mathcal{I}[v] := \int_{\mathbb{R}^d} v \left| \nabla v^{m-1} + 2\,x \right|^2\,dx$$

satisfy an *entropy – entropy production inequality*

$$\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$$

[del Pino, JD, 2002] so that

$$\mathcal{F}[v(t,\cdot)] \leq \mathcal{F}[v_0]\,e^{-4\,t}$$

The *entropy – entropy production inequality* $\mathcal{I}[v] \geq 4\,\mathcal{F}[v]$ is equivalent to the *Gagliardo-Nirenberg-Sobolev inequalities*

$$\|\nabla f\|_2^\theta \, \|f\|_{p+1}^{1-\theta} \geq \mathcal{C}_{\mathrm{GNS}}(p) \, \|f\|_{2p} \qquad \text{(GNS)}$$

with equality if and only if $|f(x)|^{2p} = \mathcal{B}(x) = \left(1 + |x|^2\right)^{\frac{1}{m-1}}$

$$p = \frac{1}{2\,m-1} \quad \Longleftrightarrow \quad m = \frac{p+1}{2\,p} \in [m_1, 1) \quad \text{with} \quad m_1 = \frac{d-1}{d}$$

$u = f^{2\,p}$ so that $u^m = f^{p+1}$ and $u\left|\nabla u^{m-1}\right|^2 = (p-1)^2\,|\nabla f|^2$

Spectral gap: sharp asymptotic rates of convergence

[Blanchet, Bonforte, JD, Grillo, Vázquez, 2009]

$$\left(C_0+|x|^2\right)^{-\frac{1}{1-m}} \le v_0 \le \left(C_1+|x|^2\right)^{-\frac{1}{1-m}} \qquad \text{(H)}$$

Let $\Lambda_{\alpha,d}>0$ be the best constant in the *Hardy–Poincaré inequality*

$$\Lambda_{\alpha,d}\int_{\mathbb{R}^d} f^2\,\mathrm{d}\mu_{\alpha-1} \le \int_{\mathbb{R}^d}|\nabla f|^2\,\mathrm{d}\mu_\alpha \quad \forall\, f\in \mathrm{H}^1(\mathrm{d}\mu_\alpha)\,, \quad \int_{\mathbb{R}^d} f\,\mathrm{d}\mu_{\alpha-1}=0$$

with $\mathrm{d}\mu_\alpha := (1+|x|^2)^\alpha\,dx$, for $\alpha<0$

Lemma

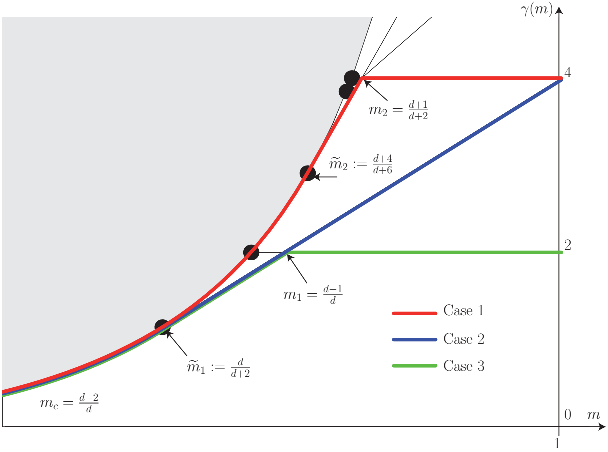
Under assumption (H)

$$\mathcal{F}[v(t,\cdot)] \le C\,e^{-2\,\gamma(m)\,t} \quad \forall\, t\ge 0$$

with $\gamma(m) := 2$ *if* $m_1\le m<1$ $(d\ge 2)$

It is possible to improve on $\gamma(m)$

Spectral gap



[Denzler, McCann, 2005]
[BBDGV, 2009] [BDGV, 2010] [JD, Toscani, 2010-2015]
Much more is know, *e.g.*, [Denzler, Koch, McCann, 2015]

Initial and asymptotic time layers

▷ Asymptotic time layer: constraint, spectral gap and improved entropy – entropy production inequality

▷ Initial time layer: the carré du champ inequality and a backward estimate

The asymptotic time layer improvement

Linearized free energy and *linearized Fisher information*

$$\mathsf{F}[g] := \frac{m}{2} \int_{\mathbb{R}^d} g^2\, \mathcal{B}^{2-m}\, dx \quad \text{and} \quad \mathsf{I}[g] := m\,(1-m) \int_{\mathbb{R}^d} |\nabla g|^2\, \mathcal{B}\, dx$$

Improved Hardy-Poincaré inequality. Under the orthogonality condition $\int_{\mathbb{R}^d} x\, g\, \mathcal{B}^{2-m}\, dx = 0$, we have

$$\mathsf{I}[g] \geq 4\,\alpha\, \mathsf{F}[g] \quad \text{where} \quad \alpha = 2 - d\,(1-m)$$

Proposition

With $m \in (0,1)$ large enough, if $\int_{\mathbb{R}^d} v\, dx = \int_{\mathbb{R}^d} \mathcal{B}\, dx$ and

$$\int_{\mathbb{R}^d} x\, v\, dx = 0 \quad \textit{and} \quad (1-\varepsilon)\,\mathcal{B} \leq v \leq (1+\varepsilon)\,\mathcal{B}$$

for some explicit ε, χ and η

$$\mathcal{I}[v] \geq (4+\eta)\,\mathcal{F}[v]$$

The initial time layer improvement: backward estimate

For some strictly convex function ψ with $\psi(0) = 0$, $\psi'(0) = 1$, we have

$$\mathcal{I} - 4\,\mathcal{F} \geq 4\,(\psi(\mathcal{F}) - \mathcal{F}) \geq 0$$

By the *carré du champ* method,we also have

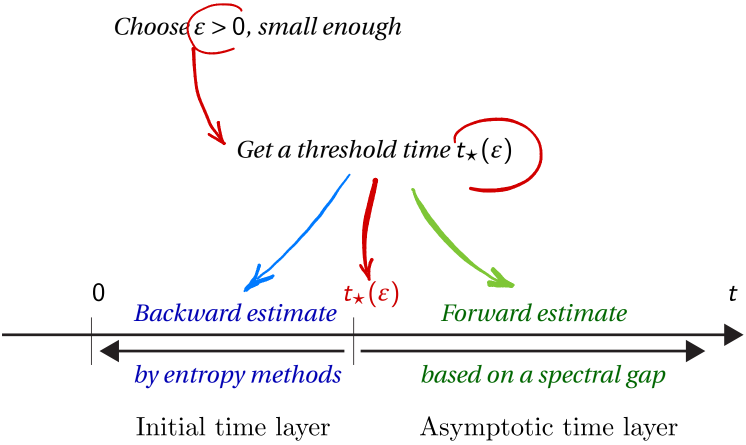
$$\frac{d\,\mathcal{Q}}{dt} \leq \mathcal{Q}\,(\mathcal{Q} - 4) \quad \text{where} \quad \mathcal{Q}[v] := \frac{\mathcal{I}[v]}{\mathcal{F}[v]}$$

Lemma

Assume that $m > m_1$ and v is a solution to $(r\,\mathrm{FDE})$ with initial datum $v_0 \geq 0$. If for some $\eta > 0$ and $t_\star > 0$, we have $\mathcal{Q}[v(t_\star,\cdot)] \geq 4 + \eta$, then

$$\mathcal{Q}[v(t,\cdot)] \geq 4 + \frac{4\,\eta\,e^{-4\,t_\star}}{4 + \eta - \eta\,e^{-4\,t_\star}} \quad \forall\,t \in [0, t_\star]$$

Our strategy



The threshold time and the uniform convergence in relative error

▷ The regularity results allow us to glue the initial time layer estimates with the asymptotic time layer estimates

The improved entropy – entropy production inequality holds for any time along the evolution along (*r* FDE)

(and in particular for the initial datum)

▷ A *global Harnack Principle*. If v is a solves (r FDE) for some nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$A[v_0] := \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} v_0\,dx < \infty \qquad (\mathrm{H}_A)$$

then

$$(1-\varepsilon)\,\mathcal{B} \le v(t,\cdot) \le (1+\varepsilon)\,\mathcal{B} \quad \forall\, t \ge t_\star$$

for some ***explicit*** $t_\star$ depending only on ε and $A[v_0]$

Cf. Matteo's lecture

Uniform convergence in relative error

Theorem

[Bonforte, JD, Nazaret, Simonov, 2021] Assume that $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/3, 1)$ if $d = 1$ and let $\varepsilon \in (0, 1/2)$, small enough, $A > 0$, and $G > 0$ be given. There exists an explicit threshold time $T \geq 0$ such that, if u is a solution of

$$\frac{\partial u}{\partial t} = \Delta u^m \tag{FDE}$$

with nonnegative initial datum $u_0 \in \mathrm{L}^1(\mathbb{R}^d)$ satisfying

$$\mathsf{A}[u_0] = \sup_{r>0} r^{\frac{d\,(m-m_c)}{(1-m)}} \int_{|x|>r} u_0\,dx \leq A < \infty \tag{H_A}$$

$\int_{\mathbb{R}^d} u_0\,dx = \int_{\mathbb{R}^d} B\,dx = \mathcal{M}$ *and* $\mathcal{F}[u_0] \leq G$, *then*

$$\sup_{x\in\mathbb{R}^d} \left| \frac{u(t,x)}{B(t,x)} - 1 \right| \leq \varepsilon \quad \forall\, t \geq T$$

Improved entropy – entropy production inequality (subcritical case)

Theorem

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. Then there is a positive number ζ such that

$$\mathcal{I}[v] \geq (4 + \zeta)\, \mathcal{F}[v]$$

for any nonnegative function $v \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v] = G$, $\int_{\mathbb{R}^d} v\, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x\, v\, dx = 0$ and v satisfies (H_A)

With $t_\star(\varepsilon) = \frac{1}{2} \log R(T)$, we have the *asymptotic time layer estimate*

$$(1 - \varepsilon)\, \mathcal{B} \leq v(t, \cdot) \leq (1 + \varepsilon)\, \mathcal{B} \quad \forall\, t \geq t_\star$$

and, as a consequence, the *initial time layer estimate*

$$\mathcal{I}[v(t, .)] \geq (4 + \zeta)\, \mathcal{F}[v(t, .)] \quad \forall\, t \in [0, t_\star] \quad \text{where} \quad \zeta = \frac{4\, \eta\, e^{-4\, t_\star}}{4 + \eta - \eta\, e^{-4\, t_\star}}$$

Two consequences

▷ Improved decay rate for the rescaled fast diffusion equation

Corollary

Let $m \in (m_1, 1)$ if $d \geq 2$, $m \in (1/2, 1)$ if $d = 1$, $A > 0$ and $G > 0$. If v is a solution of (r FDE) with nonnegative initial datum $v_0 \in \mathrm{L}^1(\mathbb{R}^d)$ such that $\mathcal{F}[v_0] = G$, $\int_{\mathbb{R}^d} v_0 \, dx = \mathcal{M}$, $\int_{\mathbb{R}^d} x \, v_0 \, dx = 0$ and v_0 satisfies (H_A), then

$$\mathcal{F}[v(t,.)] \leq \mathcal{F}[v_0] \, e^{-(4+\zeta)\, t} \quad \forall\, t \geq 0$$

▷ The ***stability in the entropy - entropy production estimate*** $\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \zeta\,\mathcal{F}[v]$ also holds in a stronger sense

$$\mathcal{I}[v] - 4\,\mathcal{F}[v] \geq \frac{\zeta}{4+\zeta}\,\mathcal{I}[v]$$

Stability results (subcritical case)

▷ We rephrase the results obtained by entropy methods in the language of stability *à la* Bianchi-Egnell

Subcritical range

$$p^* = +\infty \text{ if } d = 1 \text{ or } 2, \qquad p^* = \frac{d}{d-2} \text{ if } d \geq 3$$

$$\lambda[f]:=\left(\frac{2\,d\,\kappa[f]^{p-1}}{p^2-1}\,\frac{\|f\|_{p+1}^{p+1}}{\|\nabla f\|_2^2}\right)^{\frac{2\,p}{d-p\,(d-4)}}\,,\quad \kappa[f]:=\frac{\mathcal M^{\frac{1}{2\,p}}}{\|f\|_{2\,p}}$$

$$\mathsf A[f]:=\frac{\mathcal M}{\lambda[f]^{\frac{d-p\,(d-4)}{p-1}}\,\|f\|_{2\,p}^{2\,p}}\,\sup_{r>0}r^{\frac{d-p\,(d-4)}{p-1}}\int_{|x|>r}|f(x+x_f)|^{2\,p}\,dx$$

$$\mathsf E[f]:=\frac{2\,p}{1-p}\int_{\mathbb R^d}\left(\frac{\kappa[f]^{p+1}}{\lambda[f]^{d\,\frac{p-1}{2\,p}}}\,f^{p+1}-\mathsf g^{p+1}-\frac{1+p}{2\,p}\,\mathsf g^{1-p}\left(\frac{\kappa[f]^{2\,p}}{\lambda[f]^2}\,f^{2\,p}-\mathsf g^{2\,p}\right)\right)dx$$

$$\mathfrak S[f]:=\frac{\mathcal M^{\frac{p-1}{2\,p}}}{p^2-1}\,\frac{1}{C(p,d)}\,\mathsf Z\left(\mathsf A[f],\,\mathsf E[f]\right)$$

Theorem

Let $d\ge1$, $p\in(1,p^*)$

If $f\in\mathcal W_p(\mathbb R^d):=\left\{f\in\mathrm L^{2p}(\mathbb R^d)\,:\,\nabla f\in\mathrm L^2(\mathbb R^d)\,,\ |x|\,f^{\,p}\in\mathrm L^2(\mathbb R^d)\right\}$,

$$\left(\|\nabla f\|_2^\theta\,\|f\|_{p+1}^{1-\theta}\right)^{2\,p\,\gamma}-\left(\mathcal C_{\rm GN}\,\|f\|_{2\,p}\right)^{2\,p\,\gamma}\ge\mathfrak S[f]\,\|f\|_{2\,p}^{2\,p\,\gamma}\,\mathsf E[f]$$

With $\mathcal{K}_{\rm GNS} = C(p,d)\,\mathcal{C}_{\rm GNS}^{2\,p\,\gamma}$, $\gamma = \frac{d+2-p\,(d-2)}{d-p\,(d-4)}$, consider the *deficit functional*

$$\delta[f] := (p-1)^2\,\|\nabla f\|_2^2 + 4\,\frac{d-p\,(d-2)}{p+1}\,\|f\|_{p+1}^{p+1} - \mathcal{K}_{\rm GNS}\,\|f\|_{2\,p}^{2\,p\,\gamma}$$

Theorem

Let $d \ge 1$ and $p \in (1,p^*)$. There is an explicit $\mathcal{C} = \mathcal{C}[f]$ such that, for any $f \in \mathrm{L}^{2p}\big(\mathbb{R}^d,(1+|x|^2)\,dx\big)$ such that $\nabla f \in \mathrm{L}^2(\mathbb{R}^d)$ and $\mathsf{A}\big[f^{2p}\big] < \infty$,

$$\delta[f] \ge \mathcal{C}[f]\,\inf_{\varphi\in\mathfrak{M}}\int_{\mathbb{R}^d}\left|(p-1)\,\nabla f + f^p\,\nabla\varphi^{1-p}\right|^2 dx$$

▷ The dependence of $\mathcal{C}[f]$ on $\mathsf{A}\big[f^{2p}\big]$ and $\mathcal{F}\big[f^{2p}\big]$ is explicit and does not degenerate if $f \in \mathfrak{M}$

▷ Can we remove the condition $\mathsf{A}\big[f^{2p}\big] < \infty$?

Stability in Sobolev's inequality (critical case)

- ▷ A constructive stability result
- ▷ The main ingredient of the proof

A constructive stability result

Let $2\,p^\star = 2d/(d-2) = 2^*$, $d \geq 3$ and

$$\mathcal{W}_{p^\star}(\mathbb{R}^d) = \left\{ f \in \mathrm{L}^{p^\star+1}(\mathbb{R}^d) \,:\, \nabla f \in \mathrm{L}^2(\mathbb{R}^d)\,,\ \ |x|\, f^{\,p^\star} \in \mathrm{L}^2(\mathbb{R}^d) \right\}$$

Theorem

Let $d \geq 3$ and $A > 0$. For any nonnegative $f \in \mathcal{W}_{p^\star}(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} (1, x, |x|^2)\, f^{2^*}\, dx = \int_{\mathbb{R}^d} (1, x, |x|^2)\, \mathrm{g}\, dx \text{ and } \sup_{r>0} r^d \int_{|x|>r} f^{2^*}\, dx \leq A$$

we have

$$\delta[f] := \|\nabla f\|_2^2 - \mathsf{S}_d^2\, \|f\|_{2^*}^2 \geq \frac{\mathcal{C}_\star(A)}{4 + \mathcal{C}_\star(A)} \int_{\mathbb{R}^d} \left| \nabla f + \tfrac{d-2}{2}\, f^{\frac{d}{d-2}}\, \nabla \mathrm{g}^{-\frac{2}{d-2}} \right|^2 dx$$

$\mathcal{C}_\star(A) = \mathcal{C}_\star(0)\, \left(1 + A^{1/(2\,d)}\right)^{-1}$ *and* $\mathcal{C}_\star(0) > 0$ *depends only on* d

Peculiarities of the critical case

▷ We can remove the normalization of f, use the r.h.s. to measure the distance to the Aubin-Talenti manifold of optimal functions (in relative Fisher information) and obtain for

$$A[f] := \sup_{r>0} r^d \int_{r>0} |f|^{2^*}(x+x_f) \quad \text{and} \quad Z[f] := \left(1+\mu[f]^{-d}\,\lambda[f]^d\,A[f]\right)$$

the *Bianchi-Egnell type result*

$$\delta[f] \geq \frac{\mathfrak{C}_\star\,Z[f]}{4+Z[f]} \inf_{g\in\mathfrak{M}} \mathcal{J}[f|g]$$

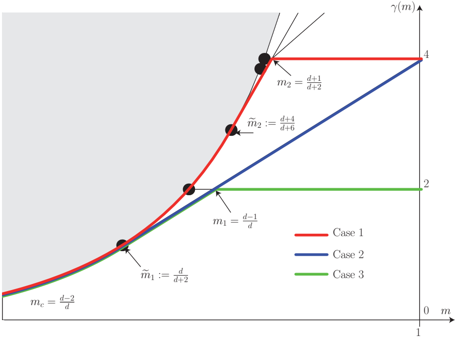
with x_f, $\lambda[f]$ and $\mu[f]$ as in the subcritical case

▷ Notion of time delay [JD, Toscani, 2014, 2015]

Extending the subcritical result in the critical case

To improve the spectral gap for $m = m_1$, we need to adjust the Barenblatt function $\mathcal{B}_\lambda(x) = \lambda^{-d/2}\,\mathcal{B}\left(x/\sqrt{\lambda}\right)$ in order to match $\int_{\mathbb{R}^d}|x|^2\,v\,dx$ where the function v solves (r FDE) or to further rescale v according to

$$v(t,x) = \frac{1}{\mathfrak{R}(t)^d}\,w\left(t+\tau(t),\frac{x}{\mathfrak{R}(t)}\right),$$



$$\frac{d\tau}{dt} = \left(\frac{1}{\mathcal{K}_\star}\int_{\mathbb{R}^d}|x|^2\,v\,dx\right)^{-\frac{d}{2}(m-m_c)} - 1\,,\quad \tau(0)=0 \quad \text{and} \quad \mathfrak{R}(t) = e^{2\,\tau(t)}$$

Lemma

$t \mapsto \lambda(t)$ and $t \mapsto \tau(t)$ are bounded on $\mathbb{R}^+$

A constructive Bianchi-Egnell stability result

- ▷ A constructive estimate for the Bianchi-Egnell stability result
- ▷ Competing symmetries and the construction of a flow
- ▷ Explicit estimates close to the manifold of optimizers

Stability for Sobolev

With $d \geq 3$, $2^* = 2\,d/(d-2)$, we consider the *stability* inequality

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq c_{\mathrm{BE}} \inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

for functions in $\dot{\mathrm{H}}^1(\mathbb{R}^d) = \left\{f \in \mathrm{L}^q(\mathbb{R}^d)\,:\,\nabla f \in \mathrm{L}^2(\mathbb{R}^d)\right\}$
$S_d = \frac{1}{4}\,d\,(d-2)\,|\mathbb{S}^d|^{2/d}$ is the optimal constant in Sobolev's inequality
$\mathcal{M}$ is the manifold of the optimal *Aubin-Talenti* functions

$$f(x) = c\,\left(a + |x-b|^2\right)^{-\frac{d-2}{2}}$$

$c_{\mathrm{BE}} = 0$: [Rodemich, 1966], [Aubin, 1976], [Talenti, 1976]
$c_{\mathrm{BE}} > 0$? [Brezis, Lieb, 1985]
$c_{\mathrm{BE}} > 0$: [Bianchi, Egnell, 1991]
$c_{\mathrm{BE}} >?$

Results in collaboration with
M.J. Esteban, A. Figalli, R.L. Frank, M. Loss

Main result

$$\mathcal{E}[f] := \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}\,, \quad \nu(\delta) := \sqrt{\frac{\delta}{1-\delta}}$$

Theorem

Let $d \ge 3$, $q = 2\,d/(d-2)$. If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ is a *non-negative* function, then

$$\mathcal{E}[f] \ge \kappa_d := \sup_{0<\delta<1} \delta\,\mu(\delta)$$

where $\mu(\delta) \ge \mathsf{m}\big(\nu(\delta)\big)$ and

$$\mathsf{m}(\nu) := \tfrac{4}{d+4} - \tfrac{2}{q}\,\nu^{q-2} \quad \text{if} \quad d \ge 6$$

$$\mathsf{m}(\nu) := \tfrac{4}{d+4} - \tfrac{1}{3}\,(q-1)\,(q-2)\,\nu - \tfrac{2}{q}\,\nu^{q-2} \quad \text{if} \quad d = 4,5$$

$$\mathsf{m}(\nu) := \tfrac{4}{7} - \tfrac{20}{3}\,\nu - 5\,\nu^2 - 2\,\nu^3 - \tfrac{1}{3}\,\nu^4 \quad \text{if} \quad d = 3$$

Comments

Upper bound: $C_{\mathrm{BE}} \le \inf_{0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}} \mathcal{E}[f] < \frac{4}{d+4}$ [König]

We also have a *lower bound* on $C_{\mathrm{BE}} := \inf_{f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}} \mathcal{E}[f]$

...work in progress, arXiv: 22209.08651

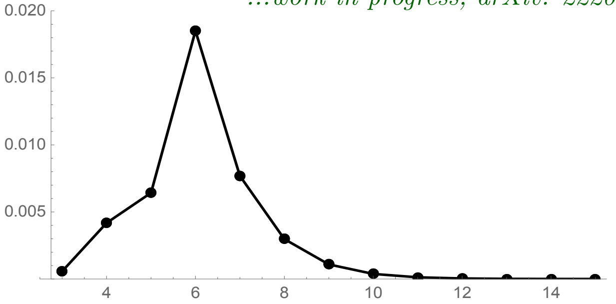


Figure: Plot of $d \mapsto \kappa(d)$ for $d = 3, 4, \ldots, 15$

Large dimensions: $\kappa(d) \sim 2^{d+1} d^{-1} (d+4)^{-\frac{d}{2}}$ as $d \to +\infty$

Strategy: two regions

- *Taylor expansion, spectral estimates:* in the region $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \le \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, prove that

$$\mathcal{E}[f] \ge \mu(\delta)$$

- *Continuous flow argument:* [Christ, 2017] if $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \ge \delta\, \|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, build a *flow* $(f_\tau)_{0\le\tau<\infty}$ s.t.

$$f_0 = f\,, \quad \|f_\tau\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)} = \|f\|_{\mathrm{L}^{2^*}(\mathbb{R}^d)}\,, \quad \tau \mapsto \|\nabla f_\tau\|_{\mathrm{L}^2(\mathbb{R}^d)} \text{ is } \searrow$$

$$\lim_{\tau\to\infty} \inf_{g\in\mathcal{M}} \|\nabla (f_\tau - g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = 0$$

$$\mathcal{E}[f] \ge \frac{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\, \|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} = 1 - S_d\, \frac{\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)}} \ge \frac{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\, \|f_{\tau_0}\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)}}{\|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}}$$

for some τ_0 (it exists ?) s.t. $\inf_{g\in\mathcal{M}} \|\nabla (f_{\tau_0} - g)\|^2_{\mathrm{L}^2(\mathbb{R}^d)} = \delta\, \|\nabla f_{\tau_0}\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$

... then $\mathcal{E}[f] \ge \mathcal{E}(f_{\tau_0}) \ge \delta\, \mu(\delta)$

Inverse stereographic projection

Denote by $s=(s_1,s_2,\ldots,s_{d+1})$ the coordinates in $\mathbb{R}^{d+1}$: $\mathbb{S}^d\subset\mathbb{R}^{d+1}$ can be parametrized in terms of stereographic coordinates by

$$s_j=\frac{2\,x_j}{1+|x|^2}\,,\quad j=1,\ldots,d\,,\quad s_{d+1}=\frac{1-|x|^2}{1+|x|^2}$$

We set

$$F(s)=\left(\tfrac{1+|x|^2}{2}\right)^{\frac{d-2}{2}}\,f(x)$$

$$\mathcal{E}[f]=\frac{\|\nabla f\|_2^2-\mathsf{S}_d\,\|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla f-\nabla g\|_2^2}=\frac{\|\nabla F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac14\,d\,(d-2)\,\|F\|^2_{\mathrm{L}^2(\mathbb{S}^d)}-\mathsf{S}_d\,\|F\|^2_{\mathrm{L}^{2^*}(\mathbb{S}^d)}}{\inf_{G\in\mathcal{M}}\left\{\|\nabla F-\nabla G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}+\frac14\,d\,(d-2)\,\|F-G\|^2_{\mathrm{L}^2(\mathbb{S}^d)}\right\}}$$

where $G(s)=c\,(a+b\cdot s)^{-\frac{d-2}{2}}$, $a>0$, $b\in\mathbb{R}^d$ and $c\in\mathbb{C}$ are constants

Competing symmetries

[Carlen, Loss, 1990]

- *Conformal rotation*

$$(UF)(s)=F(s_1,s_2,\ldots,s_{d+1},-s_d)$$

On $\mathbb{R}^d$, the function that corresponds to UF on $\mathbb{R}^d$ is given by

$$(Uf)(x)=\left(\frac{2}{|x-e_d|^2}\right)^{\frac{d-2}{2}} f\left(\frac{x_1}{|x-e_d|^2},\ldots,\frac{x_{d-1}}{|x-e_d|^2},\frac{|x|^2-1}{|x-e_d|^2}\right)$$

where $e_d=(0,\ldots,0,1)\in\mathbb{R}^d$ and $\mathcal{E}(Uf)=\mathcal{E}[f]$

- *Symmetric decreasing rearrangement*: if $f\geq 0$, let

$$\mathcal{R}f(x)=f^*(x)$$

f and f^* are equimeasurable and $\|\nabla f^*\|_2\leq\|\nabla f\|_2$

... *continuous Steiner symmetrization*

On $\mathbb{R}^d$, let

$$g_*(x) := |\mathbb{S}^d|^{-\frac{d-2}{2\,d}} \left(\frac{2}{1+|x|^2}\right)^{\frac{d-2}{2}}$$

Theorem

[Carlen, Loss] Let $f \in \mathrm{L}^{2^*}(\mathbb{R}^d)$ be a non-negative function. Consider the sequence $(f_n)_{n\in\mathbb{N}}$ of functions

$$f_n = (\mathcal{R}U)^n f$$

Then $h_f = \|f\|_{2^*}\, g_* \in \mathcal{M}$ and

$$\lim_{n\to\infty} \|f_n - h_f\|_{2^*} = 0$$

If $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$, then $(\|\nabla f_n\|_2)_{n\in\mathbb{N}}$ is a non-increasing sequence

Define $\mathcal{M}_1$ to be the set of the elements in $\mathcal{M}$ with 2^*-norm equal to 1

$$\inf_{g\in\mathcal{M}} \|\nabla f-\nabla g\|_2^2=\|\nabla f\|_2^2-S_d\,\sup_{g\in\mathcal{M}_1}\Big(f,g^{2^*-1}\Big)^2$$

Lemma

For the sequence $(f_n)_{n\in\mathbb{N}}$ of the Theorem of [Carlen, Loss] we have that

$n\mapsto\inf_{g\in\mathcal{M}}\|\nabla f_n-\nabla g\|_{2^*}^2$ *is strictly decreasing*

$$\lim_{n\to\infty}\inf_{g\in\mathcal{M}}\|\nabla f_n-\nabla g\|_2^2=\lim_{n\to\infty}\|\nabla f_n\|_2^2-S_d\,\|f\|_{2^*}^2$$

Two alternatives

Lemma

Let $0 \le f \in \dot{\mathrm{H}}^1(\mathbb{R}^d) \setminus \mathcal{M}$ s.t. $\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2 \ge \delta\,\|\nabla f\|_2^2$

One of the following alternatives holds:

(a) for all $n = 0, 1, 2 \ldots$ $\inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 \ge \delta\,\|\nabla f_n\|_2^2$

(b) $\exists n_0 \in \mathbb{N}$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla f_{n_0} - \nabla g\|_2^2 \ge \delta\,\|\nabla f_{n_0}\|_2^2 \quad \text{and} \quad \inf_{g\in\mathcal{M}} \|\nabla f_{n_0+1} - \nabla g\|_2^2 < \delta\,\|\nabla f_{n_0+1}\|_2^2$$

In case (a) we have

$$\mathcal{E}[f] = \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2^2} \ge \frac{\|\nabla f\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f\|_2^2} \ge \frac{\|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2}{\|\nabla f_n\|_2^2} \ge \delta$$

because by the Theorem of [Carlen, Loss]

$$\lim_{n\to\infty} \|\nabla f_n\|_2^2 \le \frac{1}{\delta} \lim_{n\to\infty} \inf_{g\in\mathcal{M}} \|\nabla f_n - \nabla g\|_2^2 = \frac{1}{\delta}\left(\lim_{n\to\infty} \|\nabla f_n\|_2^2 - S_d\,\|f\|_{2^*}^2\right)$$

Continuous rearrangement

Let $\mathsf{f}_0 = U f_{n_0}$ and denote by $(\mathsf{f}_\tau)_{0\le\tau\le\infty}$ the continuous rearrangement starting at f_0 and ending at $\mathsf{f}_\infty = f_{n_0+1}$
We find $\tau_0 \in [0,\infty)$ such that

$$\inf_{g\in\mathcal{M}} \|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2 = \delta\,\|\nabla \mathsf{f}_{\tau_0}\|_2^2$$

and conclude using

$$\mathcal{E}(\mathsf{f}_0) \ge 1 - S_d\,\frac{\|\mathsf{f}_0\|_{2^*}^2}{\|\nabla \mathsf{f}_0\|_2^2} \ge 1 - S_d\,\frac{\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\|\nabla \mathsf{f}_{\tau_0}\|_2^2} = \delta\,\frac{\|\nabla \mathsf{f}_{\tau_0}\|_2^2 - S_d\,\|\mathsf{f}_{\tau_0}\|_{2^*}^2}{\inf_{g\in\mathcal{M}}\|\nabla \mathsf{f}_{\tau_0} - \nabla g\|_2^2} \ge \delta\,\mu(\delta)$$

Existence of τ_0 not granted: a discussion is needed !

Remark. We can build a ***flow*** by gluing continuous symmetrization at each step of the sequence $(f_n)_{n\in\mathbb{N}}$

Analysis close to the manifold of optimizers

Proposition

Let X be a measure space and $u, r \in \mathrm{L}^q(X)$ for some $q \geq 2$ with $u \geq 0$ and $u + r \geq 0$. Assume also that $\int_X u^{q-1}\, r\, dx = 0$. If $2 \leq q \leq 3$, then

$$\|u+r\|_q^2 \leq \|u\|_q^2 + \|u\|_q^{2-q}\left((q-1)\int_X u^{q-2}\, r^2\, dx + \frac{2}{q}\int_X r_+^q\, dx\right)$$

$2 \leq q = \frac{2\,d}{d-2} \leq 3$ means $d \geq 6$ and is the most difficult case for Taylor

Corollary

Let $q = 2^*$, $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ which realizes

$$\inf_{g \in \mathcal{M}} \|\nabla f - \nabla g\|_2$$

Set $r := f - u$ and $\sigma := \|r\|_q / \|u\|_q$. If $d \geq 6$, we have

$$\|\nabla f\|_2^2 - S_d\, \|f\|_q^2 \geq \int_{\mathbb{R}^d}\left(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\, u^{q-2}\, r^2\right)dx - \frac{2}{q}\,\|\nabla r\|_2^2\, \sigma^{q-2}$$

Spectral gap estimate

Cf. [Rey, 1990] and [Bianchi, Egnell, 1991]

Lemma

Let $d \geq 3$, $q = 2^*$, $f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$ and $u \in \mathcal{M}$ be such that $\|\nabla f - \nabla u\| = \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|$. Then $r := f - u$ satisfies

$$\int_{\mathbb{R}^d} \left(|\nabla r|^2 - S_d\,(q-1)\,\|u\|_q^{2-q}\,|u|^{q-2}\,r^2\right) dx \geq \frac{4}{d+4} \int_{\mathbb{R}^d} |\nabla r|^2\,dx$$

Corollary

Let $q = 2^*$ and $0 \leq f \in \dot{\mathrm{H}}^1(\mathbb{R}^d)$. Set $\mathcal{D}[f] := \inf_{g\in\mathcal{M}} \|\nabla f - \nabla g\|_2$ and $\tau := \mathcal{D}[f]/(\|\nabla f\|_2^2 - \mathcal{D}[f]^2)^{1/2}$. If $d \geq 6$, we have

$$\|\nabla f\|_2^2 - S_d\,\|f\|_q^2 \geq \left(\tfrac{4}{d+4} - \tfrac{2}{q}\,\tau^{q-2}\right) \mathcal{D}[f]^2$$

One more step: removing the positivity assumption

The *Bianchi-Egnell stability estimate*

$$\|\nabla f\|^2_{\mathrm{L}^2(\mathbb{R}^d)} - S_d\,\|f\|^2_{\mathrm{L}^{2^*}(\mathbb{R}^d)} \geq c_{\mathrm{BE}}\,\inf_{g\in\mathcal{M}}\|\nabla f - \nabla g\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$$

Nonnegative functions: $c_{\mathrm{BE}}^{\mathrm{pos}} \geq \kappa_d$ and $c_{\mathrm{BE}} \leq c_{\mathrm{BE}}^{\mathrm{pos}} \leq \frac{4}{d+4}$

Sign-changing solutions. Take $m := \|u_-\|_{2^*}^{2^*}$ and assume that $1-m = \|u_+\|_{2^*}^{2^*}$. We argue that $2\,h(1/2)\,m \leq h(m)$ if

$$h(m) := m^{1-\frac{2}{d}} + (1-m)^{1-\frac{2}{d}} - 1$$

With $\mathrm{D}(v) := \|\nabla v\|_2^2 - S_d\,\|v\|_{2^*}^2$ and $(\ldots)$, we obtain

$$\mathrm{D}(u) \geq c_{\mathrm{BE}}^{\mathrm{pos}}\,\|\nabla u_+ - \nabla g_+\|_2^2 + \frac{2\,h(1/2)}{2\,h(1/2)+\xi_d}\,\|\nabla u_-\|_2^2$$

$$\boxed{c_{\mathrm{BE}} \geq \tfrac{1}{2}\,\kappa_d}$$

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Thank you for your attention !