Symmetry and symmetry breaking of extremal functions in some interpolation inequalities

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IN COLLABORATION WITH

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CALCULUS OF VARIATIONS AND NONLINEAR PDES

SWANSEA, 20 MAY 2011

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Outline

- Introduction: a symmetry breaking mechanism
- Symmetry results (moving planes): some simple remarks
- Caffarelli-Kohn-Nirenberg inequalities (Part I): results on symmetry and proofs
- Caffarelli-Kohn-Nirenberg inequalities (Part II) and logarithmic Hardy inequalities
- Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities
- Radial symmetry and symmetry breaking

What is based on spectral methods ? What is not ?

- The slides of this talk:

  `http://www.ceremade.dauphine.fr/∼dolbeaul/Conferences/`

- A review of known results:
  Jean Dolbeault and Maria J. Esteban
  About existence, symmetry and symmetry breaking for extremal functions of some interpolation functional inequalities

  `http://www.ceremade.dauphine.fr/∼dolbeaul/Preprints/`

Introduction

Symmetry and symmetry breaking in PDEs

Symmetry in PDEs has been widely used to understand the uniqueness or multiplicity properties of the solutions. The standard scheme goes as follows:

- prove some symmetry properties by symmetrization or comparison techniques of the solutions (ground states) of an (Euler-Lagrange) equation
- prove uniqueness by ODE techniques

but also: bifurcation analysis, branches of solutions within certain classes of symmetry, direct analysis of the solution set,...

- This talk will be focused on a very simple case (equality cases in some inequalities with homogeneous weights and homogeneous nonlinearities): Caffarelli-Kohn-Nirenberg (CKN) and "weighted logarithmic Hardy inequalities" (WLH)

- Almost all results of symmetry / symmetry breaking can be related to some spectral properties... a new result

A symmetry breaking mechanism

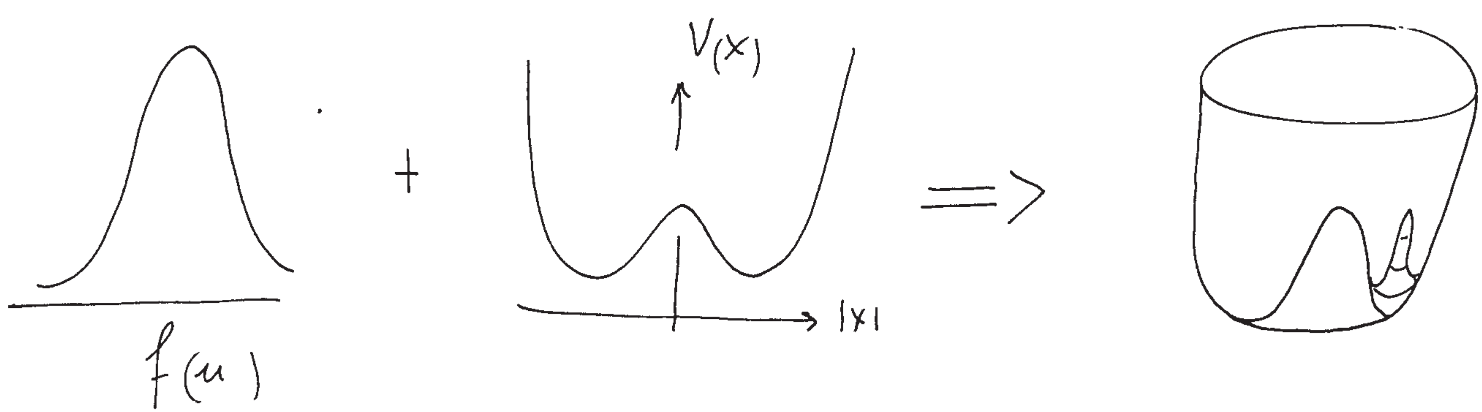
Various techniques have been developed to prove symmetry

Much less is known concerning symmetry breaking. Known results are based on

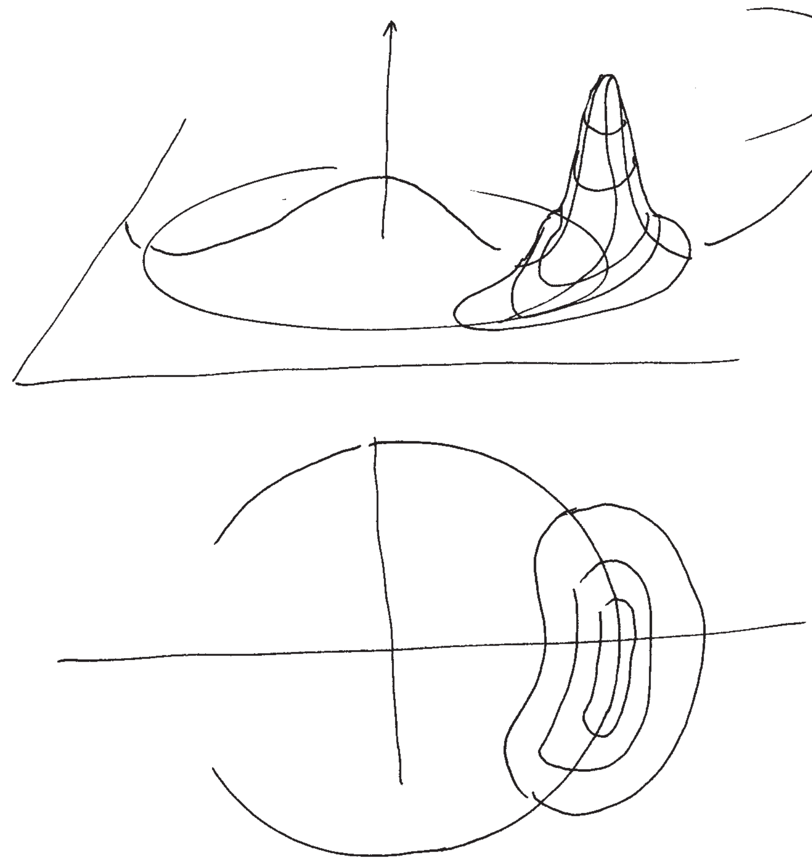
- energy considerations + linear analysis
- characterization of some asymptotic regimes

What is the reason for symmetry breaking ?

- Typical source of symmetry breaking is the competition of two effects: a potential and a nonlinearity the competition of a nonlinearity which tends to aggregate or concentrate the solution and of an (external) potential term which “prefers”

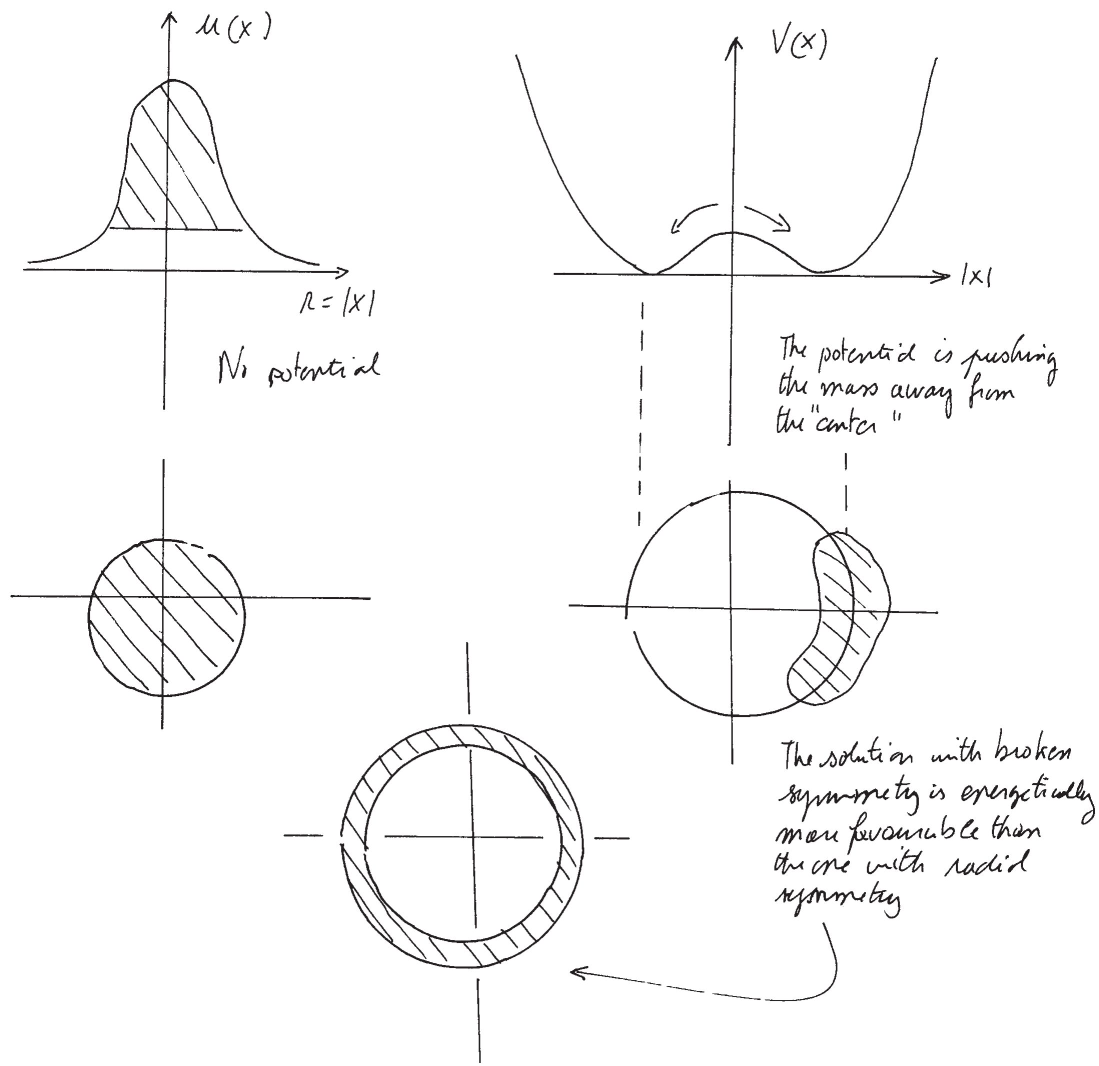


The solution with broken symmetry



Can we understand the transition from a regime of ground states with symmetry to a regime where symmetry is broken ? Can we quantify this phenomenon ? Do the non-radial solutions bifurcate from the radial ones (if one increases the strength of the potential, for instance)

The energy point of view (ground state)



Symmetry results (moving planes)

Some simple remarks

The theorem of Gidas, Ni and Nirenberg - extensions

Theorem 1. [Gidas, Ni and Nirenberg, 1979 and 1980] *Let* $u \in C^2(B)$*,* $B = B(0,1) \subset \mathbb{R}^d$*, be a solution of*

$$\Delta u + f(u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that f *is Lipschitz. If* u *is positive, then it is radially symmetric and decreasing along any radius:* $u'(r) < 0$ *for any* $r \in (0,1]$

Extension: $\Delta u + f(r,u) = 0$, $r = |x|$ if $\frac{\partial f}{\partial r} \leq 0$... a “cooperative" case

Theorem 2. [JD, Felmer, 1999] *Consider solutions of*

$$\Delta u + \lambda\, f(r,u) = 0 \text{ in } B\,, \quad u = 0 \text{ on } \partial B$$

and assume that $f \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ *(no assumption on the sign of* $\frac{\partial f}{\partial r}$*). There exists* λ_1*,* λ_2 *with* $0 < \lambda_1 \leq \lambda_2$ *such that*

- *(i) Monotonicity: if* $\lambda \in (0, \lambda_1)$*, then* $\frac{d}{dr}(u - \lambda\, u_0) < 0$ *where* u_0 *is the solution of* $\Delta u_0 + \lambda f(r,0) = 0$
- *(ii) Symmetry: if* $\lambda \in (0, \lambda_2)$*, then* u *is radially symmetric*

Proof: spectral issues

The counter-example of Gidas, Ni and Nirenberg if $\frac{\partial f}{\partial r} \geq 0$ is based on eigenfunctions and eigenvalues

A sketch of the proof of (ii): $\bar{x} = (-x_1, x')$, $|x'| = |x|$, $\bar{u}(x) = u(\bar{x})$

$$\Delta \bar{u} + \lambda\, f(r, \bar{u}) = 0$$

$v = \bar{u} - u$, $c = (f(r, \bar{u}) - f(r, u))/(u - \bar{u})$ and $\Delta v + \lambda\, c\, v = 0$
$\lambda_1 = \sup\{\lambda > 0 \,:\, \Delta v + \lambda\, c\, v = 0 \Longrightarrow v = 0\}$
$\lambda_2 = \sup\{\lambda > 0 \,:\, \Delta v + \lambda\, c\, v = 0$ and v changes sign $\Longrightarrow v = 0\}$

If $\lambda < \lambda_2$,

- either $\lambda = \lambda_1$ and v is nonnegative... but $v(\bar{x}) = u(x) - u(\bar{x}) = -v(x)$ and so $v \equiv 0$: $u = \bar{u}$
- or $\lambda \neq \lambda_1$: $v \equiv 0$, same conclusion $\square$

We learned (vague) that symmetry by the moving planes has to do with spectral properties

Caffarelli-Kohn-Nirenberg inequalities (Part I)

Joint work(s) with M. Esteban, M. Loss and G. Tarantello

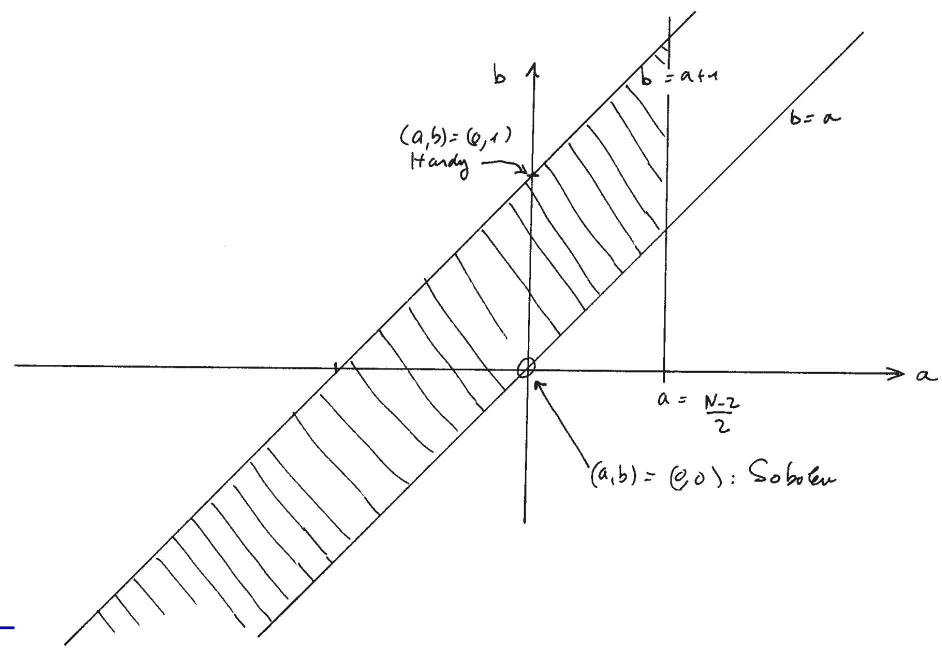
Caffarelli-Kohn-Nirenberg (CKN) inequalities

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\, u \in \mathcal{D}_{a,b}$$

with $a \le b \le a+1$ if $d \ge 3$, $a < b \le a+1$ if $d = 2$, and $a \neq \frac{d-2}{2} =: a_c$

$$p = \frac{2\,d}{d-2+2\,(b-a)}$$

$$\mathcal{D}_{a,b} := \left\{ |x|^{-b}\,u \in L^p(\mathbb{R}^d, dx) \;:\; |x|^{-a}\,|\nabla u| \in L^2(\mathbb{R}^d, dx) \right\}$$



The symmetry issue

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{2/p} \le \mathsf{C}_{a,b} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx \qquad \forall\; u \in \mathcal{D}_{a,b}$$

$\mathsf{C}_{a,b} =$ best constant for general functions u
$\mathsf{C}^*_{a,b} =$ best constant for radially symmetric functions u

$$\mathsf{C}^*_{a,b} \le C_{a,b}$$

Up to scalar multiplication and dilation, the optimal radial function is

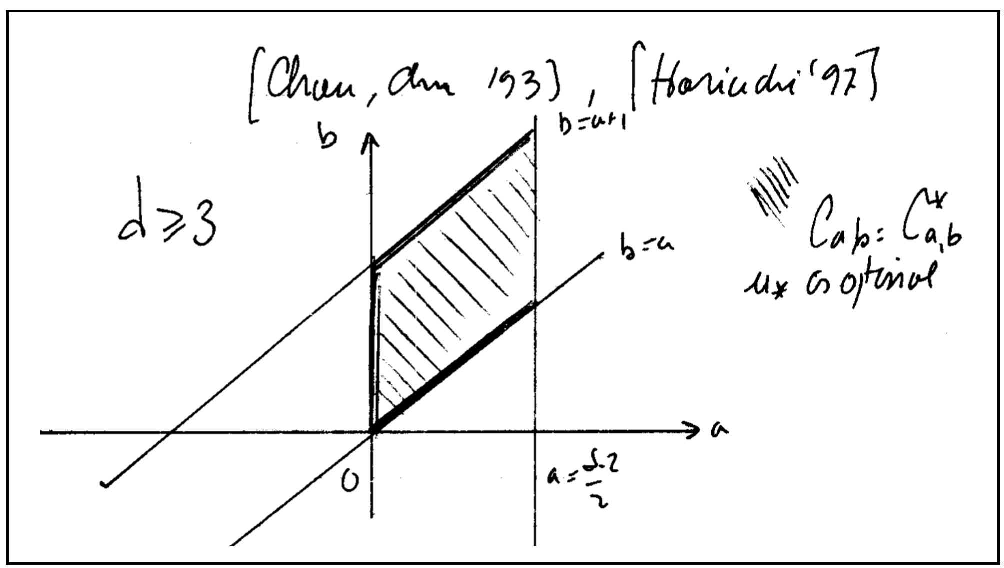
$$u^*_{a,b}(x) = |x|^{a+\frac{d}{2}\,\frac{b-a}{b-a+1}} \left(1+|x|^2\right)^{-\frac{d-2+2(b-a)}{2(1+a-b)}}$$

Questions: is optimality (equality) achieved ? do we have $u_{a,b} = u^*_{a,b}$?

Known results

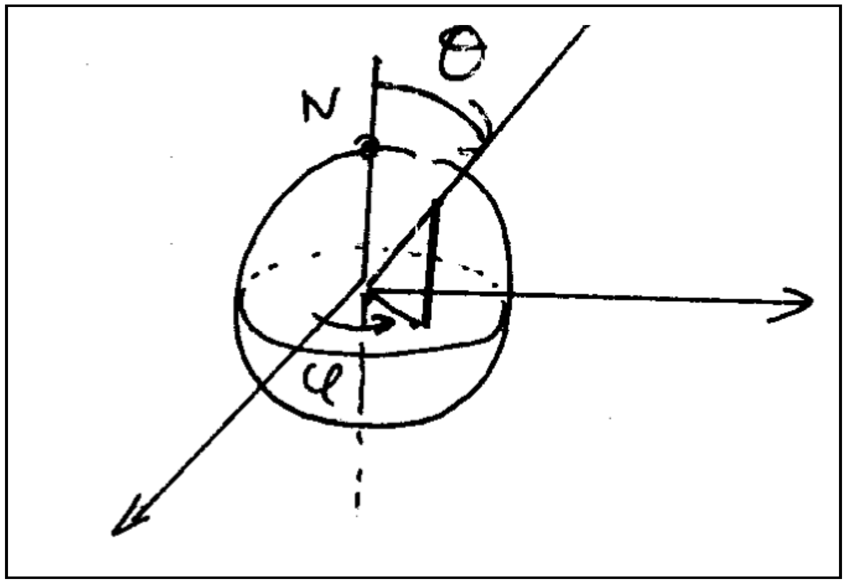
[Aubin, Talenti, Lieb, Chou-Chu, Lions, Catrina-Wang, ...]

- Extremals exist for $a < b < a+1$ and $0 \leq a \leq \frac{d-2}{2}$, for $a \leq b < a+1$ and $a < 0$ if $d \geq 2$
- Optimal constants are never achieved in the following cases
  - "critical / Sobolev" case: for $b = a < 0$, $d \geq 3$
  - "Hardy" case: $b = a+1$, $d \geq 2$
- If $d \geq 3$, $0 \leq a < \frac{d-2}{2}$ and $a \leq b < a+1$, the extremal functions are radially symmetric ... $u(x) = |x|^a\, v(x)$ + Schwarz symmetrization



More results on symmetry

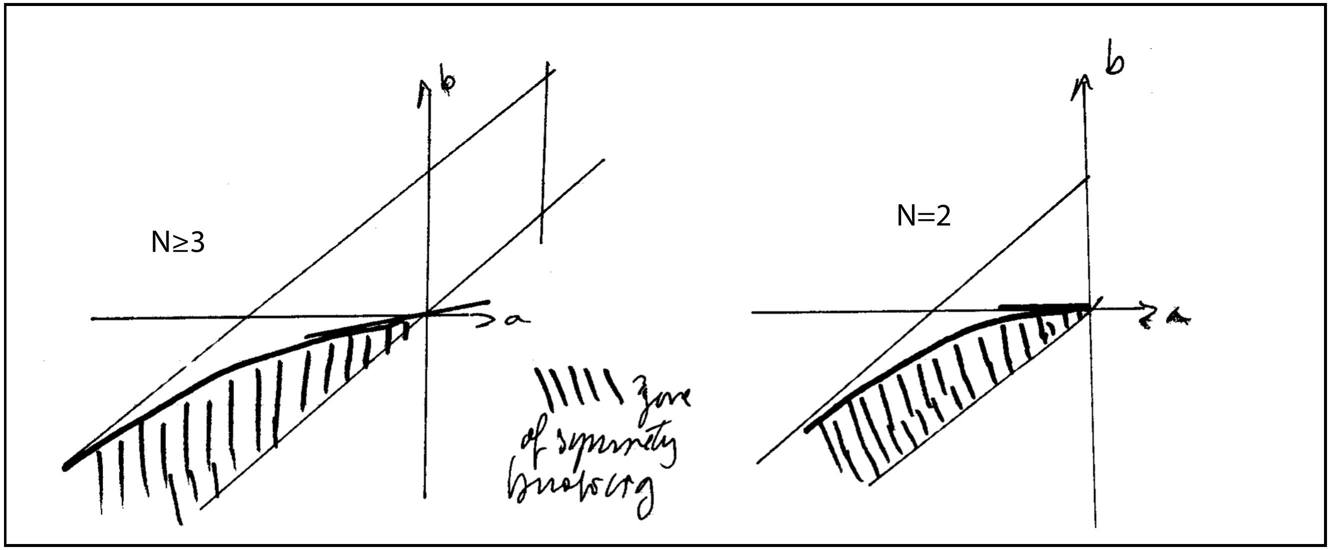
- Radial symmetry has also been established for $d \geq 3$, $a < 0$, $|a|$ small and $0 < b < a + 1$: [Lin-Wang, Smets-Willem]
- Schwarz foliated symmetry [Smets-Willem]



$d = 3$: optimality is achieved among solutions which depend only on the “latitude" θ and on r. Similar results hold in higher dimensions

Symmetry breaking

- [Catrina-Wang, Felli-Schneider] if $a < 0$, $a \le b < b^{FS}(a)$, the extremal functions ARE NOT radially symmetric !



$$b^{FS}(a) = \frac{d\,(d-2-2a)}{2\,\sqrt{(d-2-2a)^2+4(d-1)}} - \frac{1}{2}\,(d-2-2a)$$

- [Catrina-Wang] As $a \to -\infty$, optimal functions look like some decentered optimal functions for some Gagliardo-Nirenberg interpolation inequalities (after some appropriate transformation)

Approaching Onofri's inequality ($d = 2$)

- [J.D., M. Esteban, G. Tarantello] A generalized Onofri inequality

On $\mathbb{R}^2$, consider $d\mu_\alpha = \frac{\alpha+1}{\pi} \frac{|x|^{2\alpha}\,dx}{(1+|x|^{2\,(\alpha+1)})^2}$ with $\alpha > -1$

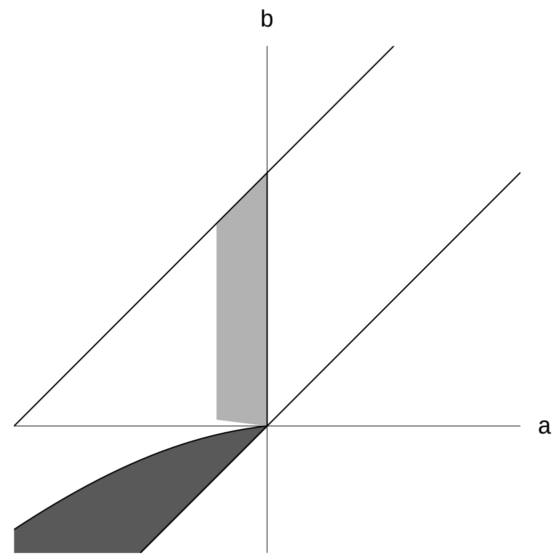
$$\log\left(\int_{\mathbb{R}^2} e^v \, d\mu_\alpha\right) - \int_{\mathbb{R}^2} v \, d\mu_\alpha \le \frac{1}{16\,\pi\,(\alpha+1)} \|\nabla v\|^2_{L^2(\mathbb{R}^2,\,dx)}$$

- For $d = 2$, radial symmetry holds if $-\eta < a < 0$ and $-\varepsilon(\eta)\,a \le b < a+1$

Theorem 3. [J.D.-Esteban-Tarantello] *For all* $\varepsilon > 0 \; \exists \, \eta > 0$ *s.t. for* $a < 0$, $|a| < \eta$

(i) *if* $|a| > \frac{2}{p-\varepsilon}\,(1+|a|^2)$, *then*
$\mathsf{C}_{a,b} > \mathsf{C}^*_{a,b}$ *(symmetry breaking)*

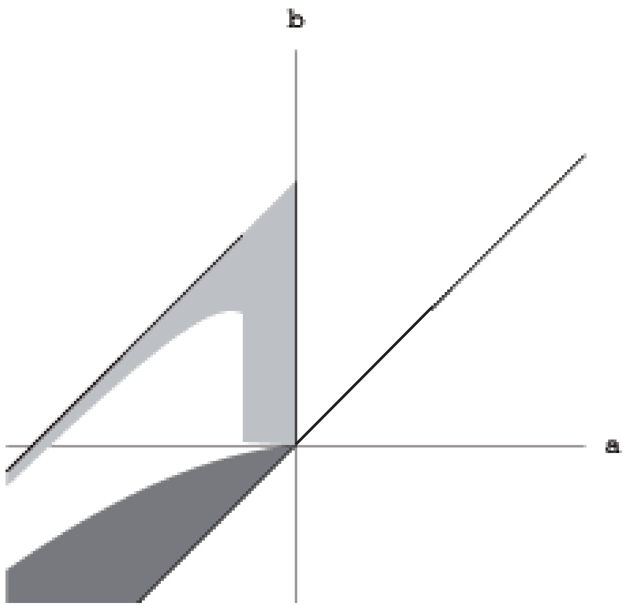
(ii) *if* $|a| < \frac{2}{p+\varepsilon}\,(1+|a|^2)$, *then*
s $\mathsf{C}_{a,b} = \mathsf{C}^*_{a,b}$ *and* $u_{a,b} = u^*_{a,b}$



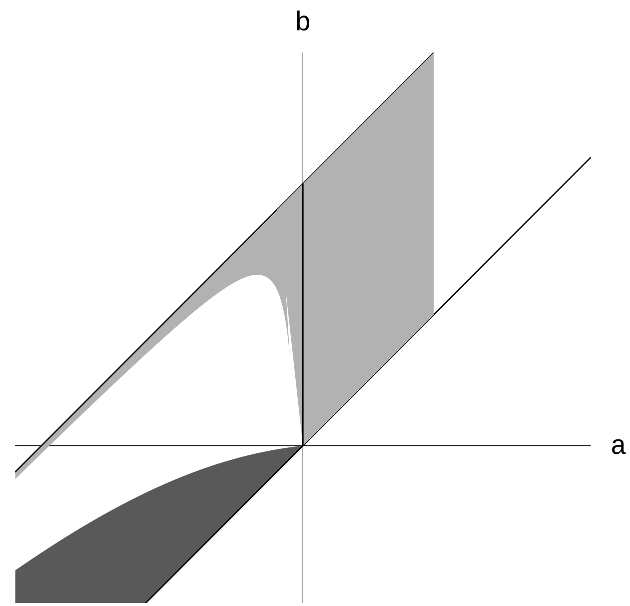
A larger symetry region

- For $d \geq 2$, radial symmetry can be proved when b is close to $a+1$

Theorem 4. [J.D.-Esteban-Loss-Tarantello] *Let $d \geq 2$. For every $A < 0$, there exists $\varepsilon > 0$ such that the extremals are radially symmetric if $a+1-\varepsilon < b < a+1$ and $a \in (A,0)$. So they are given by $u^*_{a,b}$, up to a scalar multiplication and a dilation*



$d = 2$



$d \geq 3$

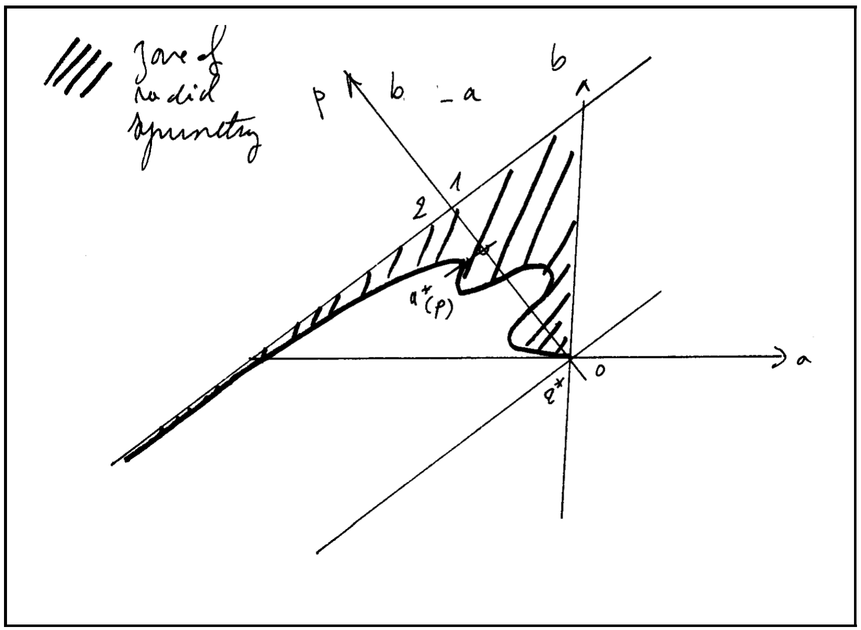
Two regions and a curve

- The symmetry and the symmetry breaking zones are simply connected and separated by a continuous curve

Theorem 5. [J.D.-Esteban-Loss-Tarantello] *For all* $d \geq 2$*, there exists a continuous function* $a^* \colon (2, 2^*) \longrightarrow (-\infty, 0)$ *such that* $\lim_{p \to 2^*_-} a^*(p) = 0$*,* $\lim_{p \to 2_+} a^*(p) = -\infty$ *and*

(i) *If* $(a, p) \in \left(a^*(p), \frac{d-2}{2}\right) \times (2, 2^*)$*, all extremals radially symmetric*

(ii) *If* $(a, p) \in (-\infty, a^*(p)) \times (2, 2^*)$*, none of the extremals is radially symmetric*



Open question. Do the curves obtained by Felli-Schneider and ours coincide ?

Emden-Fowler transformation and the cylinder $\mathcal{C} = \mathbb{R} \times \mathbb{S}^{d-1}$

$$t = \log|x| \,, \quad \omega = \frac{x}{|x|} \in \mathbb{S}^{d-1} \,, \quad w(t,\omega) = |x|^{-a}\, v(x) \,, \quad \Lambda = \frac{1}{4}\,(d-2-2a)^2$$

- Caffarelli-Kohn-Nirenberg inequalities rewritten on the cylinder become standard interpolation inequalities of Gagliardo-Nirenberg type

$$\|w\|^2_{L^p(\mathcal{C})} \le C_{\Lambda,p} \left[\|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\, \|w\|^2_{L^2(\mathcal{C})} \right]$$

$$\mathcal{E}_\Lambda[w] := \|\nabla w\|^2_{L^2(\mathcal{C})} + \Lambda\, \|w\|^2_{L^2(\mathcal{C})}$$

$$C^{-1}_{\Lambda,p} := \mathsf{C}^{-1}_{a,b} = \inf \left\{ \mathcal{E}_\Lambda(w) \;:\; \|w\|^2_{L^p(\mathcal{C})} = 1 \right\}$$

$$a < 0 \implies \Lambda > a_c^2 = \tfrac{1}{4}\,(d-2)^2$$

"critical / Sobolev" case: $b - a \to 0 \iff p \to \dfrac{2d}{d-2}$

"Hardy" case: $b - (a+1) \to 0 \iff p \to 2_+$

Methods for proving symmetry breaking

- Strategy of [Catrina-Wang, Felli-Schneider]

Expand $\mathcal{E}_\Lambda[w]$ around $w^*_{\Lambda,p}$ with w in an appropriate orthogonal space to $w_{\Lambda,p}$. This amounts to study the spectrum of

$$-\Delta + \Lambda - (p-1)\,|w^*_{\Lambda,p}|^{p-2}$$

in $H^1(\mathcal{C})$, make en expansion in spherical harmonics and compute the lowest eigenvalue associated to the first non-constant spherical harmonic function

Details will be given later

- Alternative proof in dimension $d = 2$ close to $(a, b) = (0, 0)$: [J.D.-Esteban-Tarantello]

- Energy comparison: at the end of this talk [Catrina, Wang] [JD, Esteban, Tarantello, Tertikas]

Auxiliary results for symmetry proofs

- Euler-Lagrange equations: Multiplication by constants does not affect optimality (no more scaling invariance in $\mathcal{C}$): we normalize so that the optimal functions solve $-\Delta w + \Lambda w = w^{p-1}$, that is

$$\int_{\mathcal{C}} |\nabla w|^2 \, dx + \Lambda \int_{\mathcal{C}} |w|^2 \, dx = \int_{\mathcal{C}} |w|^p \, dx$$

- Normalization: With $1/C_{\Lambda,p} = \mathcal{E}[w]/\|w\|^2_{L^p(\mathcal{C})}$, this determines $\|w\|_{L^p(\mathcal{C})}$

Lemma 6. [JD, Esteban, Loss, Tarantello] *Let $d \geq 2$, $p \in (2, 2^*)$. For any $\Lambda \neq 0$, we have*

$$\left(\mathsf{C}^d_{\Lambda,p}\right)^{-\frac{p}{p-2}} = \|w_{\Lambda,p}\|^p_{L^p(\mathcal{C})} \leq \|w^*_{\Lambda,p}\|^p_{L^p(\mathcal{C})} = 4\,|\mathbb{S}^{d-1}|\,(2\,\Lambda\,p)^{\frac{p}{p-2}}\;\frac{c_p}{2\,p\,\sqrt{\Lambda}}$$

where $p \mapsto c_p$ is increasing and $\lim_{p\to 2_+} 2^{\frac{2\,p}{p-2}}\sqrt{p-2}\,c_p = \sqrt{2\pi}$

- The extremals can be chosen to satisfy: $w_{\Lambda,p}$ depends only on r and the azimuthal angle θ, $\max_{\mathcal{C}} w_{\Lambda,p} = w_{\Lambda,p}(0,\omega_0)$ for some $\omega_0 \in \mathbb{S}^{d-1}$ and $\partial_t w_{\Lambda,p} < 0$ for any $t > 0$

“Hardy” regime (b close to $a+1$, $d \geq 2$)

- Let $(\Lambda_n)_{n\in\mathbb{N}}$ and $(p_n)_{n\in\mathbb{N}}$ be such that

$$\lim_{n\to+\infty} \Lambda_n = \Lambda \geq (d-2)^2/4 \quad \text{and} \quad \lim_{n\to+\infty} p_n = 2_+$$

such that the corresponding global minimizer $w_n := w_{\Lambda_n,\,p_n}$ satisfies

$$\mathcal{F}_{\Lambda,p}[w_{\Lambda_n,\,p_n}] < \mathcal{F}_{\Lambda,p}[w^*_{\Lambda_n,\,p_n}] \quad \text{and} \quad -\Delta_y w_n + \Lambda_n\, w_n = w_n^{p-1} \quad \text{in} \quad \mathcal{C}$$

- Define $c_n^2 := (\Lambda_n\, p_n)^{-\frac{p_n}{p_n-2}}\, 2^{\frac{p_n}{p_n-2}}\, \sqrt{p_n-2}$ and $W_n := c_n\, w_n$. We have $\lim_{n\to+\infty} c_n^{2-p_n} = \Lambda$ and

$$\limsup_{n\to+\infty} \int_{\mathcal{C}} |\nabla W_n|^2\, dy + \Lambda_n \int_{\mathcal{C}} W_n^2\, dy = \limsup_{n\to+\infty} c_n^2 \int_{\mathcal{C}} w_n^{p_n}\, dy \leq |\mathbb{S}^{d-1}|\, \sqrt{2\,\pi/\Lambda}$$

so that $(W_n)_{n\in\mathbb{N}}$ is bounded in $H^1(\mathcal{C})$. By elliptic estimates, $W_n \to W$ and $-\Delta W + \Lambda W = \Lambda W \implies W \equiv 0$

“Hardy” regime (continued)

Let $\chi_n := \nabla_\omega w_n := \sin\theta^{2-d}\,\frac{\partial}{\partial\theta}\big(\sin\theta^{d-2}\,w_n\big)$. By differentiating $-\Delta W_n + \Lambda_n\,W_n = c_n^{2-p_n}\,W_n^{p_n-1}$ with respect to θ, we get

$$-\Delta\chi_n + \Lambda_n\,\chi_n = (p_n-1)\,c_n^{2-p_n}\,W_n^{p_n-2}\,\chi_n$$

$$0 = \int_{\mathcal C}|\nabla\chi_n|^2\,dy + \Lambda_n\int_{\mathcal C}|\chi_n|^2\,dy - (p_n-1)\,c_n^{2-p_n}\int_{\mathcal C}W_n^{p_n-2}\,|\chi_n|^2\,dy$$

Since $\int_{\mathbb S^{d-1}}\chi_n\,d\omega = 0$

$$\int_{\mathcal C}|\nabla\chi_n|^2\,dy \ge (d-1)\int_{\mathcal C}|\chi_n|^2\,dy$$

by the Poincaré inequality. But W_n is bounded by $W_n(0,\omega_0)$, we get

$$0 \ge \Big(d-1+\underbrace{\Lambda_n-(p_n-1)\,c_n^{2-p_n}\,W_n(0,\omega_0)^{p_n-2}}_{\to 0 \text{ as } n\to\infty}\Big)\int_{\mathcal C}|\chi_n|^2\,dy$$

This proves that $\chi_n\equiv 0$ for n large enough □

Scaling and consequences

- A scaling property along the axis of the cylinder ($d \geq 2$)
let $w_\sigma(t,\omega) := w(\sigma\,t,\omega)$ for any $\sigma > 0$

$$\mathcal{F}_{\sigma^2\Lambda,p}(w_\sigma) = \sigma^{1+2/p}\,\mathcal{F}_{\Lambda,p}(w) - \sigma^{-1+2/p}\,(\sigma^2-1)\,\frac{\int_{\mathcal{C}}|\nabla_\omega w|^2\,dy}{\left(\int_{\mathcal{C}}|w|^p\,dy\right)^{2/p}}$$

Lemma 7. [JD, Esteban, Loss, Tarantello] *If* $d \geq 2$, $\Lambda > 0$ *and* $p \in (2, 2^*)$

(i) If $\mathsf{C}^d_{\Lambda,p} = \mathsf{C}^{d,*}_{\Lambda,p}$*, then* $C^d_{\lambda,p} = \mathsf{C}^{d,*}_{\lambda,p}$ *and* $w_{\lambda,p} = w^*_{\lambda,p}$*, for any* $\lambda \in (0,\Lambda)$

(ii) If there is a non radially symmetric extremal $w_{\Lambda,p}$*, then* $\mathsf{C}^d_{\lambda,p} > \mathsf{C}^{d,*}_{\lambda,p}$ *for all* $\lambda > \Lambda$

A curve separates symmetry and symmetry breaking regions

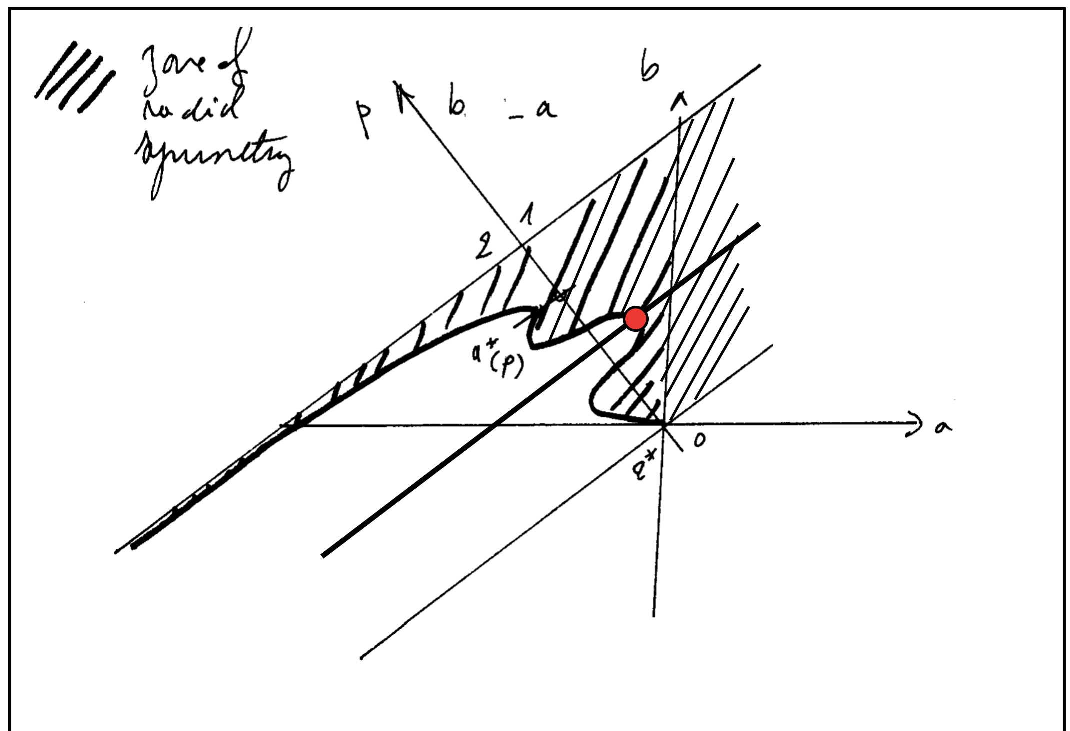
Corollary 8. [JD, Esteban, Loss, Tarantello] *Let* $d \geq 2$*. For all* $p \in (2, 2^*)$*,* $\Lambda^*(p) \in (0, \Lambda^{\rm FS}(p)]$ *and*

(i) *If* $\lambda \in (0, \Lambda^*(p))$*, then* $w_{\lambda,p} = w^*_{\lambda,p}$ *and clearly,* $C^d_{\lambda,p} = C^{d,*}_{\lambda,p}$

(ii) *If* $\lambda = \Lambda^*(p)$*, then* $C^d_{\lambda,p} = C^{d,*}_{\lambda,p}$

(iii) *If* $\lambda > \Lambda^*(p)$*, then* $C^d_{\lambda,p} > C^{d,*}_{\lambda,p}$

Upper semicontinuity
is easy to prove
For continuity,
a delicate spectral
analysis is needed



Caffarelli-Kohn-Nirenberg inequalities (Part II) and Logarithmic Hardy inequalities

Joint work with M. del Pino, S. Filippas and A. Tertikas

Generalized Caffarelli-Kohn-Nirenberg inequalities (CKN)

Let $2^* = \infty$ if $d = 1$ or $d = 2$, $2^* = 2d/(d-2)$ if $d \geq 3$ and define

$$\vartheta(p,d) := \frac{d\,(p-2)}{2\,p}$$

Theorem 9. [Caffarelli-Kohn-Nirenberg-84] *Let $d \geq 1$. For any $\theta \in [\vartheta(p,d),1]$, with $p = \frac{2\,d}{d-2+2\,(b-a)}$, there exists a positive constant $\mathsf{C}_{\rm CKN}(\theta,p,a)$ such that*

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}_{\rm CKN}(\theta,p,a) \left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta} \left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

In the radial case, with $\Lambda = (a-a_c)^2$, the best constant when the inequality is restricted to radial functions is $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$ and

$$\mathsf{C}_{\rm CKN}(\theta,p,a) \geq \mathsf{C}^*_{\rm CKN}(\theta,p,a) = \mathsf{C}^*_{\rm CKN}(\theta,p)\,\Lambda^{\frac{p-2}{2p}-\theta}$$

$$\mathsf{C}^*_{\rm CKN}(\theta,p) = \left[\frac{2\,\pi^{d/2}}{\Gamma(d/2)}\right]^{2\,\frac{p-1}{p}} \left[\frac{(p-2)^2}{2+(2\,\theta-1)\,p}\right]^{\frac{p-2}{2\,p}} \left[\frac{2+(2\,\theta-1)\,p}{2\,p\,\theta}\right]^{\theta} \left[\frac{4}{p+2}\right]^{\frac{6-p}{2\,p}} \left[\frac{\Gamma\left(\frac{2}{p-2}+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{2}{p-2}\right)}\right]$$

Weighted logarithmic Hardy inequalities (WLH)

- A “logarithmic Hardy inequality”

Theorem 10. [del Pino, J.D. Filippas, Tertikas] *Let* $d \geq 3$*. There exists a constant* $\mathsf{C}_{\rm LH} \in (0, \mathsf{S}]$ *such that, for all* $u \in \mathcal{D}^{1,2}(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2}\,dx = 1$*, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^2} \log\left(|x|^{d-2}|u|^2\right) dx \leq \frac{d}{2} \log\left[\mathsf{C}_{\rm LH} \int_{\mathbb{R}^d} |\nabla u|^2\,dx\right]$$

- A “weighted logarithmic Hardy inequality” (WLH)

Theorem 11. [del Pino, J.D. Filippas, Tertikas] *Let* $d \geq 1$*. Suppose that* $a < (d-2)/2$*,* $\gamma \geq d/4$ *and* $\gamma > 1/2$ *if* $d = 2$*. Then there exists a positive constant* $\mathsf{C}_{\rm WLH}$ *such that, for any* $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ *normalized by* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx = 1$*, we have*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}} \log\left(|x|^{d-2-2\,a}\,|u|^2\right) dx \leq 2\,\gamma\,\log\left[\mathsf{C}_{\rm WLH} \int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

Weighted logarithmic Hardy inequalities: radial case

Theorem 12. [del Pino, J.D. Filippas, Tertikas] *Let* $d \geq 1$, $a < (d-2)/2$ *and* $\gamma \geq 1/4$. *If* $u = u(|x|) \in \mathcal{D}^{1,2}_a(\mathbb{R}^d)$ *is radially symmetric, and* $\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx = 1$, *then*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,\log\left(|x|^{d-2-2\,a}\,|u|^2\right)dx \leq 2\,\gamma\,\log\left[\mathsf{C}^*_{\rm WLH}\int_{\mathbb{R}^d}\frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

$$\mathsf{C}^*_{\rm WLH} = \frac{1}{\gamma}\,\frac{\left[\Gamma\left(\frac{d}{2}\right)\right]^{\frac{1}{2\,\gamma}}}{(8\,\pi^{d+1}\,e)^{\frac{1}{4\,\gamma}}}\left(\frac{4\,\gamma-1}{(d-2-2\,a)^2}\right)^{\frac{4\,\gamma-1}{4\,\gamma}} \quad \text{if} \quad \gamma > \frac{1}{4}$$

$$\mathsf{C}^*_{\rm WLH} = 4\,\frac{\left[\Gamma\left(\frac{d}{2}\right)\right]^2}{8\,\pi^{d+1}\,e} \quad \text{if} \quad \gamma = \frac{1}{4}$$

If $\gamma > \frac{1}{4}$, *equality is achieved by the function*

$$u = \frac{\tilde u}{\int_{\mathbb{R}^d}\frac{|\tilde u|^2}{|x|^2}\,dx} \quad \text{where} \quad \tilde u(x) = |x|^{-\frac{d-2-2\,a}{2}}\,\exp\left(-\frac{(d-2-2\,a)^2}{4\,(4\,\gamma-1)}\left[\,\log|x|\,\right]^2\right)$$

Extremal functions for Caffarelli-Kohn-Nirenberg and logarithmic Hardy inequalities

Joint work with Maria J. Esteban

First existence result: the sub-critical case

Theorem 13. [J.D. Esteban] *Let $d \geq 2$ and assume that $a \in (-\infty, a_c)$*

(i) *For any $p \in (2, 2^*)$ and any $\theta \in (\vartheta(p,d), 1)$, the Caffarelli-Kohn-Nirenberg inequality (CKN)*

$$\left(\int_{\mathbb{R}^d} \frac{|u|^p}{|x|^{b\,p}}\,dx\right)^{\frac{2}{p}} \leq \mathsf{C}(\theta,p,a)\left(\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right)^{\theta}\left(\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,dx\right)^{1-\theta}$$

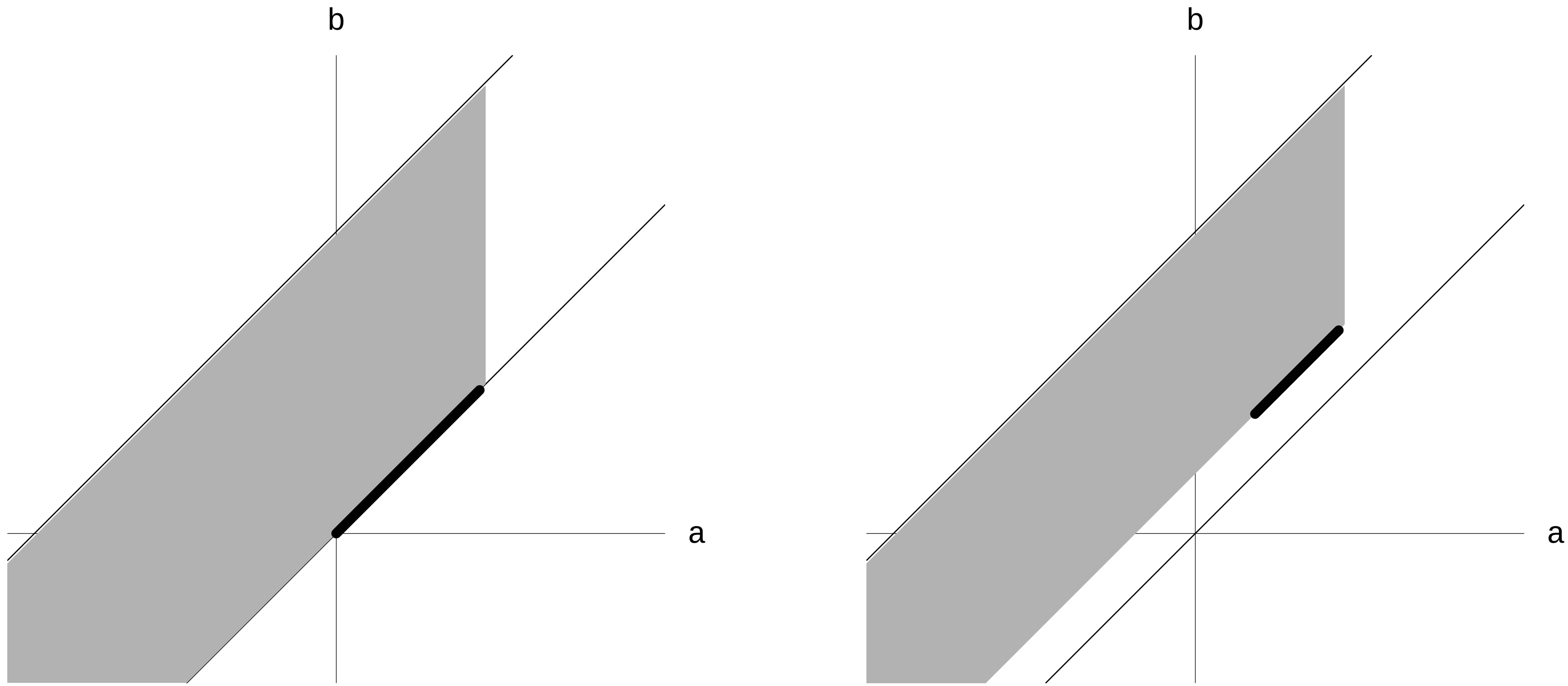
admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$
Critical case: there exists a continuous function $a^* : (2, 2^*) \to (-\infty, a_c)$ such that the inequality also admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$ if $\theta = \vartheta(p,d)$ and $a \in (a^*(p), a_c)$

(ii) *For any $\gamma > d/4$, the weighted logarithmic Hardy inequality (WLH)*

$$\int_{\mathbb{R}^d} \frac{|u|^2}{|x|^{2\,(a+1)}}\,\log\left(|x|^{d-2-2\,a}\,|u|^2\right)\,dx \leq 2\,\gamma\,\log\left[\mathsf{C}_{\mathrm{WLH}}\int_{\mathbb{R}^d} \frac{|\nabla u|^2}{|x|^{2\,a}}\,dx\right]$$

admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$
Critical case: idem if $\gamma = d/4$, $d \geq 3$ and $a \in (a^\star, a_c)$ for some $a^\star \in (-\infty, a_c)$

Existence for CKN



$d = 3,\ \theta = 1$ $\qquad$ $d = 3,\ \theta = 0.8$

A possible loss of compactness

- Gagliardo-Nirenberg interpolation inequalities: if $p \in (2, 2^*)$,

$$\|u\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \le \mathsf{C}_{\mathrm{GN}}(p)\, \|\nabla u\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\, \|u\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d)$$

If u is a radial minimizer for $1/\mathsf{C}_{\mathrm{GN}}(p)$ and $u_n(x) := u(x + n\,\mathsf{e})$, $\mathsf{e} \in \mathbb{S}^{d-1}$

$$\frac{1}{\mathsf{C}_{\mathrm{CKN}}(\vartheta(p,d),p,a)} \le \frac{\||x|^{-a}\,\nabla u_n\|^{2\,\vartheta(p,d)}_{\mathrm{L}^2(\mathbb{R}^d)}\, \||x|^{-(a+1)}\,u_n\|^{2\,(1-\vartheta(p,d))}_{\mathrm{L}^2(\mathbb{R}^d)}}{\||x|^{-b}\,u_n\|^2_{\mathrm{L}^p(\mathbb{R}^d)}}$$

$$= \frac{1}{\mathsf{C}_{\mathrm{GN}}(p)}\, \left(1 + \mathcal{R}\,n^{-2} + O(n^{-4})\right)$$

- Gross' logarithmic Sobolev inequality in Weissler's form

$$e^{\frac{2}{d}\int_{\mathbb{R}^d}|u|^2\,\log|u|^2\,dx} \le \mathsf{C}_{\mathrm{LS}}\, \|\nabla u\|^2_{\mathrm{L}^2(\mathbb{R}^d)} \quad \forall\, u \in \mathrm{H}^1(\mathbb{R}^d) \text{ such that } \|u\|_{\mathrm{L}^2(\mathbb{R}^d)} = 1$$

$$\mathsf{C}_{\mathrm{LS}} \le \mathsf{C}_{\mathrm{WLH}}$$

Second existence result

Let

$$a_\star := a_c - \sqrt{(d-1)\, e\, (2^{d+1}\, \pi)^{-1/(d-1)}\, \Gamma(d/2)^{2/(d-1)}}$$

Theorem 14 (Critical cases). [J.D. Esteban]

(i) *if* $\theta = \vartheta(p,d)$ *and* $\mathsf{C}_{\rm GN}(p) < \mathsf{C}_{\rm CKN}(\theta,p,a)$, *then (CKN) admits an extremal function in* $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$,

(ii) *if* $\gamma = d/4$, $d \geq 3$, *and* $\mathsf{C}_{\rm LS} < \mathsf{C}_{\rm WLH}(\gamma,a)$, *then (WLH) admits an extremal function in* $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$

If $a \in (a_\star, a_c)$ *then*

$$\mathsf{C}_{\rm LS} < \mathsf{C}_{\rm WLH}(d/4,a)$$

Strategy of the proofs (1/2)

- Emden-Fowler transformation and minimization on the cylinder of the functionals

$$\mathcal{E}_\theta[v] := \left(\|\nabla v\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\,\|v\|^2_{\mathrm{L}^2(\mathcal{C})}\right)^\theta \|v\|^{2\,(1-\theta)}_{\mathrm{L}^2(\mathcal{C})}$$

$$\mathcal{F}_\gamma[w] := \left(\|\nabla w\|^2_{\mathrm{L}^2(\mathcal{C})} + \Lambda\right)\,\exp\left[-\frac{1}{2\,\gamma}\int_{\mathcal{C}} |w|^2\,\log|w|^2\,dy\right]$$

  under the constraints $\|v\|_{\mathrm{L}^p(\mathcal{C})} = 1$ and $\|w\|_{\mathrm{L}^2(\mathcal{C})} = 1$

- Convergence of minimizing sequences if they are bounded in $H^1(\mathcal{C})$: *no vanishing*

Lemma 15. [Lions 1984,Catrina-Wang 2001] *Let* $r > 0$ *and* $q \in [2, 2^*)$. *If* $(f_n)_n$ *is bounded in* $\mathrm{H}^1(\mathcal{C})$ *and if* $\limsup_{n\to\infty}\int_{B_r(y)\cap\mathcal{C}} |f_n|^q\,dy = 0$ *for any* $y \in \mathcal{C}$, *then* $\lim_{n\to\infty}\|f_n\|_{\mathrm{L}^p(\mathcal{C})} = 0$ *for any* $p \in (2, 2^*)$.

Up to translations, the sequences converge to a nontrivial limit

Strategy of the proofs (2/2)

- For any $\mathsf{x}, \mathsf{y} > 0$, $\eta \in (0,1)$, we have: $(1+\mathsf{x})^\eta\,(1+\mathsf{y})^{1-\eta} \geq 1+\mathsf{x}^\eta\,\mathsf{y}^{1-\eta}$ with strict inequality unless $\mathsf{x}=\mathsf{y}$
- Passing to the limit

$$\frac{1}{\mathsf{C}_{\mathrm{CKN}}(\theta,p,a)} = \lim_{n\to\infty} \mathcal{E}_\theta[v_n] \geq \mathcal{E}_\theta[v] + \lim_{n\to\infty} \mathcal{E}_\theta[v_n - v]$$
$$\geq \frac{1}{\mathsf{C}_{\mathrm{CKN}}(\theta,p,a)} \left(\|v\|^2_{\mathrm{L}^p(\mathcal{C})} + \lim_{n\to\infty} \|v_n - v\|^2_{\mathrm{L}^p(\mathcal{C})} \right)$$

+ Brezis-Lieb Lemma

$$1 = \|v_n\|^p_{\mathrm{L}^p(\mathcal{C})} = \|v\|^p_{\mathrm{L}^p(\mathcal{C})} + \lim_{n\to\infty} \|v_n - v\|^p_{\mathrm{L}^p(\mathcal{C})}$$

- Concavity of the function $f(\mathsf{z}) := \mathsf{z}^{2/p} + (1-\mathsf{z})^{2/p}$, $\mathsf{z} \in [0,1]$, $f(\mathsf{z}) \geq 1$ with strict inequality unless $\mathsf{z}=0$ or $\mathsf{z}=1$: *no splitting*

For (WLH) use: $\eta\,\mathsf{x}^{1/\eta} + (1-\eta)\,\mathsf{y}^{1/(1-\eta)} \geq \mathsf{x}\,\mathsf{y}$
with strict inequality unless $\mathsf{x}=\mathsf{y}$ and $\eta = 1/2$

Boundedness in $H^1(\mathcal{C})$?

- (CKN) with $\theta > \vartheta(p,d)$ and (WLH) with $\gamma > d/4$: by interpolation, any minimizing sequence is bounded in $H^1(\mathcal{C})$
- Critical cases: (CKN) with $\theta = \vartheta(p,d)$ and (WLH) with $\gamma = d/4$
  - An approach by contradiction
  - An unbounded minimizing sequence concentrates
  - We take as a special minimizing sequence a sequence of minimizers corresponding to a $\theta_n > \vartheta(p,d)$ or $\gamma_n > d/4$, use the Euler-Lagrange equations and perform a blow-up analysis
  - we find a contradiction with $\mathsf{C}_{\rm GN}(p) < \mathsf{C}_{\rm CKN}(\theta,p,a)$ for $\theta = \vartheta(p,d)$, or with $\mathsf{C}_{\rm LS} < \mathsf{C}_{\rm WLH}(\gamma,a)$ if $\gamma = d/4$, $d \geq 3$

If $a \in (a_\star, a_c)$, $d \geq 3$, then $\mathsf{C}_{\rm LS} < \mathsf{C}^*_{\rm WLH}(d/4,a) \leq \mathsf{C}_{\rm WLH}(d/4,a)$ and the logarithmic Hardy inequality admits a minimizer in the critical case $\gamma = d/4$, $d \geq 3$

Radial symmetry and symmetry breaking

Joint work with
M. del Pino, S. Filippas and A. Tertikas (symmetry breaking)
Maria J. Esteban, Gabriella Tarantello and Achilles Tertikas

Symmetry breaking for (CKN) inequalities

$$\Theta(a,p,d) := \tfrac{p-2}{32\,(d-1)\,p}\;\left[(p+2)^2\,(d^2+4\,a^2-4\,a\,(d-2))-4\,p\,(p+4)\,(d-1)\right]$$

$$a_-(p) := \tfrac{d-2}{2}-\tfrac{2\,(d-1)}{p+2}$$

Theorem 16. [del Pino, J.D. Filippas, Tertikas] *Let $d\ge 2$, $2<p<2^*$ and $a<a_-(p)$. Then $\mathsf{C}(\theta,p,a)>\mathsf{C}^*(\theta,p,a)$ if either*

$$\vartheta(p,d)\le\theta<\Theta(a,p,d)\quad\textit{when}\quad a\ge\frac{d-2}{2}-\frac{2\sqrt{d-1}}{\sqrt{(p-2)(p+2)}}$$

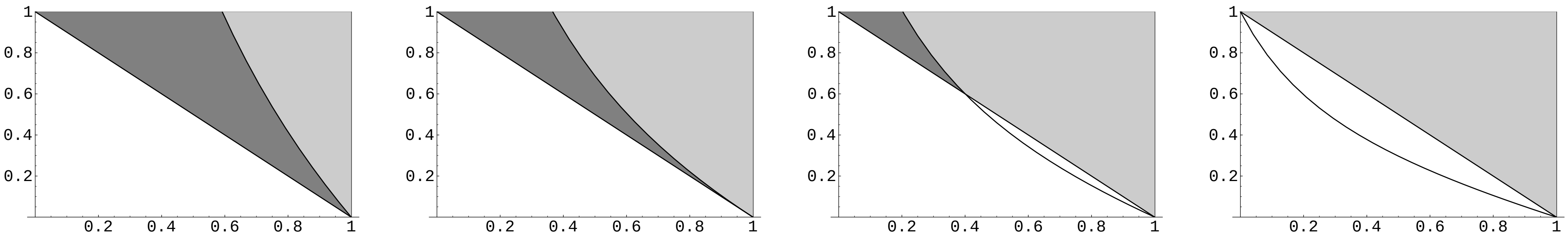
or

$$\vartheta(p,d)\le\theta\le 1\quad\textit{when}\quad a<\frac{d-2}{2}-\frac{2\sqrt{d-1}}{\sqrt{(p-2)(p+2)}}$$

In other words, symmetry breaking occurs in (CKN) if a, θ and p are in any of the two above regions

“Close” to (WLH): if $a<-1/2$, there exists $\varepsilon>0$, $\gamma_1>d/4$ and $\gamma_2>\gamma_1$ such that symmetry breaking occurs if $\theta=\gamma\,(p-2)$ for any $\gamma\in(\gamma_1,\gamma_2)$ and any $p\in(2,2+\varepsilon)$

Plots (1/2)



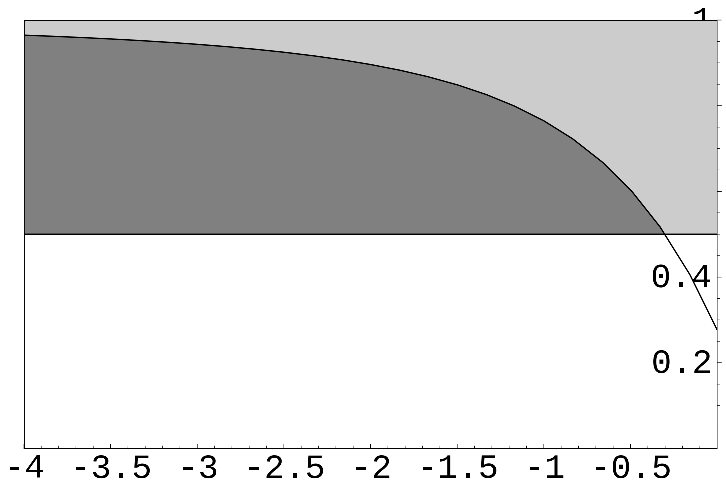
$a=-1$ $\quad a=-0.5$ $\quad a=-0.25$ $\quad a=0$

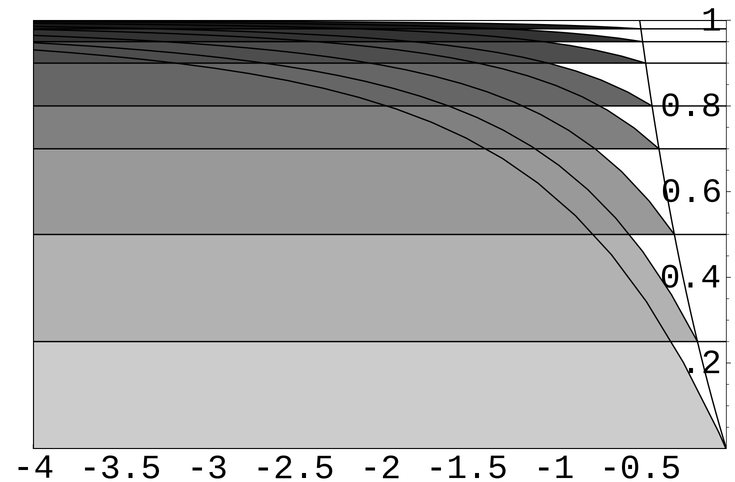
- Plots in (η,θ) coordinates, with $\eta := b-a$,
- Admissible regions appear in grey
- Symmetry breaking regions appear in dark grey
- The symmetry breaking region touches $(\eta,\theta)=(1,0)$ for $a\le -1/2$

Another parametrization: symmetry breaking holds for any

$$a < a_c - \frac{2\sqrt{d-1}}{p+2}\sqrt{\frac{2\,p\,\theta}{p-2}-1}$$

Plots (2/2)





- (a, η) representation of admissible areas: $\eta = b - a \geq 1 - \theta$
- Symmetry breaking region:

  $\eta < g(a, \theta)$ (dark grey)
- Here $d = 3$ and $\theta = 0.5$
- $\eta < g(a, 1)$ corresponds to the condition found by Felli and Schneider

- Regions of symmetry breaking:

  $$1 - \theta \leq \eta < g(a, \theta)$$

  $\theta = 1, 0.75, 0.5, 0.3, 0.2, 0.1, 0.05, 0.02$
- The enveloppe determines a curve $\eta = h(a)$
- The limit case $\eta = 0 = h(0)$ is the Felli and Schneider case
- $h(-1/2) = 1$ is consistent with symmetry breaking for WLH

Implementing the method of Catrina-Wang / Felli-Schneider

Among functions $w \in H^1(\mathcal{C})$ which depend only on s, the minimum of

$$\mathcal{J}[w] := \int_{\mathcal{C}} \left(|\nabla w|^2 + \tfrac{1}{4}\,(d-2-2\,a)^2\,|w|^2 \right) dy - [\mathsf{C}^*(\theta,p,a)]^{-\frac{1}{\theta}} \, \frac{\left(\int_{\mathcal{C}} |w|^p \; dy \right)^{\frac{2}{p\,\theta}}}{\left(\int_{\mathcal{C}} |w|^2 \; dy \right)^{\frac{1-\theta}{\theta}}}$$

is achieved by $\overline{w}(y) := \left[\cosh(\lambda\,s)\right]^{-\frac{2}{p-2}}$, $y=(s,\omega) \in \mathbb{R}\times\mathbb{S}^{d-1} = \mathcal{C}$ with $\lambda := \frac{1}{4}\,(d-2-2\,a)\,(p-2)\,\sqrt{\frac{p+2}{2\,p\,\theta-(p-2)}}$ as a solution of

$$\lambda^2\,(p-2)^2\,w'' - 4\,w + 2\,p\,|w|^{p-2}\,w = 0$$

Spectrum of $\mathcal{L} := -\Delta + \kappa\,\overline{w}^{p-2} + \mu$ is given for $\sqrt{1+4\,\kappa/\lambda^2} \geq 2\,j+1$ by

$$\lambda_{i,j} = \mu + i\,(d+i-2) - \frac{\lambda^2}{4}\left(\sqrt{1+\frac{4\,\kappa}{\lambda^2}} - (1+2\,j) \right)^2 \quad \forall\, i\,,\; j \in \mathbb{N}$$

- The eigenspace of $\mathcal{L}$ corresponding to $\lambda_{0,0}$ is generated by $\overline{w}$
- The eigenfunction $\phi_{(1,0)}$ associated to $\lambda_{1,0}$ is not radially symmetric and such that $\int_{\mathcal{C}} \overline{w}\,\phi_{(1,0)}\;dy = 0$ and $\int_{\mathcal{C}} \overline{w}^{p-1}\,\phi_{(1,0)}\;dy = 0$
- If $\lambda_{1,0} < 0$, *optimal functions for (CKN) cannot be radially symmetric* and

$$\mathsf{C}(\theta,p,a) > \mathsf{C}^*(\theta,p,a)$$

Logarithmic Hardy inequalities: symmetry breaking

Theorem 17. [del Pino, J.D. Filippas, Tertikas] *Let $d \geq 2$ and $a < -1/2$. Assume that $\gamma > 1/2$ if $d = 2$. If, in addition,*

$$\frac{d}{4} \leq \gamma < \frac{1}{4} + \frac{(d-2\,a-2)^2}{4\,(d-1)}$$

then the optimal constant $\mathsf{C}_{\rm WLH}$ is not achieved by a radial function and $\mathsf{C}_{\rm WLH} > C^*_{\rm WLH}$

- Same method, but applied to the functional

$$\mathcal{F}[w] := \frac{\int_{\mathcal{C}} |\nabla w|^2\;dy}{\int_{\mathcal{C}} |w|^2\;dy} + \sigma^2 - |\mathbb{S}^{d-1}|^{\frac{1}{2\,\gamma}}\;\exp\left[\frac{\mathcal{K}(\gamma,\sigma)}{2\,\gamma} + \frac{1}{2\,\gamma}\int_{\mathcal{C}} \frac{|w|^2}{\int_{\mathcal{C}} |w|^2\;dy}\,\log\left(\frac{|w|^2}{\int_{\mathcal{C}} |w|^2\;dy}\right)\,dy\right]$$

which has gaussian minimizers among functions $w \in H^1(\mathcal{C})$ which depend only on s

Surface / curve of separation

- Caffarelli-Kohn-Nirenberg inequality (CKN)

Theorem 18. [J.D. Esteban, Tarantello, Tertikas] *For all $d \geq 2$, there exists a continuous function a^* defined on the set $\{(\theta, p) \in (0,1] \times (2, 2^*) \; : \; \theta > \vartheta(p,d)\}$ with values in $(-\infty, a_c)$ such that $\lim_{p \to 2_+} a^*(\theta, p) = -\infty$ and*

(i) If $(a,p) \in (a^*(\theta,p), a_c) \times (2, 2^*)$, (CKN) has only radially symmetric extremals

(ii) If $(a,p) \in (-\infty, a^*(\theta,p)) \times (2, 2^*)$, none of the extremals of (CKN) is radially symmetric

(iii) For every $p \in (2, 2^*)$, $\underline{a}(\theta,p) \leq a^*(\theta,p) \leq \bar{a}(\theta,p) < a_c$

- Weighted logarithmic Hardy inequality (WLH)

Theorem 19. [J.D. Esteban, Tarantello, Tertikas] *Let $d \geq 2$, there exists a continuous function a^{**} defined on $(d/4, \infty)$, with values in $(-\infty, a_c)$ such that for any $\gamma > d/4$ and $a \in [a^{**}(\gamma), a_c)$, there is a radially symmetric extremal for WLH, while for $a < a^{**}(\gamma)$ no extremal of (WLH) is radially symmetric. Moreover, $a^{**}(\gamma) \geq \tilde{a}(\gamma)$ for any $\gamma \in (d/4, \infty)$*

The proof is similar to the case $\theta = 1$ but requires delicate estimates

Schwarz' symmetrization

With $u(x) = |x|^a \, v(x)$, (CKN) is then equivalent to

$$\||x|^{a-b} \, v\|^2_{\mathrm{L}^p(\mathbb{R}^d)} \leq \mathsf{C}_{\mathrm{CKN}}(\theta, p, \Lambda) \, (\mathcal{A} - \lambda \, \mathcal{B})^\theta \; \mathcal{B}^{1-\theta}$$

with $\mathcal{A} := \|\nabla v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$, $\mathcal{B} := \||x|^{-1} \, v\|^2_{\mathrm{L}^2(\mathbb{R}^d)}$ and $\lambda := a \, (2 \, a_c - a)$. We observe that the function $B \mapsto h(\mathcal{B}) := (\mathcal{A} - \lambda \, \mathcal{B})^\theta \; \mathcal{B}^{1-\theta}$ satisfies

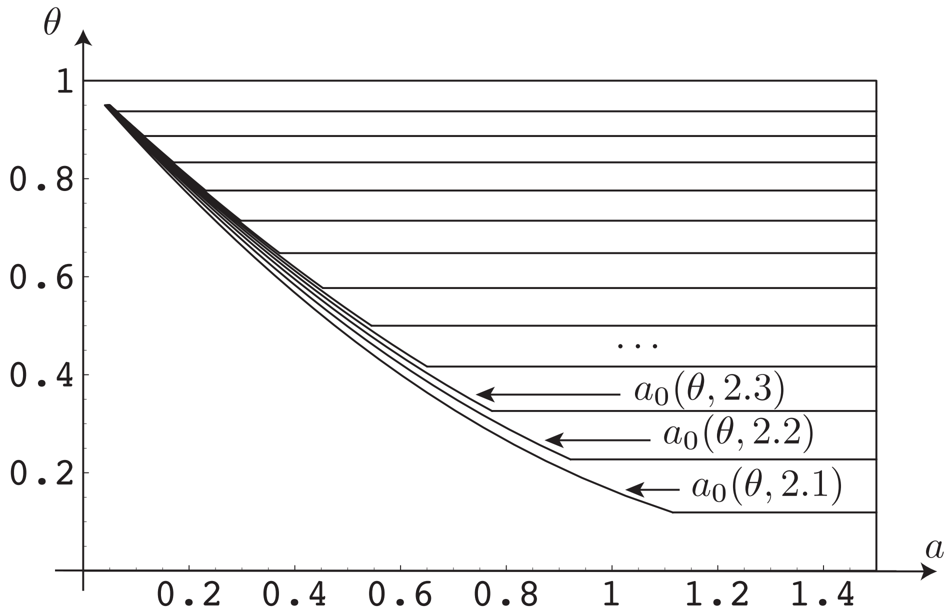
$$\frac{h'(\mathcal{B})}{h(\mathcal{B})} = \frac{1-\theta}{\mathcal{B}} - \frac{\lambda \, \theta}{\mathcal{A} - \lambda \, \mathcal{B}}$$

By Hardy's inequality ($d \geq 3$), we know that

$$\mathcal{A} - \lambda \, \mathcal{B} \geq \inf_{a>0} \big(\mathcal{A} - a \, (2 \, a_c - a) \, \mathcal{B} \big) = \mathcal{A} - a_c^2 \, \mathcal{B} > 0$$

and so $h'(\mathcal{B}) \leq 0$ if $(1-\theta) \, \mathcal{A} < \lambda \, \mathcal{B} \Longleftrightarrow \mathcal{A}/\mathcal{B} < \lambda/(1-\theta)$
By interpolation $\mathcal{A}/\mathcal{B}$ is small if $a_c - a > 0$ is small enough, for $\theta > \vartheta(p,d)$ and $d \geq 3$

Regions in which Schwarz' symmetrization holds



- Here $d = 5$, $a_c = 1.5$ and $p = 2.1$, 2.2, $\ldots 3.2$
- Symmetry holds if $a \in [a_0(\theta, p), a_c)$, $\theta \in (\vartheta(p, d), 1)$
- Horizontal segments correspond to $\theta = \vartheta(p, d)$
- Hardy's inequality: the above symmetry region is contained in $\theta > (1 - \frac{a}{a_c})^2$

Alternatively, we could prove the symmetry by the moving planes method *in the same region*

New results on symmetry breaking: WLH inequalities

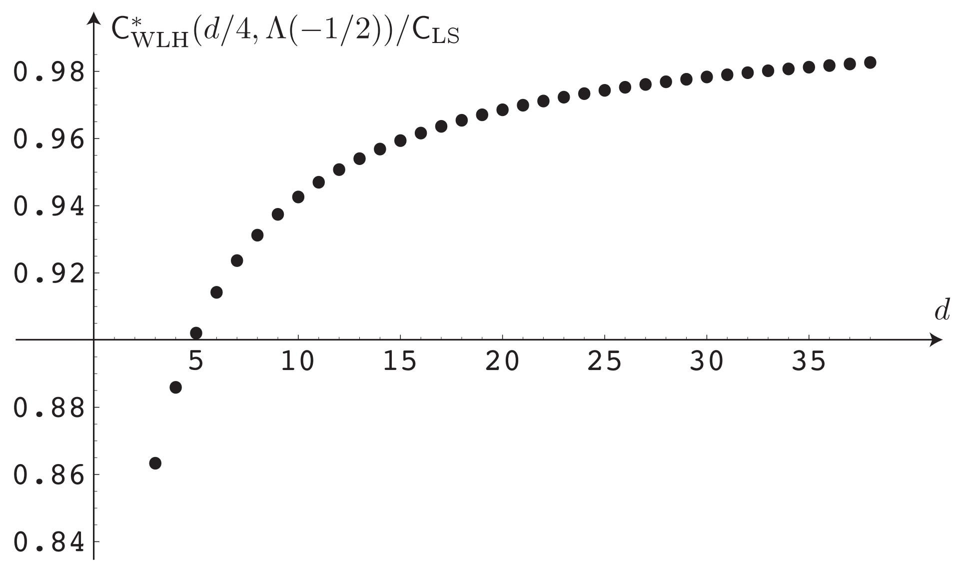
Recall that: for $\gamma = d/4$, $d \geq 3$, (WLH) admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$

If $a \in (a_\star, a_c)$ with $a_\star := a_c - \sqrt{(d-1)\, e\, (2^{d+1}\, \pi)^{-1/(d-1)}\, \Gamma(d/2)^{2/(d-1)}}$

- $\mathsf{C}_{\rm LS} < \mathsf{C}_{\rm WLH}(d/4, a)$ and for $\gamma = d/4$, $d \geq 3$, (WLH) admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$
- for $\gamma > d/4$, $d \geq 3$, (WLH) always admits an extremal function
- If $a_\star > a_{FS}(\gamma)$, for γ close enough to $d/4$ and $a \in (a_{FS}(\gamma), a_\star)$, the minimizer cannot be symmetric

The new symmetry breaking result for WLH

- $\mathsf{C}^*_{\rm WLH}(\gamma,\Lambda) < \mathsf{C}_{\rm LS}$ if and only if $\Lambda(a) > \Lambda_{\rm SB}(\gamma,d)$
- Notice that $\mathsf{C}^*_{\rm WLH}(d/4,\Lambda(-1/2)) < \mathsf{C}_{\rm LS}$ if $d \geq 3$, while, for $d=2$, we have $\lim_{\gamma\to(1/2)_+} \mathsf{C}^*_{\rm WLH}(\gamma,\Lambda(-1/2)) < \mathsf{C}_{\rm LS}$

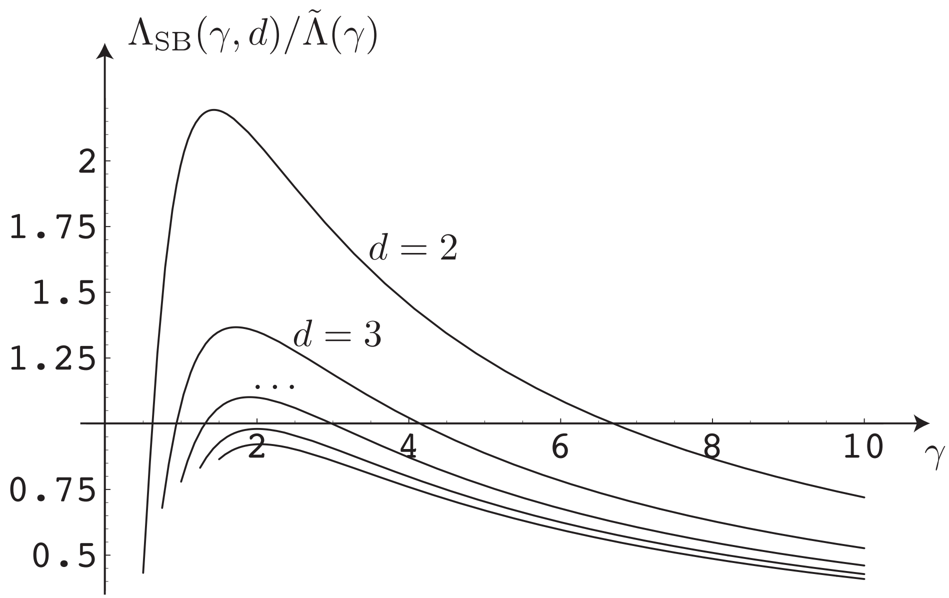


By continuity, the inequality $\mathsf{C}^*_{\rm WLH}(\gamma,\tilde\Lambda(\gamma)) < \mathsf{C}_{\rm LS}$ remains valid for $\gamma > d/4$, provided $\gamma - d/4 > 0$ is small enough... how small ?

Symmetry breaking for WLH: numerical range

By numerical calculations: $\Lambda(a) > \tilde{\Lambda}(\gamma)$ is more restrictive than $\Lambda(a) > \Lambda_{\rm SB}(\gamma, d)$ except if

- $d = 2$ and $\gamma \in [0.621414\ldots, 6.69625\ldots]$
- $d = 3$ and $\gamma \in [0.937725\ldots, 4.14851\ldots]$
- $d = 4$ and $\gamma \in [1.31303\ldots, 2.98835\ldots]$
- For $d \geq 5$, we observe that $\Lambda_{\rm SB}(\gamma, d) < \tilde{\Lambda}(\gamma)$



Symmetry breaking for WLH: a statement

$$\Lambda_{\rm SB}(\gamma,d) := \frac{1}{8}\,(4\,\gamma-1)\,e\,\left(\frac{\pi^{4\,\gamma-d-1}}{16}\right)^{\frac{1}{4\,\gamma-1}}\,\left(\frac{d}{\gamma}\right)^{\frac{4\,\gamma}{4\,\gamma-1}}\,\Gamma\left(\frac{d}{2}\right)^{\frac{2}{4\,\gamma-1}} \tag{1}$$

Theorem 20. [J.D. Esteban, Tarantello, Tertikas] *Let $d \geq 2$ and assume that $\gamma > 1/2$ if $d = 2$. If $\Lambda(a) > \Lambda_{\rm SB}(\gamma,d)$, then there is symmetry breaking: no extremal for (WLH) corresponding to the parameters (γ,a) is radially symmetric. As a consequence, there exists an $\varepsilon > 0$ such that, if $a \in [\tilde a(\gamma), \tilde a(\gamma)+\varepsilon)$ and $\gamma \in [d/4, d/4+\varepsilon)$, with $\gamma > 1/2$ if $d = 2$, there is symmetry breaking*

Recall that $\Lambda = (a-a_c)^2$, $\Lambda_{\rm SB}(d/4,d) = \Lambda_\star$ and $\underline{a}(\theta,p)$, $\tilde a(\gamma)$ are given by the method of Felli & Schneider

- $a < a_\star$ and $\gamma = d/4$: symmetry breaking
- $a \in (a_\star, a_c)$ and $\gamma = d/4$: existence of a minimizer for WLH

New results on symmetry breaking: CKN inequalities

Method:

- if $\theta = \vartheta(p,d)$ and $\mathsf{C}_{\rm GN}(p) < \mathsf{C}_{\rm CKN}(\theta,p,a)$, then (CKN) admits an extremal function in $\mathcal{D}^{1,2}_a(\mathbb{R}^d)$
- it is enough to find a good test function if we know the value of $\mathsf{C}_{\rm GN}(p)$ and if we evaluate from below $\mathsf{C}_{\rm CKN}(\theta,p,a)$ by $\mathsf{C}^*_{\rm CKN}(\theta,p,a)$

$$\mathsf{C}_{\rm GN}(p) \le \text{[test function]} \le \mathsf{C}^*_{\rm CKN}(\theta,p,a) \le \mathsf{C}_{\rm CKN}(\theta,p,a)$$

- $\mathsf{C}_{\rm GN}(p)$ converges to $\mathsf{C}_{\rm LS}$ as $p \to 2_+$, and gaussian functions are optimal for the logarithmic Sobolev inequality

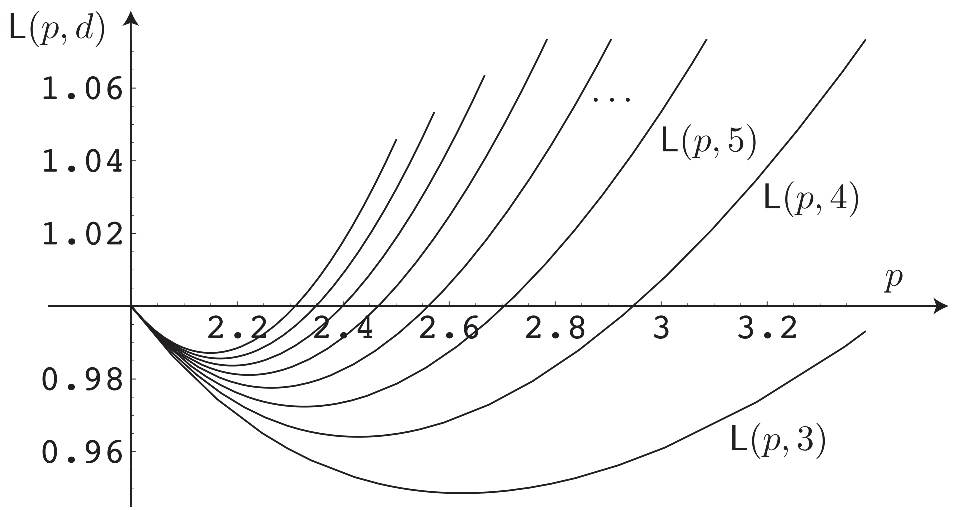
Use Gaussians as test functions, for p close enough to 2

The new symmetry breaking result for CKN

Let $\mathsf{g}(x) := (2\,\pi)^{-d/4}\,\exp(-|x|^2/4)$

$$\frac{1}{\mathsf{C}_{\mathrm{CKN}}(\vartheta(p,d),p,\Lambda(a_-(p)))} \le \frac{1}{\mathsf{C}_{\mathrm{GN}}(p)} \le h(p,d) := \frac{\|\nabla \mathsf{g}\|_{\mathrm{L}^2(\mathbb{R}^d)}^{2\,\vartheta(p,d)}\,\|\mathsf{g}\|_{\mathrm{L}^2(\mathbb{R}^d)}^{2\,(1-\vartheta(p,d))}}{\|\mathsf{g}\|_{\mathrm{L}^p(\mathbb{R}^d)}^2}$$

where $a_-(p) = \underline{\mathsf{a}}(\vartheta(p,d),p)$



Symmetry breaking occurs if

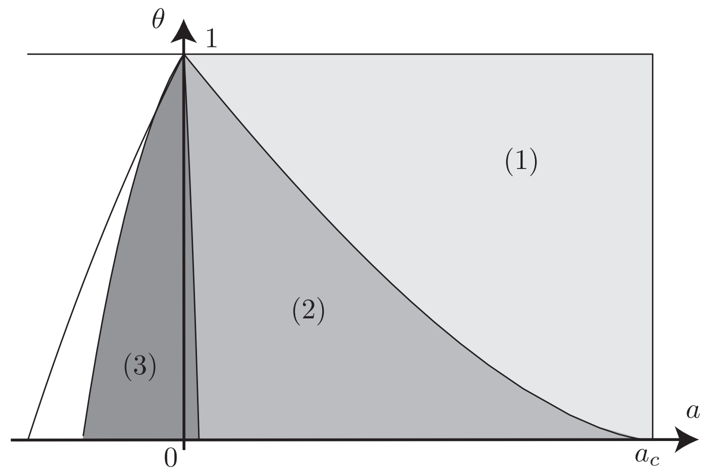
$$\mathsf{L}(p,d) := h(p,d)\,\mathsf{C}^*_{\mathrm{CKN}}\big(\vartheta(p,d),p,\Lambda(a_-(p))\big) < 1$$

Symmetry breaking for CKN: a statement

Theorem 21. [J.D. Esteban, Tarantello, Tertikas] *Let* $d \geq 2$*. There exists* $\eta > 0$ *such that for every* $p \in (2, 2+\eta)$ *there exists an* $\varepsilon > 0$ *with the property that for* $\theta \in [\vartheta(p,d), \vartheta(p,d)+\varepsilon)$ *and* $a \in [\underline{a}(\theta,p), \underline{a}(\theta,p)+\varepsilon)$*, no extremal for (CKN) corresponding to the parameters* (θ, p, a) *is radially symmetric*

There is always an extremal function for (CKN) if $\theta > \vartheta(p,d)$, and also in some cases if $\theta = \vartheta(p,d)$

Summary (1/2): Existence for (CKN)



The zones in which existence is known are:

(1) $a \geq a_0$: extremals are achieved among radial functions, by the Schwarz symmetrization method

(1)+(2) $a > a_1$: this follows from the explicit *a priori* estimates; $\Lambda_1 = (a_c - a_1)^2$

(1)+(2)+(3) $a > a_\star^{\mathrm{CKN}}$: this follows by comparison of the optimal constant for (CKN) with the optimal constant in the corresponding Gagliardo-Nirenberg-Sobolev inequality

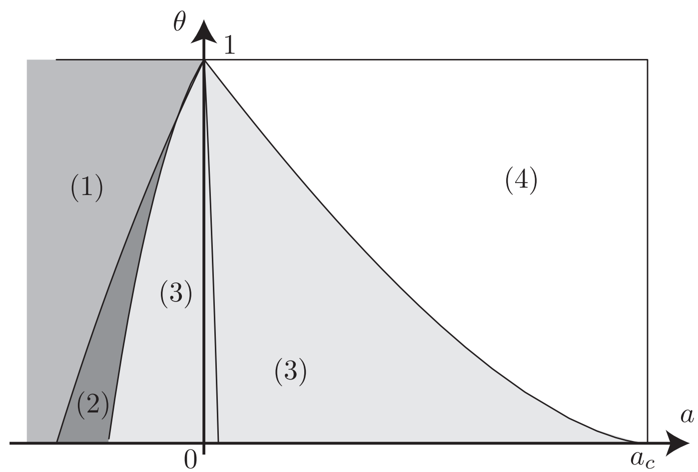
Summary (2/2): Symmetry and symmetry breaking for (CKN)

The zone of symmetry breaking contains:

(1) $a < \underline{a}(\theta, p)$: by linearization around radial extremals

(1)+(2) $a < a_\star^{\mathrm{CKN}}$: by comparison with the Gagliardo-Nirenberg-Sobolev inequality

In (3) it is not known whether symmetry holds or if there is symmetry breaking, while in (4), that is, for $a_0 \le a < a_c$, symmetry holds by the Schwarz symmetrization



Thank you !